Idiosyncratic risk and economic policy

Andrés Carvajal [1]   Herakles Polemarchakis [2]

December 2, 2009

[1]Department of Economics and CRETA, University of Warwick; a.m.carvajal@warwick.ac.uk

[2]Department of Economics and CRETA, University of Warwick; h.polemarchakis@warwick.ac.uk

Abstract

In economies subject to aggregate as well as uninsurable idiosyncratic risks, competitive equilibrium allocations are constrained inefficient: reallocations of assets support Pareto superior allocations. This is the case even if the asset market for the allocation of aggregate risks is complete.

Key words: uninsurable idiosyncratic risks; constrained suboptimality.

JEL classification numbers: D52, D60, H20

Idiosyncratic risk does not affect the allocation of resources at Pareto optimal allocations. [1] Competitive equilibria inherit this property if the asset market for the insurance of idiosyncratic risk is complete. But if realizations of idiosyncratic shocks are publicly unobservable or unverifiable, idiosyncratic risks may well not be insurable; indeed, this is a standard assumption in macroeconomics. [2]

We show, here, that in economies subject to aggregate as well as uninsurable idiosyncratic risk, competitive equilibrium allocations are constrained inefficient: reallocations of assets support Pareto superior allocations. This is the case even if the asset market for the allocation of aggregate risks is complete.

Our argument extends the argument for constrained suboptimality in economies with an incomplete asset market given in Geanakoplos and Polemarchakis (1986). [3] There, it is shown that when financial markets are incomplete competitive equilibrium allocations are constrained inefficient, generically in the spaces of economies. That result, however, does not imply that constrained inefficiency holds for economies with idiosyncratic risk: the particular structure of these economies is non-generic. [4] Here, we show that, nevertheless, generically in the set of economies with idiosyncratic risk, equilibrium allocations are, indeed, constrained suboptimal. Our result is in line with Dávila et al.(2005), where it is shown that the level of capital accumulation of a competitive economy with uninsurable idiosyncratic shocks can differ from the efficient level.

The identification of Pareto improving asset reallocations from market data remains an issue: because of the particular aggregation structure of idiosyncratic shocks, neither the positive identification results of Kübler et al (2002) and of Carvajal and Riascos (2008), nor the negative results of Carvajal and Polemarchakis (2007) apply.

The demonstration that competitive equilibria in economies with uninsurable idiosyncratic shocks are constrained suboptimal makes an important methodological point relevant for economic policy. Intervention is often said

[1] Arrow and Lind (1970), Malinvaud (1973a and 1973b.

[2] Krebs (2003), Krussell and Smith (1999).

[3] Also, Citanna, Kajii and Villanacci (1998).

[4] Put another way, the existing results allow for individual perturbations of preferences and endowments. If one is to distinguish idiosyncratic and aggregate risk, though, one must impose that some individuals, given an aggregate shock, face exactly the same idiosyncratic shocks and have the same preferences.

to be counterproductive because competitive equilibrium cannot be Pareto improved; our theorem shows that such a view is untenable.

The paper is organized as follows: In the first section, we present two examples that illustrate the mechanism by which the general result of constrained suboptimality holds; the first of these examples is for an economy exactly like the one for which we will prove general results, while the second example extends the analysis for the case of a storage economy. The following two sections introduce the general kind of economies in which our analysis holds, and define competitive equilibrium and Pareto efficiency for this kind of economies. Section 4 introduces the definition of constrained inefficiency for economies with uninsurable idiosyncratic risk, and states the main theorem, whose proof requires a construction and argument that are given in Section 5. Then, Section 6 presents two more examples of constrained suboptimality; the first extends the analysis to an economy with a production technology more general than storage, while the second example considers an economy of overlapping generations, and briefly assesses the extent to which the classical *Golden Rule* applies in the presence of idiosyncratic risk. A technical appendix completes the paper.

1 Some examples

In this section we introduce two simple examples that illustrate the main result of the paper: in the absence of insurance opportunities for idiosyncratic risk, competitive markets typically induce constrained suboptimal allocations of commodities. The first example considers a simplified economy with two types of individuals, one of whom faces uninsurable idiosyncratic risk; the example is simple enough to allow for the computation of a closed-form solution that illustrates the mechanism by which a Pareto improvement may be induced after an asset reallocation policy. The second example illustrates how this result can be extended to the case where a storage technology is available; this example is presented for general preferences first, but a specific functional form is used, again, for the purpose of obtaining an explicit solution. Storage allos constrained suboptimality to prevail even in the absence of (ex ante) heterogeneity among individuals.

1.1 A heterogeneous exchange economy

One commodity is exchanged and consumed in the first period, date 0, while two commodities are exchanged and consumed at date 1, the second period. We denote the quantities of the date-zero commodity by x; for simplicity, we index the two commodities of date 1 by $l = a, b$, and denote their quantities by x_a and x_b.

There are two types of individuals, indexed by $i = \alpha, \beta$, and each type consists of a continuum of individuals of unit mass. The intertemporal utility function of an individual of type β is

$$u^\beta(x, x_a, x_b) = x + (1-\gamma)\ln x_a + \gamma \ln x_b,$$

where, $0 < \gamma < 1$, and his endowment at date 1 consists of b units of commodity b.[5] The intertemporal utility function of an individual of type α is

$$u^\alpha(x, x_a, x_b) = x + \gamma \ln x_a + (1-\gamma)\ln x_b,$$

and his endowment at date 1 consists only of commodity a; but, importantly, it is subject to idiosyncratic shocks: it is $a \pm \varepsilon$, with equal probability.[6]

Importantly, at date 0 individuals can trade only in the consumption good and a risk-free bond that matures at date 1. Letting the consumption good be numèraire in the first period, we denote by q the price of the bond. At date 1, individuals only trade on the two commodities; letting commodity a be numèraire, we denotre the price of commodity b by p.

By direct computation, if holdings of the bond are y for individuals of type α and $-y$ for individuals of type β, the equilibrium price at date 1 is

$$p(y) = \frac{(1-\gamma)a + (1-2\gamma)y}{(1-\gamma)b},$$

which depends non-trivially on asset holdings as long as $\gamma \neq 1/2$, a condition that we now impose. At date 1, then, the marginal utility of revenue for individuals of type β is

$$\lambda^\beta = \frac{1}{pb - y},$$

[5] With quasi-linear preferences, it is not necessary to specify the endowments of individuals at date 0.

[6] At date 1 equal proportions of individuals of type α have endowments $a+\varepsilon$ and $a-\varepsilon$, and, as a consequence, there is no aggregate risk.

while for individuals of type α it varies with the realization of the idiosyncratic shock – the personal state of each individual – and is

$$\lambda^\alpha(\varepsilon) = \frac{1}{a+\varepsilon+y}$$

or

$$\lambda^\alpha(-\varepsilon) = \frac{1}{a-\varepsilon+y},$$

with equal probability. The optimization of individuals of type β at date 0 requires, therefore, that

$$q = \frac{1}{pb-y} = \frac{(1-\gamma)}{(1-\gamma)a-\gamma y},$$

while optimization of individuals of type α requires that

$$q = (\frac{1}{2})\frac{1}{a+\varepsilon+y} + (\frac{1}{2})\frac{1}{a-\varepsilon+y} = \frac{a+y}{(a+y)^2-\varepsilon^2}.$$

As a consequence, at equilibrium,

$$y = \frac{-a+\sqrt{a^2+4\varepsilon^2(1-\gamma)}}{2}.$$

A policy intervention perturbs assets holdings and makes revenue transfers at date 0: policy is a pair $(\mathrm{d}x, \mathrm{d}y)$ of transfers of revenue and bonds to individuals of type α. The welfare effects of a policy are

$$\mathrm{d}u^\alpha = \mathrm{d}x + q\mathrm{d}y - \frac{1}{2}(\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + \lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon))p'\mathrm{d}y,$$

and

$$\mathrm{d}u^\beta = -\mathrm{d}x - q\mathrm{d}y - \lambda^\beta(x_b^\beta - b)p'\mathrm{d}y.$$

Pareto improving interventions exist if the matrix

$$\begin{pmatrix} 1 & q - \frac{1}{2}(\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + \lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon))p' \\ -1 & -q - \lambda^\beta(x_b^\beta - b)p' \end{pmatrix}$$

is nonsingular, which is the case for *any* $\varepsilon \neq 0$.[7]

[7] Singularity of the matrix would occur if, and only if,

$$\frac{1}{2}\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + \frac{1}{2}\lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon) = -\lambda^\beta(x_b^\beta - b),$$

In order to find the type of policy that induces a Pareto improvement, we write

$$\mathrm{d}y = \frac{\mathrm{d}u^\alpha + \mathrm{d}u^\beta}{(\Lambda^\alpha + \Lambda^\beta)p'},$$

where

$$\Lambda^\alpha = -\frac{1}{2}(\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + \lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon))$$

and

$$\Lambda^\beta = -\lambda^\beta(x_b^\beta - b).$$

By direct computation, $\Lambda^\alpha + \Lambda^\beta > 0$ if, and only if,

$$\frac{1}{p} < \frac{b}{pb - y},$$

which is the case since $y > 0$. It follows, then, that the sign of a Pareto improving $\mathrm{d}y$ is the same as the sign of p', namely positive if $\gamma < 1/2$, and negative if $\gamma > 1/2$.

1.2 A storage economy

The setting is again of two-periods, but, now, that there is only a continuum of ex-ante identical individuals. Suppose that one commodity is available for exchange and consumption in date 0, and assume that this good is storable. At date 1 there are two commodities: individuals trade and consume what they have stored of the first commodity, along with any further endowment they may have, and they also exchange and consume a second commodity.

Denote by k the amount of commodity that is stored by each individual at date 0, and by e_2 the endowment of commodity 2 they all receive at date 1. But assume the endowment of commodity 1 is subject to idiosyncratic risk: for an individual in personal state s, the endowment of commodity 1 is $e_{1,s}$. Let $\pi_{i,s}$ be the proportion of individuals in state s, and, for simplicity of notation, let $e_1 = \mathrm{E}[e_{1,s}]$.[8]

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which is equivalent to

$$\frac{1-\gamma}{p} = -\frac{1}{pb - y}\left(\frac{\gamma(pb - y)}{p} - b\right),$$

and, hence, to $y = 0$, which occurs only in the absence of idiosyncratic shocks, with $\varepsilon = 0$.

[8]Throughout the paper, the probability measure with respect to which the expectation is taken is left implicit. That is, by $\mathrm{E}[e_{1,s}]$ we mean $\mathrm{E}_\pi[e_{1,s}]$.

