The identification of attitudes towards ambiguity and risk from asset demand [1]

Herakles Polemarchakis[2] Larry Selden[3] Xinxi Song[4]

December 29, 2016
Current: December 05, 2017

[1]We thank Andrés Carvajal, Peter Hammond and Ludovic Renou for helpful discussions. We are particularly indebted to Xiao Wei for his detailed comments and suggestions. Xinxi Song acknowledges financial support from the UK-China Scholarships for Excellence program, and financial support from the Research and Innovation Center of Metropolis Economic and Social Development, Capital University of Economics and Business..

[2]Department of Economics, University of Warwick;
Email: h.polemarchakis@warwick.ac.uk

[3]Columbia Business School, Columbia University and University of Pennsylvania; Email: ls49@columbia.edu

[4]International School of Economics and Management, Capital University of Economics and Business; Email: songxinxi@cueb.edu.cn

Abstract

Individuals behave differently when they know the objective probability of events and when they do not. The smooth ambiguity model accommodates both ambiguity (uncertainty) and risk. For an incomplete, competitive asset market, we give sufficient conditions for the asset demand function generated by smooth ambiguity preferences to identify the ambiguity and risk indices as well as the ambiguity probability measure. We do not require ambiguity beliefs to be observable: in a generalized specification, they may not even be defined. An ambiguity free asset plays an important role for identification. Subsequently, we show that if observations from the asset demand function passing the revealed preference test become dense, then the asset demand function can be rationalized by some ambiguity preferences, and ambiguity preferences constructed from finite observations converge to the underlying ambiguity preferences.

Keywords: risk; uncertainty; identification.

JEL Classification Number: D11; D80; D81.

1 Introduction

Ambiguity preferences distinguish between uncertainty, where an individual cannot assign unambiguous probabilities to specific events, and risk, where such an assignment is possible.[1] Indeed, over the years following the critical contributions of Ellsberg (1961) in response to von Neumann and Morgenstern (1947) and Savage (1954), laboratory data have demonstrated that individuals often do not conform to expected utility that does not distinguish between risk and uncertainty;[2] and, recently, there has been a significant increase in experimental tests that focus on this and related questions.

Even though the vast majority of studies of attitudes towards risk have considered lottery experiments, an alternative empirical approach considers an asset demand rather than a lottery setting. Two important applications of this approach to ambiguity preferences are Bayer, Bose, Polisson, and Renou (2013) and Ahn, Choi, Gale, and Kariv (2014). In the former, the authors derived testable inequality conditions for the data to be consistent with ambiguity preferences. Both papers have been confined to the case of complete asset markets. We assume the asset demand function is observed and generated by some smooth ambiguity preferences, and derive sufficient conditions for the identification of an individual's distinct preferences over uncertainty and risk. We validate the assumption on existence of ambiguity preferences by revealed preference test, and demonstrate that the smooth ambiguity preferences determined by finite observations from asset demand function will converge to the true ambiguity preferences when the observations become dense. Importantly, both the identification process and convergence argument apply to incomplete asset markets.

A number of alternative models distinguish between uncertainty and risk: the seminal formulation of multiple priors and maxmin preferences of Gilboa and Schmeidler (1989), multiplier preferences of Anderson, Hansen, and Sargent (2003) and variational preferences of Maccheroni, Marinacci, and Rustichini (2009). We choose to focus on the model of smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005) for several reasons.[3] First, as the authors note, the model (i) achieves a separation of ambiguity as characterized by their uncertainty beliefs and their aversion to uncertainty, and it (ii) generates smooth indifference curves, rather than kinked indifference curves that may obfuscate the argument. In addition, the approach applies to first- and second-order distributions and, as a result, we can readily relate the analysis to the familiar expected utility case. Finally, the smooth ambiguity model has been used in important asset demand analyses, such as Gollier (2011). Mukerji and Tallon (2001) argued

[1] Ghirardato (2004), p. 36.

[2] Camerer and Weber (1992) and Attanasi, Gollier, Montesano, and Pace (2014) and the references cited therein.

[3] An interesting extension of this model is in Seo (2009).

that competitive markets in which investors maximize ambiguity preferences display properties that are both empirically relevant and excluded by expected utility.[4]

The identification of fundamentals from observable market data can be posed, most simply, in the context of certainty; there, Mas-Colell (1977) demonstrated that the demand function identifies the preferences of the consumer. Importantly, the argument for identification is local: if prices are restricted to an open neighborhood, they identify fundamentals in an associated neighborhood. Evidently, the arguments extend to economies under pure risk, but with a complete system of markets in elementary securities. Identification becomes problematic, and more interesting, when the set of observations is restricted. Under pure risk, this arises when the asset market is incomplete and the payoffs to investors are restricted to a subspace of possible payoffs. Nevertheless, Green, Lau, and Polemarchakis (1979), Dybvig and Polemarchakis (1981) and Geanakoplos and Polemarchakis (1990) demonstrated that identification is possible as long as the utility function has an expected utility representation with a state-independent cardinal utility index, and the distribution of asset payoffs is known. Polemarchakis (1983) extended the argument to the joint identification of tastes and beliefs; but, the argument relies crucially on the presence of a risk free asset and, more importantly, does not allow risk due to future endowments. Recently, Kübler and Polemarchakis (2017) derived conditions that guarantee identification with no knowledge either of the cardinal utility index (attitudes towards risk) or of the distribution of future endowments or payoffs of assets; the argument applies even if the asset market is incomplete and demand is observed only locally. Here, assuming the revealed preference test confirms that asset demands are indeed consistent with smooth ambiguity preferences, we derive sufficient conditions such that the uncertainty and risk indices can be identified from asset demand. One key innovation in the extension of prior results under pure risk is the introduction of an ambiguity free asset with payoff distributions that coincide across ambiguity or uncertainty states. As a result, the identification process can be conducted for both the smooth ambiguity model and its extended version, where for the latter existence of subjective probabilities is not required. The portfolio indifference correspondence is an alternative to asset demand for identification.

The above identification arguments assume the existence of underlying preferences, and try to acertain its uniqueness from the demand function. An alternative approach initiated by Afriat (1967) starts with finite observations of demand and price, and show that if these observations satisfy the strong axiom of revealed preference, then there exists a preference relation

[4] The smooth ambiguity model has not been without controversy – Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012).

rationalizing the data, and some piece-wise linear concave utility function can be explicitly constructed. Needless to say, the preference relation or utility function is not unique. However, Mas-Colell (1978) showed that the preference relations determined by finite observations in Afriat (1967) will converge to the unique true preference if the observations from a continuous demand function satifying the strong axiom of revealed preference, the boundary condition and being income Lipschitzian become dense. For a finite set of observations, Varian (1983a), in an extension of Afriat (1967), provided conditions necessary and sufficient for portfolio choices to be generated by expected utility maximization with a known distribution of asset payoffs in incomplete markets. For the case of complete financial markets, Kübler, Selden, and Wei (2014) eliminated quantifiers under the assumption that the probability distribution over states of the world is known and can vary; as did Echenique and Saito (2015) for subjective expected utility, under the assumption that beliefs are unknown. Kübler and Polemarchakis (2017) proved the convergence of preferences and beliefs constructed in Varian (1983a) or Echenique and Saito (2015) to a unique profile as the number of observations becomes dense. Here we demonstrate the convergence of preferences and beliefs generated in the revealed preference argument to the unique underlying characteristics. It is important to note that convergence bridges the gap between the recoverability and the identification of ambiguity preferences, and it answers the question whether demand is indeed generated by ambiguity preferences. Identification refers to the uniqueness of unobservable characteristics; recoverability refers to a method by which these characteristics can be known.[5]

The rest of the paper is organized as follows. The next section introduces notation and the portfolio optimization problem. In Section 3, we first review the identification of the risk index in the traditional expected utility model, and then develop the identification of the risk and ambiguity indices for ambiguity preferences, assuming the asset demand function is generated by some smooth ambiguity preferences. In Section 4, we argue for the existence of smooth ambiguity preferences using revealed preference tests, and show that smooth ambiguity preferences constructed from finite observations converge to the unique true underlying preferences when the number of observations becomes dense. In the Appendix, we give examples where identification fails, the revealed preference tests for ambiguity preferences, and the identification argument from the portfolio indifference correspondence.

[5]Recently, a quite difference approach to convergence has been developed in Chambers, Echenique, and Lambert (2017). They derive conditions such that a close approximation of an individual's preferences can be obtained from a finite set of binary comparisons rather than demand observations.

2 Setup

States of the world are $\omega \in \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is a finite set and has the following product structure: $\boldsymbol{\Omega} = \boldsymbol{A} \times \boldsymbol{S}$, where $a \in \boldsymbol{A}$ are ambiguity states, and $s \in \boldsymbol{S}$ are risk states. $\boldsymbol{\Omega}$ can be interpreted as a set of possible outcomes of two-stage lotteries; in this case, elements in $\boldsymbol{A}$ and $\boldsymbol{S}$ are, respectively, outcomes of first and second stage lotteries.[6] We assume it is risk states contingent on which asset payoff and consumption occur.[7] A probability measure on the set of states of the world, $\boldsymbol{\pi} \in \Delta(\boldsymbol{\Omega})$, can be expressed as $\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$, where $\boldsymbol{\mu} \in \Delta(\mathbf{A})$ is a probability measure over states of uncertainty, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is a family of conditional probability measures over states of risk, and $\pi_{as} = \mu_a \nu_{as}$.

The ambiguity and risk states can be identified from laboratory experiments and real data. In Ellsberg's experiments, the ambiguity state is the composition of color balls, and the risk state is the color of balls contingent on which lotteries pay off. When conditional probability distributions $\boldsymbol{\nu}$ are indexed by some unknown parameter, then the ambiguity state is the value of the parameter. Ju and Miao (2012) estimated a hidden Markov regime-switching process of US consumption and equity dividends, and identified two ambiguity states, i.e., boom state and recession state, conditional on which the equity return is lognormal distribution. In the robustness theory developed by Hansen and Sargent (2001) and Hansen (2007) where model misspecification is a concern, the ambiguity state is each possible model which gives prediction of probability distribution. Chen, Ju, and Miao (2014) estimated two statistical models of stock returns in a portfolio allocation application.