With Bernoulli utility indices v_s, the individual's von Neumann-Morgestern ex-ante utility is

$$u(k, x) = -k + \mathrm{E}[v_s(x_s)],$$

where $x_s = (x_{1,s}, x_{2,s})$ denotes the individuals' consumption at date 1 in state s. And if we let commodity 1 be numèraire at date 1, so that prices can be denoted by $(1, p)$, then the nominal wealth of an individual in personal state s will be $e_{1,s} + k + pe_2$. Letting λ_s denote the marginal utility of income in personal state s, it then follows that the first-order condition for optimization at date 0 is that $\mathrm{E}[\lambda_s] = 1$, and, as a consequence, that the ex-ante utility impact of an infinitesimal perturbation to the level of savings, $\mathrm{d}k$, around its competitive equilibrium level, is

$$\mathrm{d}u = -\mathrm{d}k + \mathrm{E}[\lambda_s((-x_{2,s} + e_2)\mathrm{d}p + \mathrm{d}k)] = \mathrm{E}[\lambda_s(-x_{2,s} + e_2)]\mathrm{d}p.$$

Under certainty, market-clearing implies that $\mathrm{E}[\lambda_s(-x_{2,s} + e_2)] = 0$, so it is impossible to improve ex-ante utility by the implementation of levels of savings different from the ones chosen under competitive equilibrium. But, if date-1 spot prices depend on the effective endowment of commodity 1, then the equilibrium utility level can be improved upon if

$$\mathrm{E}[\lambda_s(-x_{2,s} + e_2)] \neq 0.$$

As in the previous example, we can impose a simple structure in order to get a closed-form solution. First, suppose that the Bernoulli indices are state-independent and equal

$$v(x_s) = \ln x_{1,s} + \ln x_{2,s}.$$

By direct computation, given savings of k, equilibrium prices are $p = (e_1 + k)/e_2$, an expression that depends positively on k. Also, then, $\lambda_s = 2/(e_{1,s} + e_1 + 2k)$ and

$$e_2 - x_{2,s} = \frac{e_2}{2}\left(\frac{e_1 - e_{1,s}}{e_1 + k}\right),$$

so if we further assume that there are only two personal states, writing that $e_{1,s} = e_1 \pm \varepsilon$, and take that these states occur with equal probability, then we get that

$$\mathrm{E}[\lambda_s(e_2 - x_{2,s})] = \frac{1}{e_1 + k}\left(\frac{\varepsilon^2}{4(e_1 + k)^2 - \varepsilon^2}\right).$$

Now, at the equilibrium level of savings, it must be true that $\mathrm{E}[\lambda_s] = 1$. By direct computation, this implies that

$$\mathrm{E}[\lambda_s(e_s - x_{2,s})] = \frac{\varepsilon^2}{4(e_1 + k)^2} > 0,$$

which means that in this economy individuals *underinvest* at competitive equilibrium.

2 The economy

Consider an economy that evolves over two dates, 0 and 1. In the society of this economy, individuals are of different types, denoted by $i = 1, \ldots, I$, and within each type there is a continuum of individuals of mass 1. Individuals of a given type are ex-ante identical, but can face different idiosyncratic shocks at date 1: each individual may find herself in any one of a set of different personal states, which we index by $s = 1, \ldots, S$. We denote the distribution of individuals of type i across personal states by the vector $\pi^i = (\pi^i_1, \ldots, \pi^i_S) \gg 0$: in date 1, a fraction π^i_s of individuals of type i *will* find themselves in personal state s.

There is a finite number of commodities in the economy, denoted by $l = 1, \ldots, L$, and individuals consume these commodities in both dates. At date 0, the endowment of an individual depends on her type, and we denote it by the bundle e^i_0 of commodities. At date 1, individual endowments depend on the type and are subject to idiosyncratic risk: in personal state s, an individual of type i is endowed with a bundle e^i_s. We summarize this information by letting $e^i = (e^i_0, \ldots, e^i_s)$ denote the endowment of an individual of type i,[9] and assume that each person will have strictly positive amounts of all commodities at date 0 and in all personal states at date 1.

It is important to notice that while a person's date-1 endowment depends on her type and on the idiosyncratic shock that is realized for her, at the macroeconomic level there is no risk:[10] the aggregate endowment of the economy is $\sum_i \sum_{s=1}^S \pi^i_s e^i_s$.

[9]For simplicity of notation, we identify date 0 with state $s = 0$, whenever there is no possibility of confusion.

[10]All the results in the paper are true in the presence of aggregate risk, even if this risk is fully insurable, as long as idiosyncratic risk remains uninsurable. With aggregate risk, the presentation of the problem and the results is more difficult, but the proofs of the theorems are, in fact, simpler.

An individual's consumption plan is $x = (x_0, \ldots, x_S)$, where each x_s is a bundle of commodities. We assume that the preferences of an individual over consumption plans are represented by the utility function[11]

$$u^i(x) = u_0^i(x_0) + \sum_{s=1}^{S} \pi_s^i u_1^i(x_s),$$

where the temporal, cardinal utility indices, u_0^i and u_1^i, belong to the class of strictly monotonic, strongly concave, $\mathbf{C}^3$ functions $v : \mathbb{R}_{++}^L \to \mathbb{R}$, that satisfy the following interiority condition: if a sequence $(x_n)_{n=1}^{\infty}$ of strictly positive consumption bundles converges to a boundary bundle, then $||Dv(x_n)||^{-1} Dv(x_n) \cdot x_n \to 0$ and $||Dv(x_n)||^{-1} \to \infty$. We denote this class of functions by $\mathcal{U}$, and endow it with the topology of $\mathbf{C}^3$, uniform convergence on compacta.[12]

An *economy* is completely described by the profile of endowments and preferences, $(e, u) = ((e^1, u^1), \ldots, (e^I, u^I))$.[13] The space of economies is endowed with the product topology.

We make the following assumptions: (i) individuals are ex-ante heterogeneous, in the sense that $I \geq 2$; (ii) idiosyncratic risk exists, as $S \geq 2$; and (iii) the set of commodities is diverse, in the sense that $L \geq IS$.

3 Competitive equilibrium and Pareto efficiency

Given an economy, we consider only allocations of commodities that treat all individuals of the same type symmetrically. Thus, an *allocation* for economy (e, u) is a profile $x = (x^1, \ldots, x^I)$ that specifies a consumption plan for each type; the allocation is *feasible* if $\sum_i x_0^i = \sum_i e_0^i$ and $\sum_i \sum_{s=1}^{S} \pi_s^i x_s^i = \sum_i \sum_{s=1}^{S} \pi_s^i e_\sigma^i$. The definition of a *Pareto efficient allocation* is the usual one, restricted to the class of symmetric allocations that we allow.[14]

[11] Again, the arguments given for our results would be simpler if we did not assume that preferences are additively separable.

[12] See Aliprantis and Border (1999, §3.17).

[13] We will maintain the distributions of individuals of a given type across personal states fixed, so, for simplicity, we exclude these parameters from the definition of an economy.

[14] The usual intuition that at Pareto efficient allocations individual utility functions must have collinear gradients carries over to the present context, but applies in a strong sense: at efficient allocations idiosyncratic risk is fully shared, so (i) for a given type, consumption

We want to consider the effects of uninsurable idiosyncratic risk, so we assume that only a riskless asset can be traded: there is only one financial asset in the economy; it pays one unit of commodity 1 at date 1. We allow for trade in no other assets, in particular in assets that insure against idiosyncratic risk. Holdings of the asset will be denoted by y, while its price will be denoted as q.

Besides assets, people trade commodities in spot markets. Prices of commodities are $p_0 = (1, \ldots, p_{0,l}, \ldots)$ at date 0, and $p_1 = (1, \ldots, p_{1,l}, \ldots)$ at date 1;[15] across dates, prices of commodities are $p = (p_0, p_1)$. We identify commodity 1 as the numèraire of the economy at each spot, and normalize its price to be 1. All other prices are taken to be strictly positive, and we denote by $\mathcal{P}$ the set of normalized spot prices, so that, across date events, $p \in \mathcal{P}^2$.

Given prices p and q, an individual chooses a consumption plan that maximizes her ex-ante utility, and holdings of the asset that make her consumption plan financially feasible. That is, an individual of type i will choose a plan x and holdings y subject to the constraints that: (i) at date zero, she must be able to afford her portfolio along with current consumption: $p_0(e_0^i - x_0) = qy$; and (ii) in each personal state at date 1, the return of the portfolio, which is simply her asset holdings, must cover the value of her planned consumption, beyond her endowments: $p_1(x_s - e_s^i) = y$, for each personal state s.[16]

We can simplify the expression of the problem by using matrix notation. Define the $(S+1) \times L(S+1)$ matrix

$$\Psi(p) := \begin{pmatrix} p_0 & 0 & \ldots & 0 \\ 0 & p_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_1 \end{pmatrix}$$

must be invariant across personal states; and (ii) across types, gradients must be collinear, even for different personal states, given an aggregate state. Formally it suffices to observe that if allocation x is Pareto efficient, then there are strictly positive numbers γ^i, one for each type, such that $\gamma^i Du_0^i(x_0^i) = \gamma^{i'} Du_0^{i'}(x_0^{i'})$ and $\gamma^i Du_1^i(x_s^i) = \gamma^{i'} Du_1^{i'}(x_{s'}^{i'})$, for all pairs of types, i and i', and all pairs of personal states, s and s'.

[15]For simplicity of notation, price vectors and gradients of utility functions are taken as row vectors, whereas quantities are taken as column vectors.

[16]The unique optimizer for individuals of type i, (x^i, y^i) is characterized by the following first-order conditions: for a vector $\lambda^i = (\lambda_0^i, \ldots, \lambda_S^i) \gg 0$,
(i) $Du_0^i(x_0^i) = \lambda_0^i p_0$, and $\pi_s^i Du_1^i(x_s^i) = \lambda_s^i p_1$ for all $s = 1, \ldots, S$;
(ii) $\lambda_0^i q = \sum_{s=1}^S \lambda_s^i$; and
(iii) $p_0(e_0^i - x_0^i) = qy^i$, and $p_1(x_s^i - e_s^i) = y^i$ at all $s = 1, \ldots, S$.

and the $(S+1) \times 1$ vector $R(q) := (-q, 1, \ldots, 1)^\intercal$. Then, the optimization problem of an individual of type i is simply

$$\max_{x,y} u^i(x) : \Psi(p)(x - e^i) = R(q)y,$$

and the first-order necessary and sufficient conditions are that for some strictly positive vector $\lambda^i = (\lambda^i_0, \ldots, \lambda^i_S)$, we have that $Du^i(x^i) = \lambda^i \Psi(p)$ and $\lambda^i R(q) = 0$, while $\Psi(p)(x^i - e^i) = R(q)y^i$.

For economy (e, u), a *competitive equilibrium* consists of an allocation of commodities, one of asset holdings, a vector commodity prices and a price for the riskless asset, such that all the types of individuals optimize and all markets clear. For a more operational characterization of equilibria, it is convenient for us to make the vectors of individual marginal utilities of income part of the definition of equilibrium, and to let the first-order conditions of the optimization problems account for individual rationality. Formally, thus, we define the function[17]

$$\mathcal{F}(x, \lambda, y, p, q, e, u) := \begin{pmatrix} \vdots \\ (Du^i(x^i) - \lambda^i \Psi(p))^\intercal \\ R(q)y^i - \Psi(p)(x^i - e^i) \\ (\lambda^i R(q))^\intercal \\ \vdots \\ \sum_i (\tilde{e}^i_0 - \tilde{x}^i_0) \\ \sum_{i,s} \pi^i_s (\tilde{e}^i_s - \tilde{x}^i_s) \\ \sum_i y^i \end{pmatrix},$$

where, here and henceforth, $\tilde{e}$ and $\tilde{x}$ exclude the numèraire commodity at all states, and define an equilibrium for economy (e, u) as a 5-tuple (x, λ, y, p, q) such that $\mathcal{F}(x, \lambda, y, p, q, e) = 0$.

Under our assumptions, ompetitive equilibrium is guaranteed to exist for any economy. Importantly, the following result shows that, given any profile

[17]The domain of $\mathcal{F}$ is

$$\mathbb{R}^{I(S+1)L}_{++} \times \mathbb{R}^{I(S+1)}_{++} \times \mathbb{R}^I \times \mathcal{P}^2 \times \mathbb{R} \times \mathbb{R}^{I(S+1)L}_{++} \times \mathcal{U}^{2I}.$$

It maps into

$$(\mathbb{R}^{(S+1)L}_{++} \times \mathbb{R}^{S+1}_{++} \times \mathbb{R})^I \times \mathbb{R}^{2(L-1)}_{++} \times \mathbb{R}.$$

of preferences, there are only finitely many competitive equilibria, generically on endowments.

THEOREM **1** *For any profile of preferences, the set of profiles of endowments for which the economy has finitely many equilibria is open and has full Lebesgue measure.*

Proof: This argument is well known, so details are omitted. It follows from the fact that, given u, function $\mathcal{F}(\cdot, u)$ is transverse to 0, and from the Transversality Theorem (see, for instance, Guillemin and Pollack, 1974). *Q.E.D.*

Henceforth, a subset of a finite-dimensional Euclidean space is said to be *strongly generic* if it is open and has full Lebesgue measure (its complement has null measure), and a subset of an abstract metric space is said to be *generic* if it is open and dense.

It follows that the model under consideration preserves the standard positive properties of the GEI model with numèraire assets: equilibrium always exists, and there are only finitely many equilibria in a strongly generic set of endowments, for any given profile of preferences. Since idiosyncratic shocks are nontrivial (and uninsurable), it is reasonable to expect that competitive equilibria be inefficient. We now show that, also generically, inefficiency of competitive equilibrium holds in a strong sense: since idiosyncratic risk is nontrivial, generically in endowments equilibrium consumption depends on all idiosyncratic shocks, for all types of individuals, and is therefore Pareto inefficient.

THEOREM **2** *For any profile of preferences, there exists a strongly generic subset of profiles of endowments for which at every competitive equilibrium allocation, x, one has that $x_s^i \neq x_{s'}^i$, for all types i and all personal states $s, s' = 1, \ldots, S$, $s \neq s'$.*

Proof: For each type i, and each pair of date-1 personal states, $s \neq s'$, define the function

$$(x, y, \lambda, p, q, e) \mapsto \begin{pmatrix} \mathcal{F}(p, q, x, y, \lambda, e, u) \\ \frac{\lambda_s^i}{\pi_s^i} - \frac{\lambda_{s'}^i}{\pi_{s'}^i} \end{pmatrix}.$$

By direct computation, this mapping is transverse to 0, so, on a strongly generic set of endowments, the mapping is transverse to 0 as a function of

(x, y, λ, p, q) only. Since (x, y, λ, p, q) contains one fewer argument than the mapping has components, this implies implies that $\lambda_s^i / \pi_s^i \neq \lambda_{s'}^i / \pi_{s'}^i$ whenever $\mathcal{F}(x, y, \lambda, p, q, e, u) = 0$. By the first-order conditions of individual optimization, it follows that $x_s^i \neq x_{s'}^i$ at every equilibrium allocation for endowments in this strongly generic set. The intersection the sets constructed in this way for all (i, s, s') is strongly generic. *Q.E.D.*

4 Constrained inefficiency

The fact that an allocation is Pareto inefficient says that a reallocation of consumption plans that is feasible from the point of view of the aggregate resources available to the society, can improve ex-ante wellbeing for all types. This does not say, however, that one such reallocation exists which can be implemented through the existing financial instruments.

An allocation x is **constrained-inefficient** if there exist commodity prices $\hat{p}$, a commodity allocation $\hat{x}$, date-zero revenue transfers $(\hat{\tau}^1, \ldots, \hat{\tau}^I)$ and an asset allocation $(\hat{y}^i, \ldots, \hat{y}^I)$ such that:
1. revenue transfers are balanced: $\sum_i \hat{\tau}^i = 0$;
2. the asset allocation is feasible: $\sum_i \hat{y}^i = 0$;
3. individual consumptions are optimal (given prices and wealth): for every i, $\hat{x}_0^i$ solves the problem

$$\max_{x'} u_0^i(x') : \hat{p}_0 x' \leq \hat{p}_0 e_0^i + \hat{\tau}^i,$$

and each $\hat{x}_s^i$ solves

$$\max_{x'} u_1^i(x') : \hat{p}_1 x' \leq \hat{p}_1 e_s^i + \hat{y}^i;$$

4. all markets clear: $\sum_i (e_0^i - \hat{x}_0^i) = 0$ and $\sum_i \sum_{s=1}^S \pi_s^i (e_s^i - \hat{x}_s^i) = 0$; and
5. every individual is ex-ante better off at $\hat{x}$: for every i, $u^i(\hat{x}^i) > u^i(x^i)$.

This is, an allocation is constrained inefficient if a reallocation of wealth, via revenue at date zero and the riskless asset at date one, and competitive trade in the commodity markets can make all types of individual ex-ante better off (condition 5). Conditions 1 and 2 imply that the reallocation is balanced, condition 3 implies that all individuals are rational in the commodity markets, which clear by condition 4.

The main result of this paper is that, typically, equilibrium allocations are constrained inefficient. The theorem has the important implication that just by trading the risk-free asset differently, all types of individuals in the society could be made ex-ante strictly better off.

THEOREM **3** *There exists a generic subset of economies, $\mathcal{D}$, where every equilibrium allocation is constrained inefficient: for every $(e, u) \in \mathcal{D}$, if (x, λ, y, p, q) is an equilibrium for (e, u), then x is constrained inefficient.*

The proof of the theorem exploits the idea of Geanakoplos and Polemarchakis (1996), in the technique of Citanna et al (1998): by perturbing the Hessians of the utility functions, in a local, finite-dimensional subspace of economies, we can change the shape of the demand functions, without changing their level at given prices and endowments (which is makes the set of equilibria invariant in the subspace). These perturbations are used to imply that, generically, relative prices can induce income reallocations beyond the span of the existing asset. For the argument to hold, sufficiently many relative prices (hence commodities) are needed. The argument, which invokes the Transversality Theorem once more, is applied on the finite-dimensional subspace, locally, to obtain constrained inefficiency in a strongly generic subset of that subspace; for the global space of economies, the latter local result implies denseness and hence genericity.

5 Genericity of constrained inefficiency

We first present a series of properties that hold generically at equilibrium, and which will be useful in the proof of Theorem 3. With these results, we construct, for any economy in a generic set a lower-dimensional neighborhood of economies. In order to prove that the set of economies where all equilibrium allocations are constrained inefficient is dense, we will show that for any economy in the generic set we can find an economy in the lower-dimensional neighborhood, arbitrarily close to it, where the property holds true.

5.1 A characterization of constrained suboptimality

Define the function

$$\mathcal{H}(x,\lambda,p,y,\tau,e,u) := \begin{pmatrix} \vdots \\ u^i(x^i) \\ \vdots \\ (Du^i(x) - \lambda\Psi(p))^\mathsf{T} \\ \Psi(p)(x^i - e^i) - (\tau^i, 1, \ldots, 1)^\mathsf{T} \\ \vdots \\ \sum_i (\tilde{e}_0^i - \tilde{x}_0^i) \\ \sum_i \sum_{s=1}^S \pi_s^i (\tilde{e}_s^i - \tilde{x}_s^i) \\ \sum_i \tau^i \\ \sum_i y^i \end{pmatrix}.$$

This function plays a role similar to $\mathcal{F}$, in the sense that it will make the definition of constrained inefficiency operational, but three differences deserve to be pointed out. First, the price of the asset is not an argument, and the block of components that includes the first-order conditions of consumers does not include no-arbitrage conditions for assets: in the alternative plan that makes an allocation constrained inefficient, asset holdings are not being traded or determined by individual optimization. And secondly, the first I components of $\mathcal{H}$ are the types' utility levels, which did not appear in $\mathcal{F}$; these components, which did not appear in $\mathcal{F}$, will used to determine the welfare effects of asset reallocations. Finally, the previous-to-last component of $\mathcal{H}$ captures whether date-0 revenue transfers are balanced.

Lemma 1 *If (x,λ,y,p,q) is an equilibrium for economy (e,u) and the matrix*

$$D_{x,\lambda,p,y,\tau}\mathcal{H}(x,\lambda,p,y,(-qy^i)_{i=1}^I,e,u)$$

has full (row) rank, then allocation x is constrained inefficient.

The proofs of this and all the other lemmas in the paper are deferred to an appendix.

5.2 Finite-dimensional subspaces of economies

5.2.1 Some auxiliary results

All results so far have been independent of our assumptions that commodities are sufficiently diverse. That assumption, however, is crucial in the following lemma. Define the $(L-1)\times IS$ matrix

$$\mathbf{Z}(x,e) := (\ \tilde{e}^1_1-\tilde{x}^1_1 \quad \ldots \quad \tilde{e}^1_S-\tilde{x}^1_S \quad \tilde{e}^2_1-\tilde{x}^2_1 \quad \ldots \quad \tilde{e}^I_S-\tilde{x}^I_S\ ),$$

The assumption that commodities are diverse gives that at equilibrium, generically on endowments, there exists no non-null vector θ, orthogonal to $(\pi^1_1,\ldots,\pi^1_S,\pi^2_1,\ldots,\pi^I_S)$ such that $\mathbf{Z}(x,e)\theta=0$.

LEMMA **2** *For any profile of preferences, on a strongly generic subset of endowments, matrix $\mathbf{Z}(x,e)$ has rank $IS-1$ at every competitive equilibrium.*

Another auxiliary function will be obtained by looking only at the lower block of components of function $\mathcal{H}$, namely the conditions that define competitive equilibrium, in commodity trade only, for a given, feasible, allocation of date-0 revenue and holdings of the riskless asset:

$$\mathcal{G}(x,\lambda,p,y,\tau,e,u) := \begin{pmatrix} \vdots \\ (Du^i(x)-\lambda\Psi(p))^{\mathsf{T}} \\ \Psi(p)(x^i-e^i)-(\tau^i,1,\ldots,1)^{\mathsf{T}} \\ \vdots \\ \sum_i(\tilde{e}^i_0-\tilde{x}^i_0) \\ \sum_{i,s}\pi^i_s(\tilde{e}^i_s-\tilde{x}^i_s) \\ \sum_i \tau^i \\ \sum_i y^i \end{pmatrix}.$$

The set of economies defined in the following lemma will be the basis of our local analysis afterwards.

LEMMA **3** *There exists a generic subset of economies, $\mathcal{D}_r$, where there is only a finite number of equilibria and, at all these equilibria,*
1. matrix $D_{x,\lambda,y,p,q}\mathcal{F}(x,\lambda,y,p,q,e,u)$ has full rank;
2. for every i, s and s', with $s\neq s'$, $\lambda^i_s/\pi^i_s\neq\lambda^i_{s'}/\pi^i_{s'}$ and $x^i_s\neq x^i_{s'}$;
3. matrix $\mathbf{Z}(x,e)$ has rank $IS-1$; and
4. matrix $D_{x,\lambda,p,y,\tau}\mathcal{G}(x,\lambda,p,y,-(qy^i)^I_{i=1},e,u)$ has full rank.