A distribution of wealth across risk states is

$$\boldsymbol{x} = (..., x_s, ...) \in \mathbb{R}_+^S.$$

A utility function over distributions of wealth is

$$U(\boldsymbol{x}; \boldsymbol{\nu}) : \mathbb{R}_+^S \to \mathbb{R},$$ [8]

that is smooth, strictly monotonically increasing and strictly quasi-concave in $\boldsymbol{x}$, continuous in $\boldsymbol{\nu}$ and satisfies a boundary condition: the closure of the indifference "curve" through any strictly positive distribution is contained

[6] Segal (1990). He gave persuasive arguments for the potential superiority of Anscombe-Aumman setup over Savage setup in analyzing ambiguity attitudes.

[7] If asset payoff and consumption are contingent on both ambiguity states and risk states, all our following results go through with some notational change. The assumption made here makes our argument stronger, and is more consistent with observations in practice.

[8] We indicate explicitly the dependence of utility on the conditional probability measures $\boldsymbol{\nu}$, since we assume $\boldsymbol{\nu}$ is objectively observable and varies exogenously.

in the strictly positive orthant or

$$\boldsymbol{x} \in \mathbb{R}_{++} \;\Rightarrow\; Cl\;\{\boldsymbol{x} : U(\boldsymbol{x};\boldsymbol{\nu}) = U(\bar{\boldsymbol{x}};\boldsymbol{\nu})\} \in \mathbb{R}_{++}.$$

In Klibanoff, Marinacci, and Mukerji (2005) and Seo (2009), the probability measure $\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$ is given, and a set of axioms are necessary and sufficient for the existence of a risk index and an ambiguity index,

$$u : \mathbb{R}_{++} \to \mathbb{R}, \quad \text{and} \quad \tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R},$$

respectively, such that

$$U(\boldsymbol{x};\boldsymbol{\nu}) = E_{\boldsymbol{\mu}}\tilde{\phi}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big) \tag{1}$$

represents ambiguity preferences. Alternatively, if

$$\phi = \widetilde{\phi} \circ u, \quad \phi : \mathbb{R}_{++} \to \mathbb{R},$$

then (1) takes the form

$$U(\boldsymbol{x};\boldsymbol{\nu}) = E_{\boldsymbol{\mu}}\phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\Big). \tag{2}$$

For the representation (1), a positive affine transformation of the risk index u does not change preferences if and only if a compensating transformation is applied to $\tilde{\phi}$. In contrast, the preferences corresponding to (2) are invariant to a positive affine transformation of the risk index.[9] Under the formulation (1), an individual is strictly ambiguity averse if $\tilde{\phi}$ is strictly concave (alternatively, in (2), ϕ is more concave than u), and ambiguity neutral if $\tilde{\phi}$ is linear (alternatively, in (2), ϕ is as concave as u). As argued in Selden and Wei (2014), for (2), an individual is strictly ambiguity averse if ϕ is strictly concave, and ambiguity neutral if ϕ is linear. This difference is a matter of interpretation, since clearly $\phi = \tilde{\phi} \circ u$ establishes the equivalence of the formulations.[10]

Klibanoff, Marinacci, and Mukerji (2005) and Seo (2009) exploited the insight in Segal (1990) that non-reduction of two-stage lotteries can accommodate the Ellsberg Paradox, and they derived the same functional form (1) or (2). Since they considered different preference domains, the probability

[9] This point is discussed in Klibanoff, Marinacci, and Mukerji (2005), p. 1858.

[10] As in Example 1 in Selden and Wei (2014), suppose we interpret $\widetilde{\phi}$ and u in (1), respectively, as the ambiguity and risk indices. Consider a specific lottery with no risk and only uncertainty. Then increasing the decision maker's risk aversion produces the counter intuitive result that the certainty equivalent of the lottery decreases. A considerably more intuitive conclusion is reached if, alternatively, we follow the suggestion of Selden and Wei (2014) to use the representation (2) and interpret ϕ and u, respectively, as the ambiguity and risk indices. Then, increasing the concavity of the risk index has no impact on ϕ and the certainty equivalent of the lottery, referenced above, does not change.

measure over ambiguity states, $\boldsymbol{\mu}$, was subjective in Klibanoff, Marinacci, and Mukerji (2005), and it was objective in Seo (2009). The non-reduction of compound objective lotteries in Seo (2009) was confirmed by experimental studies in Halevy (2007) that demonstrated that ambiguity aversion and compound objective lotteries are closely related. Our results cover both formulations.

We assume that the objective probabilities $\boldsymbol{\nu}$ are observable; but, they can vary across observations. Such an assumption is reasonable. In the experiments of Ellsberg (1961) or Ahn, Choi, Gale, and Kariv (2014), for each ambiguity state, i.e., each possible composition of color balls, the conditional probabilities $\boldsymbol{\nu}_a$ are objectively known to the subjects. In the asset pricing and portfolio allocation applications, the conditional distribution of stock returns are estimated by Ju and Miao (2012) and Chen, Ju, and Miao (2014) respectively. The assumption that objective conditional probability distributions $\boldsymbol{\nu}_a$ are known or observed is allowed for in the asset setting of Varian (1983a) and the incomplete market demand tests in Kübler, Selden, and Wei (2016).

Observation of probability measure $\boldsymbol{\mu}$ is not required; but we assume they do not change across observations. Within the two-stage lotteries framework of Anscombe and Aumann (1963), it is not plausible to know the probability measure $\boldsymbol{\mu}$, since it is subjective. However, if the domain of preferences is compound objective lotteries, assuming observation of probability measure $\boldsymbol{\mu}$ is not unreasonable.

The generalization of smooth ambiguity preferences we introduce here does not require or even refer to a probability measure over ambiguity states. The certainty equivalent wealth for ambiguity state a is

$$w_a(\boldsymbol{x}) = u^{-1}(E_{\boldsymbol{\nu}_a} u(x_s)),$$

and the distribution of certainty equivalent wealth levels across states of ambiguity is

$$\boldsymbol{w}(\boldsymbol{x}) = (..., w_a(\boldsymbol{x}), ...).$$

Generalized ambiguity preferences can be represented by

$$U(\boldsymbol{x}; \boldsymbol{\nu}) = \Phi\big(\boldsymbol{w}(\boldsymbol{x})\big) = \Phi\big(..., w_a(\boldsymbol{x}), ..\big), \tag{3}$$

where $\Phi : \mathbb{R}^A_{++} \to \mathbb{R}$ is an ordinal ambiguity index defined over the distribution of certainty equivalent wealth levels across states of ambiguity. The representation (1) is a special case of the functional form (3); for instance, if $\Phi\left(u_1, ..., u_A\right) = \sum_{a=1}^{A} \mu_a \phi\left(u_a\right)$, then

$$\Phi(..., \sum_{s=1}^{S} \nu_{as} u\left(x_s\right), ...) = \sum_{a=1}^{A} \mu_a \phi\left(\sum_{s=1}^{S} \nu_{as} u\left(x_s\right)\right).$$

By using (1) or (2) and (3), respectively, we obtain in Sections 3 identification results with and without requiring existence of the probabilities $\boldsymbol{\mu}$.

We do not give an axiomatic characterization of the generalized smooth ambiguity representation (3).

Assets are $j \in \boldsymbol{J}$ that is finite. Payoffs of asset j across risk states are

$$\boldsymbol{r}_j = (..., r_{sj}, ...)',$$

a column vector; conditional on risk state s, payoffs of assets are $\mathbf{R}_s = (..., r_{sj}, ...)$, a row vector; and the matrix of asset is

$$\boldsymbol{R} = (..., \boldsymbol{r}_j, ...) = (..., \boldsymbol{R}_s, ...)'$$

that has full column rank or, equivalently, payoffs of assets, $\{\boldsymbol{r}_j\}$ are linearly independent.[11]

A portfolio of assets is $\boldsymbol{y} = (..., y_j, ...)$ and it generates the distributions of wealth across risk states $\boldsymbol{x} = \boldsymbol{Ry}$. The set of portfolios that generate strictly positive $\boldsymbol{x}$ is non-empty,

$$\boldsymbol{Y} = \{\boldsymbol{y} : \boldsymbol{Ry} \gg \mathbf{0}\} \neq \emptyset,$$

that is open. The domain of asset prices not allowing for arbitrage is

$$\boldsymbol{P} = \{\boldsymbol{p} : \boldsymbol{Ry} > 0 \Rightarrow \boldsymbol{py} > 0\} = \{\boldsymbol{p} = \boldsymbol{\pi R}, \boldsymbol{\pi} \gg \mathbf{0}\}.$$

Given the asset price vector $\boldsymbol{p}$, the optimization problem of the individual is

$$\max_{\boldsymbol{y} \in \boldsymbol{Y}} U(\boldsymbol{Ry}; \boldsymbol{\nu}), \quad s.t. \;\; \boldsymbol{p} \cdot \boldsymbol{y} \leq 1. \tag{4}$$

A solution to the optimization problem, $\boldsymbol{y}(\boldsymbol{p}; \boldsymbol{\nu})$, exists, satisfies $\boldsymbol{Ry}(\boldsymbol{p}; \boldsymbol{\nu}) \gg 0$, and it is unique; it defines the demand function for assets,

$$\boldsymbol{y} : (\boldsymbol{P}; \boldsymbol{\nu}) \rightarrow \boldsymbol{Y}.$$

Importantly, the demand function is invertible.

3 Identification

We address the following question: Suppose that data based on asset demand functions is consistent with the existence of (generalized) smooth ambiguity preferences; can the underlying ambiguity and risk indices be identified? We will justify the assumption on existence of (generalized) smooth ambiguity

[11] For notational ease, we assume asset payoffs do not change across observations; however, all our results hold if asset payoffs change and are observable in each observation.

preferences by the revealed preference tests and convergence argument in Section 4.

The demand for assets satisfies the necessary and sufficient first order conditions for the optimization problem (4),

$$\boldsymbol{D}U(\boldsymbol{R}\boldsymbol{y};\boldsymbol{\nu}) = \lambda \boldsymbol{p}, \ \ \lambda > 0,$$

$$\boldsymbol{p}\boldsymbol{y} = 1.$$

These conditions identify the family of marginal rates of substitution of assets,

$$m_{jk} : (\boldsymbol{Y}, \boldsymbol{\nu}) \to (0, \infty)$$

defined by

$$m_{jk}(\boldsymbol{y};\boldsymbol{\nu}) = \frac{\frac{\partial U(\boldsymbol{R}\boldsymbol{y};\boldsymbol{\nu})}{\partial y_j}}{\frac{\partial U(\boldsymbol{R}\boldsymbol{y};\boldsymbol{\nu})}{\partial y_k}}.$$

Before proceeding to the identification of ambiguity preferences, we review the identification of risk preferences in Green, Lau, and Polemarchakis (1979), Dybvig and Polemarchakis (1981) and Polemarchakis (1983).