5.3 Finite-dimensional subspaces of economies

For any given economy satisfying the properties of Lemma 3, we will construct a neighborhood of economies where the set of equilibria is the same.

These economies will have the same endowments, but different preferences; the change in preferences will only be in their second derivatives at equilibrium consumptions, which will imply the invariance of equilibria. The construction will first fix a set of commodity bundles at which we will perturb the utility functions; then, we will introduce the perturbations to these functions.

The construction is local, so we start by fixing an economy $(\bar{e}, \bar{u})$ in the set $\mathcal{D}_r$ defined in Lemma 3. Let $\mathcal{E}$ denote the set of competitive equilibria of this economy, and notice that, since this set is finite, we can isolate its points in open balls of sufficiently small radius: in particular, let $\epsilon > 0$ be such that the open balls of radius 2ϵ around each equilibrium contain no other equilibria.[18]

5.3.1 Step 1: Bundles where preferences will be perturbed

We will perturb the utility functions in neighborhoods of observed consumption levels. For this construction, let X_0^i and X_1^i, for each type i, be the levels of date-0 and date-1 consumption bundles obtained at some equilibrium: formally,

$$X_0^i := \{x \in \mathbb{R}^L | \bar{x}_0^i = x \text{ for some } (\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}) \in \mathcal{E}\}$$

and

$$X_1^i := \{x \in \mathbb{R}^L | \bar{x}_s^i = x \text{ for some } (\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}) \in \mathcal{E} \text{ and some } s = 1, \ldots, S\}.$$

By the properties of economy $(\bar{e}, \bar{u})$, we know that all sets X_0^i and X_1^i are finite, so one can find $\epsilon^i > 0$ such that the open balls of radius $2\epsilon^i$ isolate these points from one another.[19]

5.3.2 Step 2: Perturbations of preferences

For each $\epsilon > 0$, let $\rho_\epsilon : \mathbb{R}^L \to [0, 1]$ denote a $\mathbf{C}^\infty$ function such that $\rho_\epsilon(\delta) = 1$ in $B_\epsilon(0)$ and $\rho_\epsilon(\delta) = 0$ outside $B_{2\epsilon}(0)$.[20] Now, let $(\Delta_{\bar{x}})_{\bar{x} \in X_0^i}$ be an array of

[18] That is, that for every equilibrium $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}) \in \mathcal{E}$, one has that

$$B_{2\bar{\epsilon}}(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}) \cap \mathcal{E} = \{(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q})\}.$$

[19] As before, this means that for each $\bar{x} \in X_0^i$, one has that $B_{2\epsilon^i}(\bar{x}) \cap X_0^i = \{\bar{x}\}$, and similarly for bundles in X_1^i.

[20] See Guillemin and Pollack (1974), or Citanna et al. (1998).

symmetric, $L \times L$ matrices with norm less than $\delta > 0$;[21] this array contains one matrix for each of the equilibrium levels of consumption of individuals of type i at date 0. For δ small enough, the function

$$u_0^i(x) = \bar{u}_0^i(x) + \frac{1}{2} \sum_{\bar{x} \in X_0^i} \rho_{\epsilon^i}(x - \bar{x}) \cdot (x - \bar{x})^\top \Delta_{\bar{x}}(x - \bar{x})$$

satisfies all the assumptions we have imposed on utility functions (i.e., it lies in the class $\mathcal{U}$). Of course, we can do the same for date-1 preferences: for an array $(\Delta_{\bar{x}})_{\bar{x} \in X_1^i}$, containing a symmetric matrix for each date-1 equilibrium consumption, we can construct an admissible date-1 utility function for individuals of type i by taking

$$u_1^i(x) = \bar{u}_1^i(x) + \frac{1}{2} \sum_{\bar{x} \in X_1^i} \rho_{\epsilon^i}(x - \bar{x}) \cdot (x - \bar{x})^\top \Delta_{\bar{x}}(x - \bar{x}).$$

And as a result, we obtain for each individual a new ex-ante utility function $u^i(x) = u_0^i(x_0) + \sum_{s=1}^S \pi_s^i u_1^i(x_s)$, which we can vary smoothly simply by changing the arrays of matrices Δ.

More formally, let $\breve{\mathcal{U}}_\delta$ be the set of all profiles of ex-ante utility functions that can be obtained by perturbing the corresponding $\bar{u}_0^i$ and $\bar{u}_1^i$ function in the way we just described; this set is a finite-dimensional submanifold of $\mathcal{U}^{2I}$, parameterized by

$$\mathbb{B}_\delta := \prod_i (B_\delta(0)^{\# X_0^i} \times B_\delta(0)^{\# X_1^i}),$$

with each open ball taken in $\mathbb{R}^{L(L+1)/2}$. Importantly, notice that if one restricts $\mathcal{F}$ to profiles of preferences defined in $\breve{\mathcal{U}}_\delta$ and adopts this parameterization, then $\mathcal{F}$ is twice continuously differentiable. Also, profiles in this set satisfy, by construction, the following properties, for any individual:

1. for any bundle x, there exists at most one perturbation that will be "active," in the sense that there will be at most one $\bar{x} \in X_0^i$ such that

$$u_0^i(x) = \bar{u}_0^i(x) + \frac{1}{2} r \cdot (x - \bar{x})^\top \Delta_{\bar{x}}(x - \bar{x})$$

[21] Formally, each matrix is a real vector with $L(L+1)/2$ components; their norms are computed by treating them as vectors.

for $r > 0$, with a similar result holding for u_1^i; and
2. at every the equilibrium of $(\bar{e}, \bar{u})$ the perturbations affect only the Hessian of the utility functions, and these are perturbed exactly by the corresponding matrix: for each $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}) \in \mathcal{E}$,

$$u_0^i(\bar{x}_0^i) = \bar{u}_0^i(\bar{x}_0^i), Du_0^i(\bar{x}_0^i) = D\bar{u}_0^i(\bar{x}_0^i) \text{ and } D^2u_0^i(\bar{x}_0^i) = D^2\bar{u}_0^i(\bar{x}_0^i) + \Delta_{\bar{x}_0^i},$$

and

$$u_1^i(\bar{x}_s^i) = \bar{u}_1^i(\bar{x}_s^i), Du_1^i(\bar{x}_s^i) = D\bar{u}_1^i(\bar{x}_s^i) \text{ and } D^2u_1^i(\bar{x}_s^i) = D^2\bar{u}_1^i(\bar{x}_s^i) + \Delta_{\bar{x}_s^i}$$

for every date-1 state s.

5.3.3 Invariance of equilibria

By the Implicit Function Theorem, we have the following lemma of local invariance of equilibrium.

LEMMA 4 *There exist $\tilde{\delta} > 0$ and $\tilde{\epsilon} > 0$ such that, for every profile of preferences u in the set $\mathcal{U}_{\tilde{\delta}}$ and every equilibrium $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q})$ in $\mathcal{E}$, one has that*

$$\mathcal{F}(x, \lambda, y, p, q, \bar{e}, u) = 0 \text{ and } (x, \lambda, y, p, q) \in B_{\tilde{\epsilon}}(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q})$$

are true for, and only for, $(x, \lambda, y, p, q) = (\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q})$.

The importance of this construction is, finally, the following theorem of invariance of equilibrium with respect to our local perturbations of preferences.

PROPOSITION 1 *There exists $\delta > 0$ such that for every profile of preferences u in $\mathcal{U}_\delta$, the set of competitive equilibria of economy $(\bar{e}, u)$ is exactly $\mathcal{E}$.*

Proof: Let $\epsilon := \min\{\bar{\epsilon}, \tilde{\epsilon}\} > 0$. By continuity of $\mathcal{F}$, there exists $\bar{\delta} > 0$ such that, for every $\Delta \in B_{\bar{\delta}}(0)$, $\mathcal{F}(x, \lambda, y, p, q, \bar{e}, \Delta) = 0$ implies that

$$(x, \lambda, y, p, q) \in \cup_{(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}) \in \mathcal{E}} B_\epsilon(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}, \bar{q}).$$

Let $\delta := \min\{\bar{\delta}, \tilde{\delta}\}$. Notice that if $\Delta \in \mathbb{B}_\delta$, then, by Lemma 4, the set of equilibria of economy $(\bar{e}, \Delta)$ is a subset of $\mathcal{E}$. The other inclusion is immediate. *Q.E.D.*

5.3.4 The local subspace of economies

Given any economy in $\mathcal{D}_r$, the argument for denseness of the set of economies where all equilibrium allocations are constrained inefficient will look for economies where this property holds in the following "neighborhood" of the given economy.

Lemma 5 *Given an economy, $(\bar{e}, \bar{u})$, and a profile of utility perturbations $\Delta \in \mathbb{B}_\delta$, define the following system of equations:*
(1) for every i and every $s = 1, \ldots, S$,

$$\frac{1}{\lambda_0^i}\pi_s^i D\bar{u}_1^i(x_s^i) + \pi_s^i(D^2\bar{u}_1^i(x_s^i) + \Delta_s^i)\beta_s^i - \gamma_s^i p_1^\mathsf{T} + \pi_s^i \tilde{\mathbb{I}}^\mathsf{T}\mu = 0;$$

(2) for every i and every $s = 1, \ldots, S$, $p_1 \cdot \beta_s^i = 0$;
(3) $\sum_i \sum_{s=1}^S \lambda_s^i \tilde{\mathbb{I}}\beta_s^i + \sum_i \sum_{s=1}^S \gamma_s^i(\tilde{\bar{e}}_s^i - \tilde{\bar{x}}_s^i) = 0$; and
(4) for every i, $\sum_{s=1}^S \gamma_s^i + \eta = 0$,
where $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}$, $\beta_s^i \in \mathbb{R}^L$ and $\gamma_s^i \in \mathbb{R}$. If $(\bar{e}, \bar{u}) \in \mathcal{D}_r$ and δ is chosen as in Proposition 1, then there exists a subset of $\mathcal{U}_\delta$ of preferences that is
(i) strongly generic (as subset of $\mathcal{U}_\delta$), and
(ii) such that if (x, λ, y, p, q) is a competitive equilibrium of economy $(\bar{e}, \Delta)$, then the system above has no solution.[22]

5.4 The proof of Theorem 3

We are now ready to show that the set of economies where all equilibrium allocations are constrained inefficient is dense. Since $\mathcal{D}_r$ is generic, it suffices to show that for each $(\bar{e}, \bar{u}) \in \mathcal{D}_r$ we can find an economy (e, u), arbitrarily close to $(\bar{e}, \bar{u})$, where the property holds. For this, we are going to maintain the endowments fixed, namely $e = \bar{e}$, and are going to look for alternative preferences in the lower-dimensional "neighborhood" of $\bar{u}$ that is defined by Lemma 5. Denote this neighborhood by $\mathcal{N}$. Since $\mathcal{N}$ has full measure, as subset of the local subspace of economies constructed around $(\bar{e}, \bar{u})$, if we find that generically in $\mathcal{N}$, all equilibrium allocations of $(\bar{e}, \Delta)$ are constrained inefficient, we will have proved the Theorem.

[22] Here and henceforth, we identify the profile of matrices Δ with the profile of ex-ante preferences constructed as in §5.3.2 above, given $(\bar{e}, \bar{u}) \in \mathcal{D}_r$.