3.1 Pure risk

The probability measure over states of risk is

$$\boldsymbol{\pi} \in \Delta(\boldsymbol{S}),$$

and the utility function of the individual is

$$U(\boldsymbol{x}) = E_{\boldsymbol{\pi}} u(x_s),$$

where u is the (cardinal) risk index.

Under pure risk, for an expected utility maximizer, the demand for assets identifies the family of marginal rates of substitution

$$m_{jk}(\boldsymbol{y}) = \frac{E_{\boldsymbol{\pi}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{E_{\boldsymbol{\pi}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k} > 0.$$

In Green, Lau, and Polemarchakis (1979), (1) the risk index u is analytic on the nonnegative real line, strictly increasing and strictly concave and (2) the probability measure over states of risk, $\boldsymbol{\pi} \in \Delta(\boldsymbol{S})$, is known. Alternatively, in Dybvig and Polemarchakis (1981), (1) the risk index u is twice continuously differentiable on the positive real line, it is strictly increasing and strictly concave, (2) there is an asset that is risk free across states of risk, $\boldsymbol{r}_{s1} = 1$, and (3) the probability measure over states of risk, $\boldsymbol{\pi} \in \Delta(\boldsymbol{S})$, is known. In both cases, the demand for assets identifies the risk index u up to a positive affine transformation.

Remark 1. *With a risk free asset, identification does not require full knowledge of the distribution of payoffs* $(\boldsymbol{R}, \boldsymbol{\pi})$. *It is only necessary to know the second moment of the payoff distribution of a risky asset.*

In Polemarchakis (1983), (1) the risk index u is smooth on the positive real line, is strictly increasing and strictly concave, and, at some $\overline{x}$ in the domain of u, $u^{(n)} = d^n u/dx^n \neq 0$, $n = 1, ...,$ (2) there is an asset that is risk free, $\boldsymbol{r}_1 = 1$, across states of risk, and (3) the risk index u is known. The demand for assets identifies all moments of the distribution of asset payoffs.

Remark 2. *It suffices to know the variance of the distribution of returns of a risky asset, instead of the risk index,* u.

Remark 3. *Knowing the second moment of the return of one risky asset cannot be dispensed with. In a slightly different context, for simplicity, an investor with a CARA (constant absolute risk aversion) risk index,* $u(x) = -e^{-\rho x}$, *demands a risky asset with normally distributed payoffs,* $\boldsymbol{r}_2 \sim N(\mu, \sigma^2)$ *against a risk free asset with payoff* $\boldsymbol{r}_1 = 1$: $y_2 = (\mu - 1)/(\rho\sigma^2)$. *It follows that the simultaneous identification of the risk index and the distribution of asset payoffs is not possible, without at least partial knowledge of the distribution.*

3.2 Ambiguity

In this subsection, the ambiguity preferences of an individual are represented by the utility function (1) (or (2)), or more generally by (3).

We first consider the case of the smooth ambiguity representation (1) (or (2)). The demand for assets identifies the family of marginal rates of substitution

$$m_{jk}(\boldsymbol{y}; \boldsymbol{\nu}) = \frac{E_{\boldsymbol{\mu}} \phi'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y}))) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{R}\boldsymbol{y}) \boldsymbol{r}_j}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))}}{E_{\boldsymbol{\mu}} \phi'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y}))) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{R}\boldsymbol{y}) \boldsymbol{r}_k}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))}} > 0, \tag{5}$$

where $\boldsymbol{\mu}$ is the probability measure over ambiguity states, and $\boldsymbol{\nu}_a$ is the probability measure conditional on each ambiguity state associated with the distribution of returns for each asset.

An asset is *ambiguity free* if, conditional on each ambiguity state, it generates the same distribution of returns.

Example. *There are 3 risk states and 2 ambiguity states. An asset pays* $(1, a, a)$ *across risk state. The probability distributions conditional on ambiguity states are* $(\frac{1}{2}, 0, \frac{1}{2})$ *and* $(\frac{1}{2}, \frac{1}{2}, 0)$, *respectively. Then this asset is ambiguity free, even if* $a \neq 1$ *that would make the asset risky.*

Since the identification argument depends crucially on the existence of an ambiguity free asset, it deserves attention.[12] In particular, for arbitrary asset payoffs and conditional probabilities, an ambiguity free asset need not exist. Being ambiguity free is a joint restriction on asset payoff $\boldsymbol{r} = (r_1, ..., r_s, ..., r_S)$[13] and the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ where $\boldsymbol{\nu}_a = (\nu_{a1}, ..., \nu_{as}, ..., \nu_{aS})$. One extreme case is a risk free asset, i.e., $r_s = r_{s'}$ for all s and s'. Such an asset is ambiguity free independent of conditional probabilities.[14] The other extreme case is when $r_s \neq r_{s'}$ for any s and s'. In this case such an asset can never be ambiguity free for any conditional probabilities. A risky, yet ambiguity free asset lies in between, and its existence is not guaranteed for arbitrary asset payoffs and conditional probabilities.

To characterize the ambiguity free asset, we partition the risk states $S = \{1, ..., s, ..., S\}$ into disjoint subsets S^n, i.e., $S = \cup_n S^n$ and $S^n \cap S^m = \emptyset$ for $n \neq m$, such that

$$S^n = \{s, s' \in S : r_s = r_{s'}\},$$

that is, S^n is a set of risk states on which this asset pays off the same.

Lemma 1. *An asset with payoff $r = (r_1, ..., r_s, ..., r_S)$ is ambiguity free under conditional probability distributions $\{\nu_a\}_{a=1}^{A}$ iff $\sum_{s \in S^n} \nu_{as} = \sum_{s \in S^n} \nu_{a's}$ for all n, a and a'.*

From this lemma, we know if we restrict the space of asset payoffs and conditional probability measures, the existence of an ambiguity free asset is not a problem. For example, an ambiguity free asset is implied by the restricted probability space $\mathfrak{P} = \{\boldsymbol{\nu}_a : \nu_{11} = ... = \nu_{a1} = ... = \nu_{A1}\}$, where each conditional probability applies the same probability to the first risk state. Then any return vector $(a, b, \ldots, b)$ is ambiguity free and risky for $a \neq b$. Of course, other restrictions on the asset payoffs and conditional probability measures, which satisfy the necessary and sufficient conditions in Lemma 1, would also generate ambiguity free assets. The restricted probability space $\mathfrak{P}$ is not generic in the (non-restricted) probability space, but it is enough for us to work on, and it is widely used in experimental work. Consider the lab test setup in Ahn, Choi, Gale, and Kariv (2014) (p.196) where the subjects are informed that state 2 occurs with probability 1/3 whereas states 1 and 3 occur with unknown probabilities, which sum to 2/3. This case

[12] An ambiguity free asset appears in Klibanoff, Marinacci, and Mukerji (2005) (p.1876), where the effect of ambiguity and risk attitudes on portfolio choice is examined numerically. However, note that different from our assumption, asset payoffs in their example depend on both ambiguity states and risk states, which is hardly observed in practice.

[13] For the analysis of an ambiguity free asset, we consider a single asset and omit its index.

[14] In the remainder of this paper when we refer to an ambiguity free asset, we will mean it is ambiguity free and risky, even though we do not emphasize the latter property.

is consistent with a setting where one ambiguity free asset can be traded, where the asset pays off a in state 2, and b in states 1 and 3 with $a \neq b$.

An ambiguity free asset can also be identified in real financial markets using the factor model. Suppose a firm's stock payoffs are potentially affected by domestic policy (Left or Right) and foreign country's policy (Left or Right). Suppose the manager of a firm is familiar with domestic situation, and has objective probability for domestic policy, say $\frac{2}{3}$ for Left, $\frac{1}{3}$ for Right. However, the manager has ambiguity over the policy distribution in foreign country, depending on whether the economic condition in foreign country will be in boom or recession. If in boom, the probability is $\frac{1}{4}$ for Left, $\frac{3}{4}$ for Right; if in recession, the probability is $\frac{1}{6}$ for Left, $\frac{5}{6}$ for Right. Therefore, there are four risk states:$s_1 =$(Domestic Left, Foreign Left), $s_2 =$(Domestic Left, Foreign Right), $s_3 =$(Domestic Right, Foreign Left), and $s_4 =$(Domestic Right, Foreign Right), contingent on which firm's stock pays off. Under the first ambiguity state (i.e., in boom), the probability distribution is $(\frac{2}{12}, \frac{6}{12}, \frac{1}{12}, \frac{3}{12})$; under the second ambiguity state (i.e., in recession), the probability distribution is $(\frac{2}{18}, \frac{10}{18}, \frac{1}{18}, \frac{5}{18})$. Suppose there is some factor $\boldsymbol{f}_1$ (e.g., certain macroeconomic variable) with realizations $(2, 2, 1, 1)$ across risk states, then this factor is ambiguity free. Factors $\boldsymbol{f}_i$, where $i \neq 1$, are ambiguous. For any stock j, by projection, its payoff $\boldsymbol{r}_j$ can be written as

$$\boldsymbol{r}_j = \alpha_j + \beta_{j1} \cdot \boldsymbol{f}_1 + \sum_{i \neq 1} \beta_{ji} \cdot \boldsymbol{f}_i + \boldsymbol{\epsilon}_j.$$

If the payoff of stock j is affacted by factor f_1 only, then this asset is ambiguity free. In home bias puzzle literature, foreign stocks are considered to be ambiguous, and domestic stocks whose return is not affected by the ambiguity states in foreign country are considered to be ambiguity free.

The next theorem gives sufficient conditions for the identification of the smooth ambiguity model using the ambiguity free asset; moreover, the probability measure over states of uncertainty, $\boldsymbol{\mu}$, is identified as well.

Suppose that

(1) the smooth ambiguity utility (2) satisfies the condition that $\phi\left(u^{-1}(\cdot)\right)$ is strictly concave on $\mathbb{R}_{++}$, with the indices u and ϕ both being twice continuously differentiable, strictly increasing, and strictly concave on $\mathbb{R}_{++}$,

(2) there is an asset $j = 1$ that is risk free, where $\boldsymbol{r}_1 = 1$ across states of the world, and

(3) the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is known.

Theorem 1. *If*

(1) there is an asset $j = 2$ that is ambiguity free: its payoff distribution is invariant to the states of ambiguity, and

(2) the matrix

$$\begin{bmatrix} E_{\boldsymbol{\nu}_1}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_1}\boldsymbol{r}_J \\ ... & E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j & ... \\ E_{\boldsymbol{\nu}_A}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_A}\boldsymbol{r}_J \end{bmatrix}_{A\times(J-1)}$$

has full row rank A,

then, the demand for assets identifies the risk index u on $\mathbb{R}_{++}$ and the ambiguity index ϕ on $\mathbb{R}_{++}$, each up to a positive affine transformation, as well as the ambiguity state probability measure $\boldsymbol{\mu}$.

Proof. We argue in a series of steps.

Step 1–identification of the risk index u.

We restrict attention to the portfolios $\boldsymbol{y} = (y_1, y_2, 0, ..., 0)$, and let $\tilde{\boldsymbol{y}} = (y_1, y_2)$ be the associated truncated portfolio. Since the distribution of payoffs for assets 1 and 2 is invariant across states of ambiguity, there exists a probability measure, $\tilde{\boldsymbol{\nu}} \in \Delta(\boldsymbol{S})$, and a matrix of payoffs of assets over states of risk $\tilde{\boldsymbol{R}} = (\boldsymbol{1}_{\#\boldsymbol{S}}, \tilde{\boldsymbol{r}}_2)$,[15] such that, the distribution of payoffs of assets generated by $(\boldsymbol{\nu}_a, \boldsymbol{R}\boldsymbol{y})$, for any state of ambiguity, coincides with the distribution generated by $(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})$. As a consequence,

$$m_{12}(\tilde{\boldsymbol{y}};\tilde{\boldsymbol{\nu}}) = \frac{E_{\tilde{\boldsymbol{\nu}}}u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})}{E_{\tilde{\boldsymbol{\nu}}}u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})\tilde{\boldsymbol{r}}_2} > 0. \tag{6}$$

Identification of the cardinal risk index u on $\mathbb{R}_{++}$, then follows as under pure risk.

Step 2–identification of the probability measure $\boldsymbol{\mu}$.

If we restrict attention to the portfolio $\tilde{\boldsymbol{y}} = (x, 0, ..., 0)$, for each j ($j = 2, ..., J$), equation (5) gives

$$E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j = \frac{1}{m_{1j}(\tilde{\boldsymbol{y}};\boldsymbol{\nu})},$$

which can be written in matrix form

$$[\mu_1, ..., \mu_A]\begin{bmatrix} E_{\boldsymbol{\nu}_1}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_1}\boldsymbol{r}_J \\ ... & E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j & ... \\ E_{\boldsymbol{\nu}_A}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_A}\boldsymbol{r}_J \end{bmatrix} = [\frac{1}{m_{12}(\tilde{\boldsymbol{y}};\boldsymbol{\nu})}, ..., \frac{1}{m_{1J}(\tilde{\boldsymbol{y}};\boldsymbol{\nu})}]. \tag{7}$$

The full row rank condition (2) implies that the probability measure $\boldsymbol{\mu}$ can be uniquely identified.

Step 3–identification of the ambiguity index ϕ.

[15] $\boldsymbol{1}_{\#\boldsymbol{S}}$ is the vector of 1's of dimension $\#\boldsymbol{S}$, the cardinality of $\boldsymbol{S}$.

We restrict attention to the marginal rate of substitution between risk free asset 1 and one ambiguous asset j, $m_{1j}(\boldsymbol{y};\boldsymbol{\nu})$, in equation (5). Take the derivative on both sides of equation (5) with respect to y_j, and evaluate the resulting functional equation at $\tilde{\boldsymbol{y}} = (x, 0, ..., 0)$, we get

$$[(E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2 - E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2] \frac{\phi''(x)}{\phi'(x)} =$$

$$[E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_j)^2 - E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2] \frac{u''(x)}{u'(x)} + (E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2 \frac{\partial m_{1j}(x, 0, ..., 0; \boldsymbol{\nu})}{\partial y_j}. \quad (8)$$

Since the conditional probability measures over risk states, $\boldsymbol{\nu}$, are known, and the probability measure over ambiguity states, $\boldsymbol{\mu}$, has been identified, all the moments $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j$, $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_j)^2$ and $E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2$, can be computed. The full row rank condition implies that there exists at least one ambiguous asset j such that $(E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2 \neq E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a} \boldsymbol{r}_j)^2$, i.e., the coefficient of $\frac{\phi''(x)}{\phi'(x)}$ does not vanish. Given the risk index u identified, equation (8) in turn identifies the ambiguity index ϕ on $\mathbb{R}_{++}$, up to a positive affine transformation. □

Remark 4. *The identification in Theorem 1 does not require any knowledge of the probability measure over ambiguity states, μ, which can be identified from asset demand under the full row rank condition. And it is not necessary to know the complete conditional probability measures over risk states; in particular; if one ambiguous asset has payoffs $(\boldsymbol{r}_j)^2$ $(j \in \{3, ..., J\})$, then knowing the variance of the ambiguity free asset and the conditional means of the ambiguous assets suffices.*

Remark 5. *As under pure risk, knowing the second moment of the distribution of asset payoffs that is invariant across states of ambiguity permits identification of the risk index u, as well as identification of the asset payoffs independent of the states of ambiguity.*

Remark 6. *The full row rank condition requires variation of the conditional mean return $E_{\boldsymbol{\nu}_a} r_j$ across ambiguity states for an ambiguous asset j. That is, there is ambiguity over the expected returns of the ambiguous assets.*

Remark 7. *Theorem 1 is proved with conditional probability measures $\boldsymbol{\nu}$ being fixed, and the full row rank condition requires $A \leq (J-1)$, i.e., the number of ambiguity states being less than or equal to the number of assets J minus 1. However, since we allow conditional probabilities to vary across observations, the marginal rate of substitution, $m_{1j}(\boldsymbol{y}; \boldsymbol{\nu}^n)$, could be observed under different observations of conditional probabilities $\boldsymbol{\nu}^n$. If $\begin{bmatrix} E_{\boldsymbol{\nu}_1^1} \boldsymbol{r}_j & ... & E_{\boldsymbol{\nu}_1^N} \boldsymbol{r}_j \\ ... & E_{\boldsymbol{\nu}_a^n} \boldsymbol{r}_j & ... \\ E_{\boldsymbol{\nu}_A^1} \boldsymbol{r}_j & ... & E_{\boldsymbol{\nu}_A^N} \boldsymbol{r}_j \end{bmatrix}$, the matrix of conditional expected returns for* *an ambiguous asset j under N observations of conditional probability measures, has full row rank, then the identification argument in Theorem 1 goes through. Therefore, one ambiguous asset, in addition to the risk free and ambiguity free assets, suffices.*

If an individual is endowed with the generalized smooth ambiguity (3), then her demand for assets identifies the family of marginal rates of substitution

$$m_{jk}(\boldsymbol{y};\boldsymbol{\nu}) = \frac{\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))}}{\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))}} > 0.$$

Suppose that

(1) the generalized smooth ambiguity utility (3) is strictly quasi-concave with respect to $\boldsymbol{x}$, with the index u being twice continuously differentiable, strictly increasing, and strictly concave on $\mathbb{R}_{++}$ and the index Φ being continuously differentiable, strictly increasing, and strictly quasi-concave on $\mathbb{R}^A_{++}$,

(2) there is an asset $j = 1$ that is risk free, where $\boldsymbol{r}_1 = 1$ across states of the world, and

(3) the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is known.

Theorem 2. *If*

(1) there is an asset $j = 2$ that is ambiguity free: its payoff distribution is invariant to the states of ambiguity, and

(2) the matrix

$$\begin{bmatrix} E_{\boldsymbol{\nu}_1} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_2 & \dots & E_{\boldsymbol{\nu}_1} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_J \\ \vdots & E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j & \vdots \\ E_{\boldsymbol{\nu}_A} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_2 & \dots & E_{\boldsymbol{\nu}_A} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_J \end{bmatrix}_{A\times(J-1)}$$

has full row rank A at each portfolio $\boldsymbol{y}$,

then, the demand for assets identifies the risk index u on $\mathbb{R}_{++}$, up to a positive affine transformation, and the ordinal utility function Φ on $\mathbb{R}^A_{++}$, up to a strictly increasing transformation.

Proof. We argue in a series of steps.

Step 1—identification of the risk index u.

When we focus on the portfolio $\tilde{\boldsymbol{y}} = (y_1, y_2, 0, ..., 0)$, the marginal rate of substitution between risk free asset 1 and ambiguity free asset 2, m_{12}, will identify the cardinal risk index u on $\mathbb{R}_{++}$, as in Theorem 1.

Step 2—identification of the ambiguity index Φ.