LEMMA 6 *The function*[23]

$$\mathcal{M}(x,\lambda,y,p,q,\Delta,\theta) := \begin{pmatrix} \mathcal{F}(x,\lambda,y,p,q,\bar{e},\Delta) \\ D_{x,\lambda,p,y,\tau}\mathcal{H}(x,\lambda,p,y,(-qy^i)_{i=1}^I,\bar{e},\Delta)^\top\theta \\ \frac{1}{2}(\theta^\top\theta - 1) \end{pmatrix}.$$

is transverse to 0.

By the Transversality Theorem, Lemma 6 implies that for a generic subset of $\mathcal{N}$, one has that $\mathcal{M}(\cdot,\Delta) \pitchfork 0$. Now, the matrix $D_{x,\lambda,y,p,q,\theta}\mathcal{M}$ has

$$I(S+1)L+I(S+1)+I+2(L-1)+1+I(S+1)L+I(S+1)+2(L-1)+2I+1$$

rows, and only

$$I(S+1)L+(S+1)I+I+2(L-1)+1+I+I(S+1)L+I(S+1)+2(L-1)+1+1$$

columns, and, since $I \geq 2$, it follows that it cannot have full row rank. Then, it must be that on a strongly generic subset of $\mathcal{N}$, the function $\mathcal{M}(\cdot,\Delta)$ is never zero, which means that, in that set, whenever $\mathcal{F}(x,\lambda,y,p,q,\bar{e},\Delta) = 0$, it is also true that the matrix

$$D_{x,\lambda,p,y,\tau}\mathcal{H}(x,\lambda,p,y,(qy^i)_{i=1}^I,\bar{e},\Delta)$$

has full row rank. It follows then from Theorem 1 that x is constrained inefficient, whenever $\mathcal{F}(x,\lambda,y,p,q,\bar{e},\Delta) = 0$.

6 Further examples

We now present two more examples of constrained inefficiency caused by the presence of uninsurable idiosyncratic risk. The first of them is similar to the case presented in subsection 1.2, but introduces a more general production

[23] Mapping

$$\mathbb{R}_{++}^{I(S+1)L} \times \mathbb{R}_{++}^{(S+1)I} \times \mathbb{R}^I \times \mathcal{P}^2 \times \mathbb{R} \times \mathcal{N} \times \mathbb{R}^{I+I(S+1)L+I(S+1)+2(L-1)+1+1}$$

into

$$\mathbb{R}^{I(S+1)L} \times \mathbb{R}^{I(S+1)} \times \mathbb{R}^I \times \mathbb{R}^{2(L-1)} \times \mathbb{R} \times \mathbb{R}^{I(S+1)L} \times \mathbb{R}^{I(S+1)} \times \mathbb{R}^{2(L-1)} \times \mathbb{R}^{2I} \times \mathbb{R}.$$

technology. The second example extends the argument to an economy with overlapping generations. These examples are meant to illustrate how the general results obtained in the paper can be extended to economies with more structure.

6.1 A production economy

Consider, as in the example presented in Subsection 1.2, a two-period economy populated by a continuum of mass 1 of identical individuals. Again, a single commodity is available in the first period, and it can be either consumed or invested: the amount of this commodity that is saved by the individuals becomes their endowment for the second period; we denote this amount by k. In the second period, this level k of capital is combined with a second production factor, labor, to produce a consumption good; denote by l and c, respectively, the amounts of labor used and of consumption good produced at date 1.

In the first period, individuals only consume and save. In the second period, they are endowed with $\bar{l}$ units of labor that they supply inelastically, together with their savings k, in exchange for the consumption good. Initially, suppose that there is no risk, so that all the individuals are endowed with $\bar{l}$ at date 1. The over-all utility of the individuals is given by

$$u(k, c) = -k + v(c),$$

where v is the person's utility index for consumption in the second period. If we fix the consumption good to be the numèraire of the second period, and denote the price of capital as $1 + r$ and the price of labor as w, then the date-1 budget of the individuals is

$$\tau(k) = (1 + r)k + w\bar{l},$$

which they use to buy consumption, so $c = \tau(k)$.

The technology of production is

$$y = f(k, l) + (1 - \delta)k,$$

where function f is assumed to exhibit constant returns to scale. It is immediate, then, from the maximization of profits, that $(1 + r) = f_k + (1 - \delta)$

and $w = f_l$, while, as response to a perturbation,[24] $\mathrm{d}r = f_{kk}\mathrm{d}k$ and $\mathrm{d}w = f_{lk}\mathrm{d}k = f_{kl}\mathrm{d}k$.

From the optimization of the individuals, it must be that $\lambda(1+r) = 1$, where λ represents the marginal utility of consumption at date 1. This implies that[25]

$$\mathrm{d}u = -\mathrm{d}k + \lambda(k\mathrm{d}r + \bar{l}\mathrm{d}w + (1+r)\mathrm{d}k) = \frac{k\mathrm{d}r + \bar{l}\mathrm{d}w}{1+r} = \frac{kf_{kk} + \bar{l}f_{kl}}{1+r}\mathrm{d}k,$$

which, since f_k is homogeneous of degree 0, is $\mathrm{d}u = 0$. This equality that confirms that, under certainty, the privately determined level of investment cannot be improved upon.

Now, suppose that the endowment of labor in the second period is subject to idiosyncratic risk, and is $\bar{l}_s = \bar{l} + \varepsilon_s$ in personal state s, which occurs with probability π_s. As before, $\mathrm{E}[\varepsilon_s] = 0$, and we can let

$$\tau_s(k) = (1+r)k + w\bar{l}_s$$

be the individuals' nominal wealth in personal state s. With ex-ante preferences

$$u(k, c) = -k + \mathrm{E}[v_s(c_s)],$$

if we let λ_s be the marginal utility of revenue in state s, we have that the first-order condition of the individuals at date 0 is that $(1+r)\mathrm{E}[\lambda_s] = 1$. As a consequence,

$$\mathrm{d}u = \mathrm{E}[\lambda_s(kf_{kk} + \bar{l}_s f_{kl})]\mathrm{d}k = \mathrm{E}[\lambda_s\varepsilon_s]f_{kl}\mathrm{d}k = \mathrm{cov}(\lambda_s, \varepsilon_s)f_{kl}\mathrm{d}k,$$

where $\mathrm{cov}(\lambda_s, \varepsilon_s)$ denotes the covariance of the two random variables. The equilibrium allocation is thus constrained suboptimal, as long as $\mathrm{cov}(\lambda_s, \varepsilon_s) \neq 0$. In particular, with state-independent, strictly concave Bernoulli indices,

[24] In these expressions and henceforth, we omit the arguments, $(k, \bar{l})$, of function f, for the sake of notational simplicity.

[25] In this case, since there is only one commodity at date 1, the computation of λ is immediate. In the case of multiple commodities, one would express the indirect utility of the individual as $V(\tau) = v(c(p, \tau))$ and use $\lambda = V'$. In general, also, with a consumption plan c, endowment e and prices p, one would have

$$\mathrm{d}u = -\mathrm{d}k + \lambda(-c\mathrm{d}p + e\mathrm{d}p + (1+r)\mathrm{d}k).$$

the marginal utility of income is anti-comonotone with the level of consumption, which implies that $\text{cov}(\lambda_s, \varepsilon_s) < 0$, provided that idiosyncratic risk is non-degenerate; in this case, since $f_{kl} > 0$, the expression above implies that individuals *overinvest* at date 0: if $\mathrm{d}k < 0$, then $\mathrm{d}u > 0$.[26]

Once again, for a closed-form solution we can consider a particular functional form for the Bernoulli indices, letting

$$v_s(c_s) = \frac{\beta}{\gamma} c_s^{\gamma},$$

where $\beta > 0$ and $\gamma < 1$. In this case, the marginal utilities of income are given by

$$\lambda_s = \beta c_s^{\gamma-1} = \beta(y + \varepsilon_s f_l)^{\gamma-1},$$

so the first-order condition of the individuals' optimization problem is

$$\frac{1}{1+r} = \beta \mathrm{E}[(y + \varepsilon_s f_l)^{\gamma-1}]$$

while

$$\mathrm{d}u = \beta \mathrm{E}[(y + \varepsilon_s f_l)^{\gamma-1} \varepsilon_s] f_{kl} \mathrm{d}k.$$

If we further assume, as before, two equally probable personal states, with $\varepsilon_s = \pm\varepsilon$, then the first order condition becomes

$$\frac{1}{2}\beta(y + \varepsilon f_l)^{\gamma-1} + \frac{1}{2}\beta(y - \varepsilon f_l)^{\gamma-1} = \frac{1}{1+r}$$

which implies, since $\varepsilon > 0$ and $\gamma < 1$, that

$$\frac{1}{2}\beta(y - \varepsilon f_l)^{\gamma-1} > \frac{1}{1+r}.$$

Since in this case, by substitution,

$$\mathrm{E}[\lambda_s \varepsilon_s] = \frac{1}{2}\beta(y+\varepsilon f_l)^{\gamma-1}(\varepsilon) + \frac{1}{2}\beta(y-\varepsilon f_l)^{\gamma-1}(-\varepsilon) = \beta\varepsilon(1-(y-\varepsilon f_l)^{\gamma-1}) < 0,$$

which verifies that individuals overinvest at date 0.

[26] For instance, in the case when there only are two equally probable personal states, with $\varepsilon_s = \pm\varepsilon$, by concavity it must be that $\lambda(-\varepsilon) > \lambda(\varepsilon)$, so it follows immediately that

$$\mathbb{E}_\pi[\lambda_s \varepsilon_s] = \frac{1}{2}\varepsilon(\lambda(\varepsilon) - \lambda(-\varepsilon)) < 0.$$

6.2 Overlapping generations

Finally, consider an economy of overlapping generations, a la Diamond (1965), where each generation lives for two periods, and the population grows at a constant rate $n \geq 0$. Suppose, also, that the productive sector is as in the previous example.

Suppose, initially, that the individuals of the economy are endowed with $\bar{l}$ units of labor in *each* of the two periods they live, which they supply inelastically. Using the same notation as in the previous example, their ex-ante utility is given by

$$u(k) = \bar{l}w - k + v(\bar{l}w + (1+r)k),$$

and the effects of a perturbation are

$$\mathrm{d}u = \bar{l}\mathrm{d}w - \mathrm{d}k + \lambda(k\mathrm{d}r + \bar{l}\mathrm{d}w + (1+r)\mathrm{d}k).$$

In this case, the first-order conditions of individual optimization are that $(1+r)\lambda = 1$, and, by direct substitution,

$$\mathrm{d}u = \frac{(2+r)\bar{l}\mathrm{d}w + k\mathrm{d}r}{1+r}.$$

On the other hand, since the total supply labor is $(2+n)\bar{l}$, we get, from the fact that the production technology is of constant returns to scale, that

$$(2+n)\bar{l}\mathrm{d}w + k\mathrm{d}r = 0,$$

so

$$\mathrm{d}u = \frac{r-n}{1+r}\bar{l}f_{lk}\mathrm{d}k,$$

which establishes the *Golden Rule* criterion: if the interest rate is above (resp. below) the rate of population growth, in equilibrium the economy underinvests (resp. overinvests).

Now, suppose again that the endowment of labor in the second period is subject to idiosyncratic risk, and is $\bar{l}_s = \bar{l} + \varepsilon_s$ with probability π_s, where $\mathrm{E}[\varepsilon_s] = 0$. As before, letting λ_s be the marginal utility of income in state s, we have that the first-order condition for optimization is that $(1+r)\mathrm{E}[\lambda_s] = 1$, and, hence, the welfare effects of a perturbation are

$$\mathrm{d}u = \bar{l}\mathrm{d}w - \mathrm{d}k + \mathrm{E}[\lambda_s(k\mathrm{d}r + \bar{l}_s\mathrm{d}w + (1+r)\mathrm{d}k)] = \left(\frac{r-n}{1+r}\bar{l} + \mathrm{cov}(\lambda_s, \varepsilon_s)\right) f_{lk}\mathrm{d}k.$$

With state-independent, strictly concave Bernoulli indices, $\mathrm{cov}(\lambda_s, \varepsilon_s) < 0$, and it follows from the latter expression that when the interest rate is above the rate of population growth the competitive equilibrium implies overinvestment (as in the case of certainty). But now, in the presence of idiosyncratic risk, the second prescription of the Golden Rule *may* fail, and in an economy where the interest rate is higher than the growth of population, it may be that a Pareto improvement requires for every generation to save less.

If we assume, again, that the Bernoulli indices are state-independent and equal to

$$v(c) = \frac{\beta}{\gamma} c^\gamma,$$

as in the previous example, and

$$f(k, l) = k^\alpha l^{1-\alpha},$$

then, from the first-order condition, we can solve for the optimal level of investment as

$$k = \alpha^{\frac{1}{1-\alpha\gamma}} [(2+n)\bar{l}]^{\frac{(1-\alpha)\gamma}{1-\alpha\gamma}} \Delta^{\frac{1}{1-\alpha\gamma}},$$

where $\Delta = \beta \mathrm{E}[(\alpha + \delta_s(1-\alpha))^{\gamma-1}]$ and

$$\delta_s = \frac{\bar{l} + \varepsilon_s}{(2+n)\bar{l}}.$$

Also,

$$\lambda_s = \beta c_s^{\gamma-1} = \beta(y - (1+n)\bar{l}w + \varepsilon_s w)^{\gamma-1},$$

so if we further take that $\varepsilon_s = \pm\varepsilon$ with probability 1/2, then the first-order condition becomes

$$\mathrm{E}[\lambda_s] = \frac{1}{2}\beta(y - (1+n)\bar{l}w + \varepsilon w)^{\gamma-1} + \frac{1}{2}\beta(y - (1+n)\bar{l}w - \varepsilon w)^{\gamma-1} = \frac{1}{1+r},$$

which implies that

$$\frac{1}{2}\beta(y - (1+n)\bar{l}w - \varepsilon w)^{\gamma-1} > \frac{1}{1+r}.$$

Now, by substitution,

$$\mathrm{E}[\lambda_s \varepsilon_s] = \varepsilon \left(\frac{1}{1+r} - \beta(y - (1+n)\bar{l}w - \varepsilon w)^{\gamma-1} \right) < 0.$$

Table 1: Welfare effects of a policy perturbation

n	$\frac{r-n}{1+r}\bar{l}$	$\frac{r-n}{1+r}\bar{l} + \text{cov}(\lambda_s, \varepsilon_s)$
0	0.062004539	0.020834359
0.01	0.050169498	0.009184415
0.02	0.038357781	-0.002443666
0.03	0.026569236	-0.014050019
0.04	0.014803712	-0.02563478
0.05	0.003061059	-0.037198082
0.06	-0.008658872	-0.048740058
0.07	-0.020356227	-0.060260838
0.08	-0.03203115	-0.071760553
0.09	-0.043683786	-0.08323933

Moreover, using the equilibrium value of k, we can compute $\mathrm{E}[\lambda_s \varepsilon_s]$ to determine whether the presence of idiosyncratic risk voids the classical prescription of the golden rule. Suppose that the values of the different parameters are $\alpha = \gamma = 0.5$, $\bar{l} = 1.4$, $\varepsilon = 0.6$ and $\beta = 0.95$. Table 1 gives the policy prescription, in terms of the sign a Pareto improving perturbation $\mathrm{d}k$, for different values of n.

With these values, if $n \leq 0.01$, from the third column of the table, one has that $\mathrm{d}u/\mathrm{d}k > 0$, namely that the economy underinvests at equilibrium – and for these values, since $r > n$, the Golden Rule prescribes, similarly, that the economy should invest more. For values of $n \geq 0.06$, one has that $\mathrm{d}u/\mathrm{d}k < 0$, so that the Pareto improvement should be induced by a reduction in capital accumulation, which agrees with the recommendation of the Golden Rule, for in these cases $r < n$. But the same is not true for values of $0.02 \leq n \leq 0.05$, where the actual $\mathrm{d}u/\mathrm{d}k < 0$ implies that the economy should accumulate *less* capital, while the Golden Rule prescribes the opposite, since $r > n$.

7 Concluding remarks

The positive properties of the standard general equilibrium model are true in a model of aggregate and uninsurable idiosyncratic risks. For every profile of endowments and preferences, there exists at least one equilibrium, and for every profile of preferences, in a strongly generic set of endowments equilibria are finite (Theorem 1).

Pareto efficiency requires marginal rates of substitution for commodities to be independent of the individual and of the idiosyncratic shocks; since these shocks are uninsurable, for any profile of preferences, on a strongly generic set of endowments all competitive equilibrium allocations are Pareto inefficient (Theorem 2).

But Pareto efficiency ignores the existence of financial constraints, in particular the fact that idiosyncratic risks are uninsurable. A more interesting questions is whether the existing assets, which only allow for (perfect) insurance against aggregate shocks, can be used to induce a Pareto improvement over the equilibrium allocations, without requiring that commodity markets be closed. Following the definition of constrained suboptimality (Stiglitz, 1982, and Geanakoplos and Polemarchakis, 1986), a simple characterization of equilibria in which that type of social improvement is possible exists (Theorem 1).

Given a profile of preferences, and a profile of endowments where there are finitely many equilibria and all of them are Pareto inefficient (a strongly generic condition), a finite-dimensional space of preferences where the set of equilibria (for the fixed profile of endowments) does not change is constructed (Theorem 1). This space is parameterized by perturbations to the Hessians of the utility functions that do not change their gradients at the equilibrium points (changes to the shape of individual demands that do not change their levels at equilibrium prices). On this finite-dimensional space of preferences, a strongly generic subset has the property that all equilibria are constrained inefficient; this implies that on an open and dense subset of the space of economies, every competitive equilibrium allocation is constrained inefficient (Theorem 3). The result requires that commodities be diverse enough (Lemma 2); this is because the Pareto improvement is generated by the response of sufficiently many relative prices to the perturbation in asset portfolios, which yields transfers of revenue across states of the world which are not available directly from the existing assets: they create insurance opportunities against idiosyncratic risk.

Appendix: Proofs of the Lemmas

Proof of Lemma 1: Since the partial Jacobean has full rank, it follows from the Inverse Function Theorem that $\mathcal{H}(\cdot, e, u)$ maps a neighborhood of $(x, \lambda, p, y, (qy^i)_{i=1}^I)$ onto a neighborhood of $\mathcal{H}(x, \lambda, p, y, (qy^i)_{i=1}^I, e, u)$. It follows that for a small enough $\delta > 0$, there exists $(\hat{x}, \hat{\lambda}, \hat{p}, \hat{y}, \hat{\tau})$ such that

$$\mathcal{H}(\hat{x}, \hat{\lambda}, \hat{p}, \hat{y}, \hat{\tau}, e, u) = (u^1(x^1) + \delta, \ldots, u^I(x^I) + \delta, 0, \ldots, 0)^\mathsf{T},$$

which means that x is constrained inefficient. *Q.E.D.*

Proof of Lemma 2: Fix a profile of preferences u. Let $\pi := (\pi_1^1, \ldots, \pi_S^1, \pi_1^2, \ldots, \pi_S^I)^\mathsf{T}$, and define the function

$$(x, \lambda, y, p, q, e, \theta) \mapsto \begin{pmatrix} \mathcal{F}(x, \lambda, y, p, q, e, u) \\ \mathbf{Z}(x, e)\theta \\ \pi^\mathsf{T}\theta_i \\ \frac{1}{2}(\theta^\mathsf{T}\theta - 1) \end{pmatrix}.$$

Suppose that $(x, \lambda, y, p, q, e, \theta) \mapsto 0$. With arguments in the order

$$(x^1, \lambda^1, y^1, \ldots, x^I, \lambda^I, y^I, e^1, \ldots, e^I, \theta),$$

its (partial) Jacobean writes as

$$\begin{pmatrix}
D^2u^1(x^1) & -\Psi(p)^\mathsf{T} & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
-\Psi(p) & 0 & R(q) & \ldots & 0 & 0 & 0 & \Psi(p) & \ldots & 0 & 0 \\
0 & R(q)^\mathsf{T} & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & D^2u^I(x^i) & -\Psi(p)^\mathsf{T} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & -\Psi(p) & 0 & R(q) & 0 & \ldots & \Psi(p) & 0 \\
0 & 0 & 0 & \ldots & 0 & R(q)^\mathsf{T} & 0 & 0 & \ldots & 0 & 0 \\
\Phi^1 & 0 & 0 & \ldots & \Phi^I & 0 & 0 & -\Phi^1 & \ldots & -\Phi^I & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
\phi^1(\theta) & 0 & 0 & \ldots & \phi^I(\theta) & 0 & 0 & -\phi^1(\theta) & \ldots & -\phi^I(\theta) & \mathbf{Z}(x, e) \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \pi^\mathsf{T} \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \theta^\mathsf{T}
\end{pmatrix}$$

where, for each i,

$$\Phi^i := \begin{pmatrix} \tilde{\mathbb{I}} & 0 & \ldots & 0 \\ 0 & \pi_1^i \tilde{\mathbb{I}} & \ldots & \pi_S^i \tilde{\mathbb{I}} \end{pmatrix} \text{ and } \phi^1(\theta) := \begin{pmatrix} 0 & \theta_1^i \tilde{\mathbb{I}} & \ldots & \theta_S^i \tilde{\mathbb{I}} \end{pmatrix},$$

where $\tilde{\mathbb{I}}$ denotes the L-dimensional identity matrix, with its first row removed. We now argue that this matrix has full row rank, in three steps.
Step 1: By standard arguments, the submatrix without the last three superrows and the supercolumns $(e^2, \ldots, e^I, \theta)$ has full row rank.
Step 2: When we add the third-to-last superrow and the $(e^2, \ldots, e^I)$ supercolumns, we add $(L-1)$ rows and $(I-1)(S+1)L > L-1$ columns. Fix $l = 2, \ldots, L$, and define the vector α as follows: $\alpha(e^i_{1,l}) = \theta^i_1$, $\alpha(e^i_{1,1}) = -p_{1,l}\theta^i_l$, and $\alpha(\chi) = 0$ for every other argument χ. Since $\pi^\mathsf{T}\theta = 0$ and $\theta^\mathsf{T}\theta = 1$, it follows that if we post-multiply the submatrix without the last two superrows and the last supercolumn by α, we get 0 everywhere except at the l-th component of its last superrow, where we get 1. It follows that that submatrix has full row rank.
Step 3: If we now add the last two superrows and the last supercolumn, and since $\theta \neq 0$ and $\pi^\mathsf{T}\theta = 0$, it follows that the whole matrix has full row rank.