For assets $j = 2, ..., J$, the first order conditions for an optimum,

$$\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{R}}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))} = \lambda \boldsymbol{p}, \lambda > 0, \tag{9}$$

can be written in matrix form,

$$[\Phi_1, ..., \Phi_a, ..., \Phi_A] \begin{bmatrix} \frac{E_{\boldsymbol{\nu}_1} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_2}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_1} u(\boldsymbol{Ry})))} & \cdots & \frac{E_{\boldsymbol{\nu}_1} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_J}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_1} u(\boldsymbol{Ry})))} \\ \vdots & \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))} & \vdots \\ \frac{E_{\boldsymbol{\nu}_A} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_2}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_A} u(\boldsymbol{Ry})))} & \cdots & \frac{E_{\boldsymbol{\nu}_A} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_J}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_A} u(\boldsymbol{Ry})))} \end{bmatrix} = [\lambda p_2, ..., \lambda p_j, ..., \lambda p_J], \tag{10}$$

where $\Phi_a = \frac{\partial \Phi}{\partial w_a}$. We denote by $\boldsymbol{C}$ the matrix in equation (10), and the matrix $\boldsymbol{C}$ has dimension A times $(J-1)$. Since we have identified the index u and the conditional distribution of asset returns is known, the matrix $\boldsymbol{C}$ is computable. Under the full row rank condition (2), $\boldsymbol{C}$ has full row rank, then

$$[\Phi_1, ..., \Phi_a, ...\Phi_A] = [\lambda p_2, ..., \lambda p_j, ..., \lambda p_J]\boldsymbol{C}^T[\boldsymbol{C}\boldsymbol{C}^T]^{-1}. \tag{11}$$

So we can trace out the marginal rates of substitution $\frac{\Phi_a}{\Phi_1}$ $(a = 2, ..., A)$ uniquely. Under the assumption in the theorem, Φ is strictly quasi-concave, continuously differentiable and has strictly positive gradient everywhere on $\mathbb{R}^A_{++}$. Following Mas-Colell (1977), knowledge of the marginal rates of substitution $\frac{\Phi_a}{\Phi_1}$ $(a = 2, ..., A)$ identifies the function Φ on $\mathbb{R}^A_{++}$, up to a strictly increasing transformation. □

Remark 8. *The full row rank condition (2) is not directly observable, but, as shown in the proof, it can be checked once the risk index u is identified. Actually, the full row rank condition (2) can be equivalently stated in terms of asset demand, since the risk index u is identified from asset demand.*

Remark 9. *If the full row rank condition (2) only holds at the portfolio $\tilde{\boldsymbol{y}} = (y_1, 0, ..., 0)$, that is, the matrix of conditional expected asset returns* $\begin{bmatrix} E_{\boldsymbol{\nu}_1}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_1}\boldsymbol{r}_J \\ ... & E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j & ... \\ E_{\boldsymbol{\nu}_A}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_A}\boldsymbol{r}_J \end{bmatrix}$ *has full row rank A, then the demand for assets identifies the ordinal index Φ, on an open neighbourhood of the uncertainty free distribution $\boldsymbol{w} = (..., w, ...)$, up to a strictly increasing transformation.*

Remark 10. *The comment in Remark 7 applies here: marginal rates of substitution are observable for different conditional probability measures, and one ambiguous asset suffices for identification.*

Remark 11. *Both Theorem 1 and Theorem 2 require the existence of one risk free asset. As argued under pure risk, we can show that without a risk free asset, the marginal rate of substitution between two ambiguity free assets identifies the risk index u, so long as the underlying risk index u is analytic at $x = 0$. Once the risk index u is identified, the identification of ambiguity index follows the same argument as in Theorem 1 or Theorem 2. We do not repeat the results here.*

The above identification arguments require observing an individual's demand for assets. An equivalent way to identify the risk and ambiguity indices is to assume knowledge of the individual's portfolio indifference correspondence

$$I(\boldsymbol{y};\boldsymbol{\nu}) = \{\boldsymbol{x} \in \mathbb{R}^J : E_{\boldsymbol{\mu}}\phi(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Rx}))) = E_{\boldsymbol{\mu}}\phi(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))\}.$$

Remark 12. *In Appendix C, we show in Proposition 1 that under the same assumptions on underlying utility functions and asset returns as in Theorem 1, we can obtain identification results from the portfolio indifference correspondence. This should not be surprising, since we can trace out asset demands from the indifference correspondence $I(\boldsymbol{y};\boldsymbol{\nu})$.*

Remark 13. *It can be shown that under the same conditions as in Theorem 2, the generalized smooth ambiguity utility (3) can be identified from the portfolio indifference correspondence. Here again, we do not repeat the results.*

4 Perils of identification

It is standard in the literature on identification for pure risk expected utility models to assume that asset demand is the result of the maximization of an expected utility function; this is the case in Green, Lau, and Polemarchakis (1979), Dybvig and Polemarchakis (1981) and Polemarchakis (1983). Analogously, in the identification of ambiguity and risk indices for the (generalized) smooth ambiguity preference model from asset demands in Section 3, we assume that the demand is the result of the maximization of (generalized) smooth ambiguity preferences.

In the pure risk expected utility case, the functional form demand test in Kübler, Selden, and Wei (2016) (that allows for incomplete markets) for a given set of asset demand functions validates that the demands were generated from expected utility preferences. For ambiguity preferences, no known functional form demand test exists. Nevertheless, following our convergence result, if price-demand data set generated by an asset demand function of a given functional form satisfy the revealed preference tests in Lemma 2 or Lemma 3, and eventually become dense, then the asset demand function must be generated by (generalized) some smooth ambiguity preferences. Combined with the convergence argument, revealed preference tests could provide support for the assumption on the existence of (generalized) smooth ambiguity preferences.

Suppose one performs an identification procedure without first verifying that demand has been generated by the preferences assumed; what can go wrong? In Appendix A, we give two explicit examples where a given set of asset demands were generated by non-expected utility preferences, and, as a consequence, it is incorrect to apply the identification process in Dybvig and Polemarchakis (1981) that assumed demand is derived from expected utility. Indeed, a perfectly natural candidate risk index is obtained, but, the corresponding expected utility does not generate the observed demand.[16] This issue has not previously been stressed in the expected utility identification literature. It should also be emphasized that when generating the data from the given demand system, it is important to allow the probabilities to vary. Otherwise, as argued in Kübler, Selden, and Wei (2016), it is not possible to know whether the probabilities enter into the utility function linearly or whether the risk indices are probability dependent. Clearly the same problem of identification of an erroneous representation can plague the smooth ambiguity identification results in the above section and hence highlights the importance of the revealed preference tests of Lemma 2 or Lemma 3 discussed in Appendix B.[17]

Convergence

In the revealed preference test, we can ascertain the consistency of asset demand with the maximization of ambiguity preferences based on the finite observations of asset demands, prices and objective probabilities

$$\{\boldsymbol{p}^n, \boldsymbol{y}^n, \boldsymbol{\nu}_a^n, \boldsymbol{R}\}_{a=1,\dots,A}^{n=1,\dots,N}.$$

Certainly, it is not possible to identify uniquely the underlying preferences from finite observations. However, the convergence result we establish here, implies that if the underlying preferences and asset payoff structure satisfy conditions in the identification results of Theorem 1 or Theorem 2, and the number of observations increases to infinity and eventually becomes dense, then the associated utility indices converge to the unique true ones. When there is no risk, the problem of convergence of revealed preferences to true preferences has been investigated by Mas-Colell (1978). Our proof differs from his in that we work in the space of utility functions while he showed

[16]One example assumes that the probabilities and payoffs enter into the asset demand functions as numbers, and the other assumes they enter as symbols.

[17]The revealed preference inequalities in Lemma 2 and Lemma 3 are nonlinear, and checking the feasibility of these inequalities calls for efficient algorithms, the existence of which is out of the scope of this paper.

convergence in preferences. It is not clear how to directly apply his proof strategy and show that the limiting preferences over assets can be represented by expected utility over consumption. Recently, Kübler and Polemarchakis (2017) established the convergence of revealed risk preferences and beliefs to the unique true von Neumann-Morgenstern utility index and beliefs. Our argument follows their approach.

We explicitly prove the convergence of the constructed utility indices and probability measure from Lemma 2; we comment on the generalized case in Lemma 3 in a remark. We start with asset demand function which we do not know whether it is generated by smooth ambiguity preferences. What we know is that finite points from this demand function pass the revealed preference tests, and we can construct a smooth ambiguity utility function to rationalize these finite data. What we claim is, under certain condition, if dense subset of this demand function pass the revealed preference tests, then the demand function is generated by some true smooth ambiguity utility function, and the constructed utility functions from finite points converge to the underlying true one.

Denote by $\mathfrak{B}^N$ a set of N observations of (normalized) prices (alternatively, N budget sets) and conditional probability measures, and by $\mathfrak{B}$ an open set of (normalized) prices and conditional probability measures at which the asset demand function is well defined and invertible. Given N observations of asset prices and conditional probability measures $\mathfrak{B}^N$, if the corresponding asset demand satisfies the revealed preference test in Lemma 2, we can construct a smooth ambiguity preference; that is, a pair of a risk aversion index $u^N(\cdot)$ and an ambiguity aversion index $\widetilde{\phi}^N(\cdot)$, and a probability measure $\boldsymbol{\mu}^N$ over ambiguity states.

Theorem 3. *Let $\hat{\boldsymbol{y}}(\boldsymbol{p},\boldsymbol{\nu})$ be a continuous function on an open set $\mathfrak{B} \subset \mathbb{R}^J \times \mathbb{R}_+^{AS}$, and let $\boldsymbol{y}^{nN} = \hat{\boldsymbol{y}}(\boldsymbol{p}^{nN},\boldsymbol{\nu}^{nN})$, $n = 1,...,N$. For each n, N and data $\{\boldsymbol{p}^{nN},\boldsymbol{y}^{nN},\boldsymbol{\nu}_a^{nN},\boldsymbol{R}\}_{a=1,...,A}^{n=1,...,N}$ generated from an increasing sequence $(\mathfrak{B}^N \subset \mathfrak{B} : N = 1,2,...)$ of finite observations of (normalized) prices, and conditional probability measures with $\mathfrak{B}^N \subset \mathfrak{B}^{N+1}$ and $\cup_N \mathfrak{B}^N$ dense in $\mathfrak{B}$, suppose solutions satisfying the revealed preference test exist and are contained in some compact set $\mathfrak{K}$.*

Then there exist fundamentals $(u^*(\cdot),\phi^*(\cdot),\boldsymbol{\mu}^*)$, such that

$$\hat{\boldsymbol{y}}(\boldsymbol{p},\boldsymbol{\nu}) = \boldsymbol{y}(\boldsymbol{p},\boldsymbol{\nu};u^*(\cdot),\phi^*(\cdot),\boldsymbol{\mu}^*) \ \ \text{for all } (\boldsymbol{p},\boldsymbol{\nu}) \in \mathfrak{B}\ .[18]$$

Moreover, if these fundamentals and the asset returns satisfy the sufficient conditions in Theorem 1, then $\boldsymbol{\mu}^N \to \boldsymbol{\mu}^*$, $u^N(\cdot) \to u^*(\cdot)$, and $\widetilde{\phi}^N(\cdot) \to \phi^*(\cdot)$, where the constructed strictly concave utility functions u^N and $\widetilde{\phi}^N$ are normalized such that $u^N(1) = 0$, $u^{N\prime}(1) = 1$, $\widetilde{\phi}^N(1) = 0$ and $\widetilde{\phi}^{N\prime}(1) = 1$.

[18] Here $\boldsymbol{y}(\boldsymbol{p},\boldsymbol{\nu};u^*(\cdot),\phi^*(\cdot),\boldsymbol{\mu}^*)$ is the asset demand at observable prices $\boldsymbol{p}$ and conditional probability measure $\boldsymbol{\nu}$, generated by unobservable utility indices $u^*(\cdot)$ and $\phi^*(\cdot)$, and probability measure $\boldsymbol{\mu}^*$.