The latter implies that the mapping is transverse to 0 and, hence, that the set of endowments on which it is transverse to 0, as a function of $(x, \lambda, y, p, q, \theta)$ only, has full measure. Since the mapping has $I(S+1)L + I(S+1) + I + 2(L-1) + 1 + (L-1) + 1 + 1$ components and $(x, \lambda, y, p, q, \theta)$ contains only $I(S+1)L + I(S+1) + I + 2(L-1) + 1 + IS$ arguments, and since $L \geq IS$, it follows that this mapping can be transverse to 0 only if it never takes the value 0. It follows that, in that full measure set, the equalities $\mathcal{F}(x, \lambda, y, p, q, e, u) = 0$, $\mathbf{Z}(x, e)\theta = 0$ and $\pi^\mathsf{T}\theta = 0$ imply that $\theta = 0$. *Q.E.D.*

Proof of Lemma 3: Theorems 1 and 2 and Lemma 2 give strongly generic sets of endowments where equilibria are finite and satisfy, respectively, the first three properties. For the fourth property, as in Theorem 1, it suffices to observe that $\mathcal{G}(\cdot, u) \pitchfork 0$, and then to invoke the Transversality Theorem, to conclude that, for any profile of preferences u, in a strongly generic set of endowments, $\mathcal{G}(\cdot, e, u) \pitchfork 0$. With this result, genericity of the set $\mathcal{D}_r$ follows by taking the intersection of these four generic sets. *Q.E.D.*

Proof of Lemma 4: This follows immediately from the first property of Lemma 3, by the Implicit Function Theorem. *Q.E.D.*

Proof of Lemma 5: Consider the function

$$\begin{pmatrix} \mathcal{F}(x, \lambda, y, p, q, \bar{e}, \Delta) \\ \vdots \\ \frac{1}{\lambda_0^i}\pi_s^i D\bar{u}_1^i(x_s^i) + \pi_s^i(D^2\bar{u}_1^i(x_s^i) + \Delta_s^i)\beta_s^i - \gamma_s^i p_1^\mathsf{T} + \pi_s^i\tilde{\mathbb{I}}^\mathsf{T}\mu \\ p_1 \cdot \beta_s^i \\ \vdots \\ \sum_i \sum_{s=1}^S \lambda_s^i \tilde{\mathbb{I}}\beta_s^i + \sum_i \sum_{s=1}^S \gamma_s^i(\tilde{\bar{e}}_s^i - \tilde{\bar{x}}_s^i) \\ \vdots \\ \sum_{s=1}^S \gamma_s^i + \eta \\ \vdots \end{pmatrix}.$$

We are going to show that this matrix is transverse to 0. For this, we first establish that when the function takes value 0, $\beta_s^i \neq 0$ for every i and every $s = 1, \ldots, S$. To see that this is the case, suppose, for instance, that $\beta_1^1 = 0$. Then, it is immediate that

$$\frac{1}{\lambda_0^1}\pi_1^1 D\bar{u}_1^1(x_1^1) + \pi_1^1\tilde{\mathbb{I}}^\mathsf{T}\mu = 0,$$

and hence, since $\pi_1^1 D\bar{u}_1^1(x_1^1) = \lambda_1^1 p_1$ and $p_{1,1} = 1$, and since the first column of $\tilde{\mathbb{I}}$ is null, we have that $\frac{\lambda_1^1}{\lambda_0^1} = \gamma_1^1$ and, hence, that $\mu = 0$. Now, the latter implies that for every i and s,

$$\frac{1}{\lambda_0^i}\pi_s^i D\bar{u}_1^i(x_s^i) + \pi_s^i(D^2\bar{u}_1^i(x_s^i) + \Delta_s^i)\beta_s^i - \gamma_s^i p_1^\mathsf{T} = 0,$$

so, pre-multiplying by β_s^i, and since, by construction, $\pi_s^i D\bar{u}_1^i(x_s^i) = \lambda_s^i p_1$ and $p_1 \cdot \beta_s^i = 0$, we have that

$$\pi_s^i(\beta_s^i)^\mathsf{T}(D^2\bar{u}_1^i(x_s^i) + \Delta_s^i)\beta_s^i = 0,$$

which implies that $\beta_s^i = 0$, and then that $\lambda_s^i/\lambda_0^i = \gamma_s^i$ for all i and s. Now, by condition (3), the latter implies that $\sum_i \sum_{s=1}^S (\lambda_s^i \ \lambda_0^i)(\tilde{\bar{e}}_s^i - \tilde{\bar{x}}_s^i) = 0$, while, since $\mathcal{F}(x, \lambda, y, p, q, \bar{e}, \Delta) = 0$, we also have that $\sum_i \sum_{s=1}^S \pi_s^i(\tilde{\bar{e}}_s^i - \tilde{\bar{x}}_s^i) = 0$. By Proposition 1, it follows that $\mathcal{F}(x, \lambda, y, p, q, \bar{e}, 0) = 0$, and hence, by Lemma 3, that $\mathbb{Z}(\bar{e}, x)$ has rank $LS - 1$, so it must be that, for some scalar k, $\lambda_s^i/\lambda_0^i = k\pi_s^i$ for all i and s. It then follows that $\lambda_s^i/\pi_s^i = \lambda_{s'}^i/\pi_{s'}^i$, for all s and s', which is impossible, by Lemma 3. We conclude, then, that $\beta_s^i \neq 0$ for all i and all $s = 1, \ldots, S$.

Now, we need to show that the Jacobean of the function has full row rank when the function takes value of 0. Since, in this case, by construction and Proposition 1,

$D_\Delta \mathcal{F}(\cdot) = 0$, it suffices to show that the Jacobean of the rest of the components of the function (i.e., excluding the first component, $\mathcal{F}$) give a full-row-rank Jacobean with respect to $(\beta, \gamma, \mu, \eta, \Delta)$. With the arguments in the order

$$(\beta_1^1, \gamma_1^1, \ldots, \beta_S^I, \gamma_S^I, \Delta_1^1, \ldots, \Delta_S^I),$$

the Jacobean writes as

$$\begin{pmatrix} \pi_1^1(D^2\bar{u}_1^1(x_1^1)+\Delta_1^1) & -p_1^\top & \ldots & 0 & 0 & N(\beta_1^1) & \ldots & 0 \\ -p_1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \pi_S^I(D^2\bar{u}_1^I(x_S^I)\Delta_S^I) & -p_1^\top & 0 & \ldots & N(\beta_S^I) \\ 0 & 0 & \ldots & -p_1 & 0 & 0 & \ldots & 0 \\ \lambda_1^1\tilde{\mathbb{I}} & \tilde{\bar{e}}_1^1 - \bar{x}_1^1 & \ldots & \lambda_S^I\tilde{\mathbb{I}} & \tilde{\bar{e}}_S^I - \bar{x}_S^I & 0 & \ldots & 0 \\ 0 & \mathbf{1}^\top & \ldots & 0 & \mathbf{1}^\top & 0 & \ldots & 0 \end{pmatrix},$$

where for $t \in \mathbb{R}^L$, $N(t)$ denotes the $L \times L(L+1)/2$ matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 & \ldots & t_L & 0 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & t_1 & t_2 & \ldots & 0 & t_2 & t_3 & \ldots & t_L & \ldots & 0 \\ 0 & 0 & t_1 & \ldots & 0 & 0 & t_2 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & t_1 & 0 & 0 & \ldots & t_2 & \ldots & t_L \end{pmatrix},$$

which has full row rank if (and only if) $t \neq 0$. We argue that the Jacobean has full row rank in a series of steps.

Step 1: The submatrix consisting of the first $2(IS)$ superrows and supercolumns is invertible, by a standard arguments; when we add the other supercolums and superrows, we add more columns than rows, so it suffices to show that we can perturb the added superrows without perturbing the initial ones, which we do in the following steps.

Step 2: Fix $l = 2, \ldots, L$, and define the vector α as follows: $\alpha(\beta_{1,l}^i) = \frac{1}{\lambda_1^1}$, $\alpha(\beta_{1,1}^1) = -\frac{p_{1,l}}{\lambda_1^1}$, and $\alpha(\chi) = 0$ for every other argument except for Δ_1^1, for which we fix $\alpha(\Delta_1^1)$ such that

$$N(\beta_1^1)\alpha(\Delta_1^1) = -\pi_1^1(D^2\bar{U}_1^1(x_1^1) + \Delta_1^1)\alpha(\beta_1^1),$$

which we can do since $N(\beta_1^1)$ contains an invertible $L \times L$ submatrix, given that $\beta_1^1 \neq 0$. Then, the postmultiplication of the Jacobean by α gives 0 in every component, except in the one corresponding to the l-th commodity in the term

$$\sum_i \sum_{s=1}^S \lambda_s^i \tilde{\mathbb{I}} \beta_s^i + \sum_i \sum_{s=1}^S \gamma_s^i (\tilde{\bar{e}}_s^i - \tilde{\bar{x}}_s^i),$$

where it gives 1.
Step 3: Fix $i = 1, \dots, I$, and define the vector α as follows: $\alpha(\gamma_1^i) = 1$, for all $l = 2, \dots, L$ $\alpha(\beta_{1,l}^i) = -(\bar{e}_{1,l}^i - x_{1,l}^i)/\lambda_l^i$, while $\alpha(\beta_{1,1}^i) = \sum_{l=2}^L p_{1,l}(\bar{e}_{1,l}^i - x_{1,l}^i)/\lambda_l^i$, and $\alpha(\chi) = 0$ for every other argument, except for Δ_1^i, where $\alpha(\Delta_1^i)$ is fixed so that

$$N(\beta_1^i)\alpha(\Delta_1^i) = -\pi_1^i(D^2\bar{U}_1^i(x_1^i) + \Delta_1^i)\alpha(\beta_1^i),$$

which we can do since $N(\beta_1^i)$ contains an invertible $L \times L$ submatrix, given that $\beta_1^i \neq 0$. As before, the postmultiplication of the Jacobean by α gives 0 in every component, except in the one corresponding to the term $\sum_{s=1}^S \gamma_s^i + \eta$, where it gives 1.

It follows that the function is transverse to 0, and, hence, that for Δ fixed on subset of $\not{\mathcal{U}}_\delta$ with full Lebesgue measure (relative to $\not{\mathcal{U}}_\delta$ itself), the function is transverse to 0 in the rest of the arguments. But, as in other arguments, this function has more components than arguments: fixing Δ, it has only

$$I(S+1)L + I(S+1) + I + 2(L-1) + 1 + ISL + IS + (L-1) + 1$$

arguments, and consists of

$$I(S+1)L + I(S+1) + I + 2(L-1) + 1 + ISL + IS + (L-1) + I$$

equations. It follows, then, that the only way in which the function can be transverse to 0, for fixed Δ, is for it to never take the value 0, which proves the result. *Q.E.D.*

Proof of Lemma 6: Suppose that $\mathcal{M}(x, \lambda, y, p, q, \Delta, \theta) = 0$. By Proposition 1, it must be that $\mathcal{F}(x, \lambda, y, p, q, \bar{e}, \bar{u}) = 0$, so, by construction,

$$u_0^i(x_0^i) = \bar{u}_0^i(x_0^i), Du_0^i(x_0^i) = D\bar{u}_0^i(x_0^i) \text{ and } D^2u_0^i(x_0^i) = D^2\bar{u}_0^i(x_0^i) + \Delta_0^i,$$

and

$$u_1^i(x_s^i) = \bar{u}_1^i(x_s^i), Du_1^i(x_s^i) = D\bar{u}_1^i(x_s^i) \text{ and } D^2u_1^i(x_s^i) = D^2\bar{u}_1^i(x_s^i) + \Delta_s^i$$

for every i and s.[27]

[27]Here, for simplicity, we are adopting the notation Δ_s^i for $\Delta_{x_s^i}^i$. Also, in what follows we will only consider utility perturbations that are "active" at the given equilibrium, so we take the profile Δ as simply $((\Delta_s^i)_{s=0}^S)_{i=1}^I$.