Proof. Take a sequence $((u^N(\cdot), \widetilde{\phi}^N(\cdot), \boldsymbol{\mu}^N) : N = 1......)$; by compactness of $\boldsymbol{\mathfrak{K}}$, the $u^N(\cdot)$ and $\widetilde{\phi}^N(\cdot)$ are equicontinuous and there exists an accumulation point $(\bar{u}(\cdot), \bar{\phi}(\cdot), \bar{\boldsymbol{\mu}})$. Since $\bar{u}(\cdot)$ and $\bar{\phi}(\cdot)$ must be concave, it must be continuous. Note that each $(u^N(\cdot), \widetilde{\phi}^N(\cdot), \boldsymbol{\mu}^N)$ as well as $(\bar{u}(\cdot), \bar{\phi}(\cdot), \bar{\boldsymbol{\mu}})$ correspond to continuous, increasing and concave indirect utility functions, $v^N(\boldsymbol{y})$ and $\bar{v}(\boldsymbol{y})$, over assets.

We first prove that the limit utility indices and the limit probability measure $(\bar{u}(\cdot), \bar{\phi}(\cdot), \bar{\boldsymbol{\mu}})$ must generate a demand function that is identical to $\hat{\boldsymbol{y}}(\boldsymbol{p}, \boldsymbol{\nu})$: that is, for all $(\boldsymbol{p}, \boldsymbol{\nu}) \in \mathfrak{B}$,

$$\boldsymbol{y}(\boldsymbol{p}, \boldsymbol{\nu}; (\bar{u}(\cdot), \bar{\phi}(\cdot), \bar{\boldsymbol{\mu}})) = \hat{\boldsymbol{y}}(\boldsymbol{p}, \boldsymbol{\nu}).$$

If not, there exists $(\boldsymbol{p}^*, \boldsymbol{\nu}^*) \in \mathfrak{B}$ and $\boldsymbol{y}^* = \hat{\boldsymbol{y}}(\boldsymbol{p}^*, \boldsymbol{\nu}^*)$ as well as $\bar{\boldsymbol{y}} \in \mathbb{R}^J$ such that $\bar{v}(\bar{\boldsymbol{y}}) > \bar{v}(\boldsymbol{y}^*)$, while $\boldsymbol{p}^*\bar{\boldsymbol{y}} \leq 1$. By the continuity and concavity of $\bar{u}$ and $\bar{\phi}$, without loss of generality,

$$\boldsymbol{p}^*\bar{\boldsymbol{y}} < 1.$$

Since $\cup_N \boldsymbol{\mathfrak{B}}^N \subset \boldsymbol{\mathfrak{B}}$ is dense, there exists a sequence $(\boldsymbol{p}^N, \boldsymbol{\nu}^N) \in \boldsymbol{\mathfrak{B}}^N$: $N = 1, 2, ...$, such that $(\boldsymbol{p}^N) \to (\boldsymbol{p}^*)$ and $(\boldsymbol{\nu}^N) \to (\boldsymbol{\nu}^*)$. By the continuity of $\hat{\boldsymbol{y}}(\boldsymbol{p}, \boldsymbol{\nu})$, there is an associated sequence of demands $(\boldsymbol{y}^N) \to (\boldsymbol{y}^*)$.

Since $\bar{v}(\cdot)$ is continuous, there is an N sufficiently large such that

$$\bar{v}(\bar{\boldsymbol{y}}) > \bar{v}(\boldsymbol{y}^N),$$

and

$$\boldsymbol{p}^N\bar{\boldsymbol{y}} < 1.$$

But since the sets $\boldsymbol{\mathfrak{B}}^N$ are nested, we must have that for all $m \geq N$ $v^m(\boldsymbol{y}^N) > v^m(\bar{\boldsymbol{y}})$, which contradicts the fact that $v^m(\cdot) \to \bar{v}(\cdot)$ point-wise.

To prove the second part of the result note the fact that $\bar{u}(\cdot)$ must be differentiable almost everywhere on on its domain and $\bar{\phi}(\cdot)$ must be differentiable almost everywhere on on its domain, then the identification result in Theorem 1 implies that fundamental are unique and the accumulation point must be the unique limit of the sequence $((u^N(\cdot), \widetilde{\phi}^N(\cdot), \boldsymbol{\mu}^N) : N = 1......)$, i.e., $\bar{u}(\cdot)$ must coincide with $u^*(\cdot)$, $\bar{\phi}(\cdot)$ must coincide with $\phi^*(\cdot)$, and the probability measure $\bar{\boldsymbol{\mu}}$ must coincide with the probability measure $\boldsymbol{\mu}^*$. □

Remark 14. *Note that we do not assume the solution set of revealed preference test in Lemma 2 is compact, instead, we assume it is contained in a compact set $\boldsymbol{\mathfrak{K}}$. The construction of utility functions is a mapping from $\boldsymbol{\mathfrak{K}}$ to a set of continuous functions. Since the mapping is continuous, and $\boldsymbol{\mathfrak{K}}$ is compact, then the set of constructed utility functions is compact.*

Remark 15. *If the underlying generalized smooth ambiguity utility indices $(u^*(\cdot), \Phi^*(\cdot))$ and the asset returns satisfy the sufficient identification conditions in Theorem 2, then the arguments in Theorem 3 can be extended* *to prove the convergence of constructed generalized smooth ambiguity utility indices* $(u^N(\cdot), \Phi^N(\cdot))$ *from Lemma 3 to the true underlying utility indices* $(u^*(\cdot), \Phi^*(\cdot))$. *The details are omitted here.*

References

S. Afriat. The construction of a utility function from demand data. *International Economic Review*, 8:67–77, 1967.

D. Ahn, S. Choi, D. Gale, and S. Kariv. Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics*, 5:195–223, 2014.

E. Anderson, L. P. Hansen, and T. Sargent. A quartet of semi-groups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association*, 1:68–123, 2003.

F. Anscombe and R. J. Aumann. A definition of subjective probability. *Annals of Mathematical Statistics*, 34:199–205, 1963.

G. Attanasi, C. Gollier, A. Montesano, and N. Pace. Eliciting ambiguity aversion in unknown and in compound lotteries: a smooth ambiguity model experimental study. *Theory and Decision*, 77:485–530, 2014.

R. C. Bayer, S. Bose, M. Polisson, and L. Renou. Ambiguity revealed. Unpublished manuscript, 2013.

C. Camerer and M. Weber. Recent developments in modeling preferences: uncertainty and ambiguity. *Journal of Risk and Uncertainty*, 5:325–370, 1992.

C. Chambers, F. Echenique, and N. Lambert. Preference identification. Unpublished manuscript, 2017.

H. Chen, N. Ju, and J. Miao. Dynamic asset allocation with ambiguous return predictability. *Review of Economic Dynamics*, 17:799–823, 2014.

P. Dybvig. Recovering additive utility functions. *International Economic Review*, 24:379–396, 1983.

P. Dybvig and H. Polemarchakis. Recovering cardinal utility. *Review of Economic Studies*, 48:159–166, 1981. URL http://www.polemarchakis.org/a14-rcu.pdf.

F. Echenique and K. Saito. Savage in the market. *Econometrica*, 83:1467–1495, 2015.

D. Ellsberg. Risk, ambiguity and the savage axioms. *Quarterly Journal of Economics*, 75:643–669, 1961.

L. Epstein. A paradox for the "smooth ambiguity" model of preference. *Econometrica*, 78:2085–2099, 2010.

J. D. Geanakoplos and H. Polemarchakis. Observability and optimality. *Journal of Mathematical Economics*, 19:153–165, 1990. URL http://www.polemarchakis.org/a39-oop.pdf.

P. Ghirardato. Defining ambiguity and ambiguity attitude. In I. Gilboa, editor, *Uncertainty in Economic Theory*, pages 36–45. Routledge, 2004.

I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.

C. Gollier. Portfolio choice and asset prices: the comparative statics of ambiguity aversion. *Review of Economic Studies*, 78:141–154, 2011.

J. R. Green, L. J. Lau, and H. M. Polemarchakis. On the recoverability of the von neumann-morgenstern utility function from asset demands. In J. R. Green and J. A. Scheinkman, editors, *Equilibrium, Growth and Trade: Essays in Honor of L. McKenzie*, pages 151–161. Academic Press, 1979. URL http://www.polemarchakis.org/a12-inm.pdf.

Y. Halevy. Ellsberg revisited: an experimental study. *Econometrica*, 75: 503–536, 2007.

L.P. Hansen. Beliefs, doubts and learning: the valuation of macroeconomic risk. *American Economic Review*, 97:1–30, 2007.

L.P. Hansen and T.J. Sargent. Robust control and model uncertainty. *American Economic Review*, 91:60–66, 2001.

N. Ju and J. Miao. Ambiguity, learning and asset returns. *Econometrica*, 80:559–591, 2012.

P. Klibanoff, M. Marinacci, and S. Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73:1849–1892, 2005.

P. Klibanoff, P. Marinacci, and M. Mukerji. On the smooth ambiguity model: a reply. *Econometrica*, 80:1303–1321, 2012.

F. Kübler and H. Polemarchakis. The identification of beliefs from asset demands. *Econometrica*, 85:1219–1238, 2017. URL http://www.polemarchakis.org/a86-idb.pdf.

F. Kübler, L. Selden, and X. Wei. Asset demand based tests of expected utility maximization. *American Economic Review*, 104:3459–3480, 2014.

F. Kübler, L. Selden, and X. Wei. Integrability of demand in incomplete markets: Kreps-porteus-selden preferences. Unpublished manuscript, 2016.

F. Maccheroni, M. Marinacci, and A. Rustichini. Ambiguity aversion, malevolent nature and the variational representation of preferences. *Econometrica*, 77:1447–1498, 2009.

A. Mas-Colell. On the recoverability of consumers' preferences from demand behavior. *Econometrica*, 45:1409–1430, 1977.

A. Mas-Colell. On revealed preference analysis. *Review of Economic Studies*, 45:121–131, 1978.

S. Mukerji and J.-M. Tallon. Ambiguity aversion and incompleteness of financial markets. *Review of Economic Studies*, 68:883–904, 2001.

H. Polemarchakis. Observable probabilistic beliefs. *Journal of Mathematical Economics*, 11:65–75, 1983. URL http://www.polemarchakis.org/a20-opb.pdf.

L. J. Savage. *The Foundations of Statistics*. John Wiley and Sons, 1954.

U. Segal. Two-stage lotteries without the reduction axiom. *Econometrica*, 58:349–377, 1990.

L. Selden and X. Wei. Effects of ambiguity aversion on savings: an interpretive note. mimeo, 2014.

K. Seo. Ambiguity and second-order belief. *Econometrica*, 77:1575–1605, 2009.

H. R. Varian. Nonparametric tests of consumer behavior. *Review of Economic Studies*, 50:99–110, 1983a.

H. R. Varian. Nonparametric tests of models of investor behavior. *Journal of Financial and Quantitative Analysis*, 18:269–278, 1983b.

J. von Neumann and O. Morgenstern. *Games and Economic Behavior*. Princeton University Press, 1947.

M. Yaari. Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, 1:315–329, 1969.

Appendix A

Examples of failure of identification

In this appendix, we give two examples to demonstrate that the identification process proposed in Dybvig and Polemarchakis (1981) can go wrong if preferences are not expected utility representable.

Example 1. *Assume that there is one risky asset and one risk free asset, where the risky asset pays off r_i with probability π_i $(i = 1, 2)$ and the risk free asset always pays off 1. Suppose the demand functions for the risk free asset and risky asset are given respectively by*

$$y_1 = \frac{1}{\pi_1^2 + \pi_2^2} \left(\frac{r_1 \pi_1^2}{r_1 p_1 - p_2} - \frac{r_2 \pi_2^2}{p_2 - r_2 p_1} \right), \tag{A.1}$$

and

$$y_2 = \frac{1}{\pi_1^2 + \pi_2^2} \left(\frac{\pi_2^2}{p_2 - r_2 p_1} - \frac{\pi_1^2}{r_1 p_1 - p_2} \right), \tag{A.2}$$

where p_1 and p_2 denote the price of the risk free and risky asset, respectively. The marginal rate of substitution (MRS) between the risk free asset and the risky asset can be calculated from the inverse demands, yielding

$$m_{12}(y_1, y_2) = \frac{p_1}{p_2} = \frac{\frac{\pi_1^2}{r_1 y_2 + y_1} + \frac{\pi_2^2}{r_2 y_2 + y_1}}{\frac{r_1 \pi_1^2}{r_1 y_2 + y_1} + \frac{r_2 \pi_2^2}{r_2 y_2 + y_1}}. \tag{A.3}$$

If the demands (A.1) and (A.2) were generated by the maximization of an expected utility function, it follows from Dybvig and Polemarchakis (1981) that

$$\begin{aligned} -\frac{u''(x)}{u'(x)} &= \frac{\frac{\partial m(x,0)}{\partial y_2} ER}{m_{12}(x,0)ER^2 - ER} = \frac{\frac{\pi_1^2 \pi_2^2 (r_1 - r_2)^2 (\pi_1 r_1 + \pi_2 r_2)}{\left(\pi_1^2 r_1 + \pi_2^2 r_2\right)^2 x}}{\frac{\left(\pi_1^2 + \pi_2^2\right)\left(\pi_1 r_1^2 + \pi_2 r_2^2\right)}{\left(\pi_1^2 r_1 + \pi_2^2 r_2\right)} - (\pi_1 r_1 + \pi_2 r_2)} \\ &= \frac{\pi_1^2 \pi_2^2 (r_1 - r_2)^2 (\pi_1 r_1 + \pi_2 r_2)}{\left(\begin{array}{c} \left(\pi_1^2 + \pi_2^2\right) \left(\pi_1 r_1^2 + \pi_2 r_2^2\right) \left(\pi_1^2 r_1 + \pi_2^2 r_2\right) \\ - (\pi_1 r_1 + \pi_2 r_2) \left(\pi_1^2 r_1 + \pi_2^2 r_2\right)^2 \end{array} \right) x} \\ &= \frac{\pi_1 \pi_2 (r_1 - r_2)(\pi_1 r_1 + \pi_2 r_2)}{(\pi_2 r_1 - \pi_1 r_2)\left(\pi_1^2 r_1 + \pi_2^2 r_2\right) x}, \end{aligned} \tag{A.4}$$

implying that

$$u(x) = -\frac{x^{-\rho}}{\rho}, \tag{A.5}$$

where

$$\rho = \frac{\pi_1\pi_2 (r_1 - r_2)(\pi_1 r_1 + \pi_2 r_2)}{(\pi_2 r_1 - \pi_1 r_2)(\pi_1^2 r_1 + \pi_2^2 r_2)} - 1$$

$$= \frac{(\pi_1 - \pi_2)(\pi_1^2 + \pi_2^2) r_1 r_2}{(\pi_2 r_1 - \pi_1 r_2)(\pi_1^2 r_1 + \pi_2^2 r_2)}, \tag{A.6}$$

which is not zero. However, it can be verified that the demand functions (A.1) and (A.2) are generated by the non-expected utility function

$$\sum_{s=1}^{2} \pi_s^2 \ln (y_1 + r_s y_2). \tag{A.7}$$

In the above example, if probabilities and payoffs enter into the demand functions as numbers, then the identified NM index (A.5) is well defined and hence we can mistakenly conclude that the preferences are represented by

$$\sum_{s=1}^{2} \pi_s u (y_1 + r_s y_2). \tag{A.8}$$

But if probabilities and payoffs enter into the demand functions as variables (symbols), then since the identified NM index is probability and payoff dependent, we can conclude that the preferences are not representable by an expected utility function. The following example shows that even if probabilities and payoffs enter into the demand functions as variables (symbols), the identification process may still go wrong if the preferences are not expected utility representable.

Example 2. *Consider the following non-expected utility defined over contingent claims*

$$-A\sum_{s=1}^{S} \pi_s (x_s - 1)^2 - \sum_{s=1}^{S} \frac{1}{S}\left(x_s - \frac{1}{S}\sum_{i=1}^{S} x_i\right)^4, \tag{A.9}$$

where $A > 0$, $0 < x_s \ll 1$, and

$$x_s = \sum_{j=1}^{J} r_{js} y_j \quad (s \in \{1, ..., S\}).$$

Then along the diagonal, $x_i = x_j$ $(i, j \in \{1, ..., S\})$, we have

$$\frac{\partial}{\partial y_i}\left(-\sum_{s=1}^{S} \frac{1}{S}\left(x_s - \frac{1}{S}\sum_{i=1}^{S} x_i\right)^4\right)$$

$$= -\sum_{s=1}^{S} \frac{4}{S}\left(r_{is} - \frac{1}{S}\sum_{j=1}^{S} r_{ij}\right)\left(x_s - \frac{1}{S}\sum_{j=1}^{S} x_j\right)^3 = 0,$$

and

$$\frac{\partial^2}{\partial y_i^2}\left(-\sum_{s=1}^{S}\frac{1}{S}\left(x_s-\frac{1}{S}\sum_{i=1}^{S}x_i\right)^4\right)$$
$$=-\sum_{s=1}^{S}\frac{12}{S}\left(r_{is}-\frac{1}{S}\sum_{j=1}^{S}r_{ij}\right)^2\left(x_s-\frac{1}{S}\sum_{j=1}^{S}x_j\right)^2=0.$$

Next we want to argue that when $0<x_s\ll 1$ *and* A *is large enough, the utility function*

$$-A\sum_{s=1}^{S}\pi_s\left(x_s-1\right)^2-\sum_{s=1}^{S}\frac{1}{S}\left(x_s-\frac{1}{S}\sum_{i=1}^{S}x_i\right)^4 \tag{A.10}$$

is increasing and concave in each of the contingent claims. Since

$$\frac{\partial}{\partial x_i}\left(-A\sum_{s=1}^{S}\pi_s\left(x_s-1\right)^2-\sum_{s=1}^{S}\frac{1}{S}\left(x_s-\frac{1}{S}\sum_{i=1}^{S}x_i\right)^4\right)$$
$$=-2A\pi_i\left(x_i-1\right)-\sum_{s=1}^{S}\frac{4}{S}\left(x_s-\frac{1}{S}\sum_{j=1}^{S}x_j\right)^3\left(\delta_{si}-\frac{1}{S}\right), \tag{A.11}$$

$-2A\pi_i\left(x_i-1\right)>0,$ *and*

$$\sum_{s=1}^{S}\frac{4}{S}\left(x_s-\frac{1}{S}\sum_{j=1}^{S}x_j\right)^3\left(\delta_{si}-\frac{1}{S}\right)$$

is bounded, when A *is large enough, eqn. (A.11) is always positive. Since*

$$\frac{\partial^2}{\partial x_i^2}\left(-A\sum_{s=1}^{S}\pi_s\left(x_s-1\right)^2-\sum_{s=1}^{S}\frac{1}{S}\left(x_s-\frac{1}{S}\sum_{i=1}^{S}x_i\right)^4\right)$$
$$=-2A\pi_i-\sum_{s=1}^{S}\frac{12}{S}\left(x_s-\frac{1}{S}\sum_{j=1}^{S}x_j\right)^2\left(\delta_{si}-\frac{1}{S}\right)^2, \tag{A.12}$$

$-2A<0$ *and*

$$-\sum_{s=1}^{S}\frac{12}{S}\left(x_s-\frac{1}{S}\sum_{j=1}^{S}x_j\right)^2\left(\delta_{si}-\frac{1}{S}\right)^2<0,$$

eqn. (A.12) is always negative. As a consequence, if we apply the risk free asset identification, that uses information only on (or, since it uses *derivatives, a neighborhood of) the diagonal, in eqn. (A.9), the second term* $-\sum_{s=1}^{S}\frac{1}{S}\left(x_s-\frac{1}{S}\sum_{i=1}^{S}x_i\right)^4$ *will be invisible. Thus, we shall identify the expected utility corresponding to the first term* $-A\sum_{s=1}^{S}\pi_s\left(x_s-1\right)^2$ *that, away from the diagonal, generates a different set of asset demand functions.*

Appendix B

Revealed preference tests with ambiguity

The revealed preference results for smooth ambiguity preferences (and generalized smooth ambiguity preferences) that follow extend previous results to an incomplete asset market setting; and they support the identification results presented in Section 3.