Let us name the rows of $D\mathcal{H}$ by[28]

$$(u1,\dots,uI,f1,b1,\dots,fI,bI,c0,c1,a,t),$$

so that we can denote θ by

$$\theta^{\mathsf{T}} := (\theta_{u1},\dots,\theta_{uI},\theta_{f1}^{\mathsf{T}},\theta_{b1}^{\mathsf{T}},\dots,\theta_{fI}^{\mathsf{T}},\theta_{bI}^{\mathsf{T}},\theta_{c0}^{\mathsf{T}},\theta_{c1}^{\mathsf{T}},\theta_a,\theta_t),$$

where $\theta_{ui} \in \mathbb{R}$, $\theta_{fi} \in \mathbb{R}^{(S+1)L}$, $\theta_{bi} \in \mathbb{R}^{S+1}$, $\theta_{c0} \in \mathbb{R}^{L-1}$ $\theta_{c1} \in \mathbb{R}^{L-1}$, $\theta_t \in \mathbb{R}$ and $\theta_a \in \mathbb{R}$. For these vectors, we will further denote by a superindex the state and/or the commodity they correspond to, if applicable (for instance, $\theta_{fi}^s \in \mathbb{R}^L$ and $\theta_{fi}^{s,l} \in \mathbb{R}$).

System $D_{x,\lambda,p,y,\tau}\mathcal{H}(x,\lambda,p,y,(qy^i)_{i=1}^I,\bar{e},\Delta)^{\mathsf{T}}\theta = 0$ can be rewritten as follows:
(1) for each i,

$$\theta_{ui}D\bar{u}_0^i(x_0^i)^{\mathsf{T}} + (D^2\bar{u}_0^i(x_0^i) + \Delta_0^i)\theta_{fi}^0 - \theta_{bi}^0 p_0^{\mathsf{T}} + \tilde{\mathbb{I}}\theta_{c0} = 0,$$

and

$$\theta_{ui}\pi_s^i D\bar{u}_1^i(x_s^i)^{\mathsf{T}} + \pi_s^i(D^2\bar{u}_1^i(x_s^i) + \Delta_s^i)\theta_{fi}^s - \theta_{bi}^s p_1^{\mathsf{T}} + \pi_s^i\tilde{\mathbb{I}}^{\mathsf{T}}\theta_{c1} = 0$$

for every $s = 1,\dots,S$;
(2) for each i, $p_0\theta_{fi}^0 = 0$ and $p_1\theta_{fi}^s = 0$ for for every $s = 1,\dots,S$;
(3) at date 0,

$$\sum_i \lambda_0^i\tilde{\mathbb{I}}\theta_{fi}^0 + \sum_i \theta_{bi}^0(\tilde{\bar{e}}_0^i - \tilde{x}_0^i) = 0,$$

while

$$\sum_i\sum_{s=1}^S \lambda_s^i\tilde{\mathbb{I}}\theta_{fi}^s + \sum_i\sum_{s=1}^S \theta_{bi}^s(\tilde{\bar{e}}_s^i - \tilde{x}_s^i) = 0$$

at date 1; and
(4) for every i, $\theta_{bi}^0 + \theta_t = 0$ and $\sum_{s=1}^S \theta_{bi}^s + \theta_a = 0$.

We establish three key properties of this system, by the following claims:

Claim 1 *For at least one type of individuals i, we have that $\theta_{ui} \neq 0$.*

[28]The logic for this choice is the following: the components of vector θ are identified with the equations of function $\mathcal{H}$; then, ui refers to the utility level of type-i individuals, fi and bi to their first-order and budget-balance conditions, $c0$ and $c1$ to commodity market clearing in both dates, and a and t to the balance required for asset allocations and revenue transfers.

Proof: Suppose, by way of contradiction, that $\theta_{ui} = 0$ for every type. By substituting in condition (1) of the system, this would imply that

$$D_{x,\lambda,p,y,\tau}\mathcal{G}(x, \lambda, p, y, (qy^i)_{i=1}^I, \bar{e}, \Delta)^\top \tilde{\theta} = 0,$$

for

$$\tilde{\theta} := (\theta_{f1}^\top, \theta_{b1}^\top, \dots, \theta_{fI}^\top, \theta_{bI}^\top, \theta_{c0}^\top, \theta_{c1}^\top, \theta_a, \theta_t)^\top.$$

Since $\mathcal{G}(x, \lambda, p, y, (qy^i)_{i=1}^I, \bar{e}, \Delta) = 0$ and $(\bar{e}, \Delta) \in \mathcal{D}_r$, it follows from Lemma 3 that $\tilde{\theta} = 0$ and hence that $\theta = 0$, which contradicts the fact that $\theta^\top \theta - 1 = 0$. *Q.E.D.*

Claim **2** *For every type i and date-1 state $s = 1, \dots, S$, we have that $\theta_{fi}^s \neq 0$.*

Proof: The argument is the same as in the proof that every $\beta_s^i \neq 0$ in Lemma 5, invoking Lemma 2, so details are omitted. *Q.E.D.*

Claim **3** *For every type i, we have that $\theta_{fi}^0 \neq 0$.*

Proof: As in the proof of Claim 2, if $\theta_{fi}^0 = 0$ for some i, then $\theta_{c0} = 0$ and $\theta_{fj}^0 = 0$ for every $j = 1, \dots, I$. This implies, by condition (1), that $\theta_{ui}\lambda_0^i = \theta_{bi}^0$, and then, by condition (4), that $\theta_{ui}\lambda_0^i = -\theta_t$ for all i and $s = 1, \dots, S$. By Claim 1, it must be that $\theta_t \neq 0$, so we can define $\mu = -\theta_t^{-1}\theta_{c1}$, $\eta = -\theta_t^{-1}\theta_a$, $\beta_s^i = -\theta_t^{-1}\theta_{fi}^s$ and $\gamma_s^i = -\theta_t^{-1}\theta_{bi}^s$. By construction, since $\theta_t \neq 0$, vector $(\beta, \gamma, \mu, \eta)$ solves the system defined in Lemma 5, which is impossible. *Q.E.D.*

Now, to see that $\mathcal{M} \pitchfork 0$, notice that $D\mathcal{M}(x, \lambda, y, p, q, \Delta, \theta)$ writes as

$$\begin{pmatrix} D_{x,\lambda,y,p,q}\mathcal{F}(x, \lambda, y, p, q, \bar{e}, \Delta) & 0 & 0 \\ M & \mathbb{N}(\theta) & D_{x,\lambda,p,y,\tau}\mathcal{H}(x, \lambda, p, y, (qy^i)_{i=1}^I, \bar{e}, \Delta)^\top \\ 0 & 0 & \theta^\top \end{pmatrix},$$

where

$$\mathbb{N}(\theta) := \begin{pmatrix} N(\theta_{f1}^0) & 0 & \dots & 0 \\ 0 & N(\theta_{f1}^1) & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & N(\theta_{fI}^S) \end{pmatrix}$$

for $N(t)$ defined as in the proof of Lemma 5. Since $D_{x,\lambda,y,p,q}\mathcal{F}(x, \lambda, y, p, q, \bar{e}, \Delta)$ has full rank, because $(\bar{e}, \Delta) \in \mathcal{D}_r$, it suffices that matrix

$$\begin{pmatrix} \mathbb{N}(\theta) & D_{x,\lambda,p,y,\tau}\mathcal{H}(x, \lambda, p, y, (qy^i)_{i=1}^I, \bar{e}, \Delta)^\top \\ 0 & \theta^\top \end{pmatrix}$$

have full row rank for $D\mathcal{M}(x, \lambda, y, p, q, \Delta, \theta)$ to have full row rank. By Claims 2 and 3, it follows that $\mathbb{N}(\theta^s_{fi})$ has full row rank for all type i and all state, present and future, $s = 0, \ldots, S$. An argument similar to the one given in Lemma 5 for transversality of the mapping defined there shows that this matrix $D\mathcal{M}(x, \lambda, y, p, q, \Delta, \theta)$ has full row rank, and, hence, that the whole matrix has full row rank. *Q.E.D.*

References

[1] Aliprantis, C. and K.C. Border (1999), *Infinite dimensional analysis*, Springer.

[2] Arrow, K.J. and J. Lind (1970), "Uncertainty and the evaluation of public investment decisions," *American Economic Review* 60, 364-378.

[3] Carvajal, A. and A. Riascos (2008), "Identification of individual demands from market data under uncertainty," *The B.E. Journal of Theoretical Economics* 8, 1 (Topics), Article 9.

[4] Carvajal, A. and H. Polemarchakis (2008), "Identification of Pareto improving policies: information as the real invisible hand," *Journal of Mathematical Economics* 44, 167-79

[5] Citanna, A., A. Kajii and A. Villanacci (1998), "Constrained suboptimality in incomplete markets: a general approach and two applications," *Economic Theory* 11, 495-521.

[6] Citanna, A., H. M. Polemarchakis and M. Tirelli (2006), "The taxation of trades in assets," *Journal of Economic Theory* 126, 299-313.

[7] Diamond, P. (1965), "National debt in a neoclassical model," *American Economic Review* 55, 1126-1150.

[8] Geanakoplos, J. and H.M. Polemarchakis (1986), "Existence, regularity and constrained suboptimality of competitive allocations when assets structure is incomplete," *Essays in Honor of K.J. Arrow,* W. P. Heller, R. M. Starr and D. Starret (Eds), Vol 3, 65-95, Cambridge University Press.

[9] Guillemin, V. and A. Pollack (1974), *Differential Topology*, Prentice Hall.

[10] Krebs, T. (2003), "Growth and welfare effects of business cycles in economies with idiosyncratic human capital risk," *Review of Economic Dynamics* 6, 846-868.

[11] Krussell, P. and A. A. Smith (1999), "On the welfare effects of eliminating business cycles," *Review of Economic Dynamics* 2, 245-272.

[12] Kübler, F., P.-A. Chiappori, I. Ekeland and H. M. Polemarchakis (2002), "The identification of preferences from the equilibrium prices of commodities and assets," *Journal of Economic Theory* 102, 403-420.

[13] Malinvaud, E. (1973a), "The allocations of small risks in large markets," *Journal of Economic Theory* 4, 312-328.

[14] Malinvaud, E. (1973b), "Markets for an exchange economy with individual risks," *Econometrica* 41, 383-410.

[15] Milnor, J.W. (1965), *Topology from the Differentiable Viewpoint*, Princeton.

[16] Stiglitz, J. (1982), "The inefficiency of stock market equilibrium," *Review of Economic Studies* 49, 241-261.

[17] Villanacci, A., L. Carosi, P. Benevieri and A. Battinelli (2002), *Differential Topology and General Equilibrium with Complete and Incomplete Markets,* Kluwer.