Smooth ambiguity

To test the smooth ambiguity preferences (1) or (2), consider a data set

$$\mathfrak{D}^N=\{\boldsymbol{p}^n,\boldsymbol{y}^n,\boldsymbol{\nu}_a^n,\boldsymbol{R}\}_{a=1,\dots,A}^{n=1,\dots,N},$$

of N observations of asset prices $\boldsymbol{p}$, asset demands (portfolio choices) $\boldsymbol{y}$, families of conditional probability distributions $\boldsymbol{\nu}$, and an asset payoff matrix $\boldsymbol{R}$.

In the following lemma, we state the conditions assuming the $\boldsymbol{\nu}$ is known and can either vary or be fixed across the set of demand and price observations, but the $\boldsymbol{\mu}$ is unknown and fixed. If the probability measure $\boldsymbol{\mu}$ is observable (and variable across observations), the conditions are still necessary and sufficient for the existence of smooth ambiguity preferences.

Lemma 2. *The following conditions are equivalent:*

(i) *There exists a continuous utility function*[19]

$$U\left(\boldsymbol{R}\boldsymbol{y};\boldsymbol{\nu}\right)=\sum_{a=1}^{A}\mu_a\widetilde{\phi}\left(\sum_{s=1}^{S}\nu_{as}u\left(\sum_{j=1}^{J}r_{sj}y_j\right)\right),\tag{B.1}$$

where $\widetilde{\phi}$ *and* u *are twice continuously differentiable, strictly increasing, and strictly concave on their domain,*[20] *such that, for all* $n\in\{1,...,N\}$,

$$\boldsymbol{y}^n\in\arg\max_{\boldsymbol{y}\in\boldsymbol{Y}}U\left(\boldsymbol{R}\boldsymbol{y};\boldsymbol{\nu}^n\right)\quad s.t.\ \boldsymbol{p}^n\cdot\boldsymbol{y}\leq\boldsymbol{p}^n\cdot\boldsymbol{y}^n.\tag{B.2}$$

[19]Here, we use the representation (1) since it is not easy to deal with u^{-1} in the revealed preference test. Evidently, since, given $\widetilde{\phi}$ and u, we can simply define $\phi=\widetilde{\phi}\circ u$ to obtain the utility (2), the revealed preference test in Lemma 2 also works for the representation (2).

[20]As noted above, $\widetilde{\phi}=\phi\circ u^{-1}$.

(ii) *There exist real numbers* $(U_s^n, M_s^n)_{s=1,...,S}^{n=1,...,N} > 0$,[21] $(\Phi_a^n)_{a=1,...,A}^{n=1,...,N}$, $(K_a^n)_{a=1,...,A}^{n=1,...,N}$ > 0, $(\mu_a)_{a=1}^A > 0$ *and* $(\lambda^n)_{n=1}^N > 0$, *such that for all* $n, m \in \{1, ..., N\}$, $s, s' \in \{1, 2, ..., S\}$, $a, a' \in \{1, 2, ..., A\}$ *and* $j \in \{1, 2, ..., J\}$,

$$U_s^n - U_{s'}^m < M_{s'}^m \left(\sum_{j=1}^J r_{sj} y_j^n - \sum_{j=1}^J r_{s'j} y_j^m \right), \tag{B.3}$$

with equality if $\sum_{j=1}^J r_{sj} y_j^n = \sum_{j=1}^J r_{s'j} y_j^m$;

$$\Phi_a^n - \Phi_{a'}^m < K_{a'}^m \left(\sum_{s=1}^S \nu_{as}^n U_s^n - \sum_{s=1}^S \nu_{a's}^m U_s^m \right), \tag{B.4}$$

with equality if $\sum_{s=1}^S \nu_{as}^n U_s^n = \sum_{s=1}^S \nu_{a's}^m U_s^m$; *and*

$$\sum_{a=1}^A \left(\mu_a K_a^n \sum_{s=1}^S \nu_{as}^n M_s^n r_{sj} \right) = \lambda^n p_j^n. \tag{B.5}$$

Remark 16. *The revealed preference test stated here is a minor extension of Bayer, Bose, Polisson, and Renou (2013). The conditions in (ii) are analogous to those in traditional revealed preference tests such as in Varian (1983b). Conditions (B.3), (B.4) and (B.5) correspond, respectively, to the strict concavity inequality of the von Neumann-Morgenstern (NM) index in each ambiguity state, the strict concavity inequalities of the ambiguity index, that has as its argument* $\sum_{s=1}^S \nu_{as}\, u \left(\sum_{j=1}^J r_{sj} y_j \right)$, *and the first order conditions of the portfolio optimization. We omit the proof.*

It should be noted that in condition (ii) in Lemma 2, the strict inequalities are satisfied, implying that SARP is satisfied. It is obvious that SARP is only necessary but not sufficient for the preferences to be representable by a strictly concave smooth ambiguity model.

Generalized ambiguity

We next derive a revealed preference test for the case where the probability measure over ambiguity states, $\boldsymbol{\mu}$, is not referred to. The test is based on maximization of the generalized smooth ambiguity preference representation $U(\boldsymbol{x}; \boldsymbol{\nu}) = \Phi(..., E_{\boldsymbol{\nu}_a} u(x_s), ...)$ discussed in Section 2. The assumed data set is

$$\mathfrak{D}^N = \{\boldsymbol{p}^n, \boldsymbol{y}^n, \boldsymbol{\nu}_a^n, \boldsymbol{R}\}_{a=1,...,A}^{n=1,...,N}.$$

[21] In condition (ii), the numbers $(U_s^n)_{s=1,...,S}^{n=1,...,N}$ represent the utility levels, which are not necessarily positive. However, the translation of any negative solution by a positive constant will still be a solution, and positivity of these numbers is without loss of generality.

Lemma 3. *The following conditions are equivalent:*

(i) *There exists a continuous utility function*[22]

$$U\left(\boldsymbol{Ry};\boldsymbol{\nu}\right)=\Phi\left(...,\sum_{s=1}^{S}\nu_{as}u\left(\sum_{j=1}^{J}r_{sj}y_{j}\right),...\right), \tag{B.6}$$

where u is twice continuously differentiable, strictly increasing, and strictly concave, and Φ is continuously differentiable, strictly increasing, and strictly quasi-concave on their domain, such that for all $n \in \{1,...,N\}$,

$$\boldsymbol{y}^{n}\in\arg\max_{\boldsymbol{y}\in\boldsymbol{Y}}U\left(\boldsymbol{Ry};\boldsymbol{\nu}^{n}\right)\quad s.t.\ \boldsymbol{p}^{n}\cdot\boldsymbol{y}\leq\boldsymbol{p}^{n}\cdot\boldsymbol{y}^{n}. \tag{B.7}$$

(ii) *There exist real numbers $(U_s^n, M_s^n)_{s=1,...,S}^{n=1,...,N} > 0$, $(\Phi^n)_{n=1}^N$, $(K_a^n)_{a=1,...,A}^{n=1,...,N} > 0$, and $(\lambda^n)_{n=1}^N > 0$ such that, for all $n, m \in \{1,2,...,N\}$, $s, s' \in \{1,2,...,S\}$, $a, a' \in \{1,2,...,A\}$ and $j \in \{1,2,...,J\}$,*

$$U_{s}^{n}-U_{s'}^{m}<M_{s'}^{m}\left(\sum_{j=1}^{J}r_{sj}y_{j}^{n}-\sum_{j=1}^{J}r_{s'j}y_{j}^{m}\right), \tag{B.8}$$

with equality if $\sum_{j=1}^{J} r_{sj}y_j^n = \sum_{j=1}^{J} r_{s'j}y_j^m$;

$$\Phi^{n}-\Phi^{m}<\sum_{a=1}^{A}K_{a}^{m}\left(\sum_{s=1}^{S}\nu_{as}^{n}U_{s}^{n}-\sum_{s=1}^{S}\nu_{a's}^{m}U_{s}^{m}\right), \tag{B.9}$$

with equality if $\sum_{s=1}^{S}\nu_{as}^{n}U_s^n = \sum_{s=1}^{S}\nu_{a's}^{m}U_s^m$; and

$$\sum_{a=1}^{A}\left(K_{a}^{n}\sum_{s=1}^{S}\nu_{as}^{n}M_{s}^{n}r_{sj}\right)=\lambda^{n}p_{j}^{n}. \tag{B.10}$$

Appendix C

Identification from portfolio indifference

Knowledge of the individual's portfolio indifference correspondence $I(\boldsymbol{y};\boldsymbol{\nu})$ gives the functional form of the indifference curve. In the case without ambiguity, indifference correspondence was used to identify individuals' preferences in Dybvig and Polemarchakis (1981), Dybvig (1983) and as early

[22] Following an argument similar to that in footnote 19, we do not include u^{-1} in the following representation.

as Yaari (1969), who used the term "acceptance frontier" instead. An individual's portfolio indifference correspondence $I(\boldsymbol{y};\boldsymbol{\nu})$ can be observed or estimated if this individual can specify all the portfolios she regards as indifferent to a particular portfolio $\boldsymbol{y}$ under conditional probability distributions $\boldsymbol{\nu}$. Proposition 1 demonstrates that identification from such information is possible.

Suppose that

(1) the smooth ambiguity utility (2) satisfies the condition that $\phi\left(u^{-1}(\cdot)\right)$ is strictly concave on $\mathbb{R}_{++}$, with the indices u and ϕ both being twice continuously differentiable, strictly increasing, and strictly concave on $\mathbb{R}_{++}$,

(2) there is an asset $j = 1$ that is risk free, where $\boldsymbol{r}_1 = 1$ across states of the world, and

(3) the family of conditional probability measures over states of risk, $\boldsymbol{\nu}: \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is known.

Proposition 1. *If*

(1) there is an asset $j = 2$ that is ambiguity free: its payoff distribution is invariant to the states of ambiguity, and

(2) the matrix

$$\begin{bmatrix} E_{\boldsymbol{\nu}_1}\boldsymbol{r}_2 & \dots & E_{\boldsymbol{\nu}_1}\boldsymbol{r}_J \\ \dots & E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j & \dots \\ E_{\boldsymbol{\nu}_A}\boldsymbol{r}_2 & \dots & E_{\boldsymbol{\nu}_A}\boldsymbol{r}_J \end{bmatrix}_{A\times(J-1)}$$

has full row rank A,

then, the portfolio indifference correspondence identifies the risk index u on $\mathbb{R}_{++}$ and the ambiguity index ϕ on $\mathbb{R}_{++}$, each up to a positive affine transformation, as well as the ambiguity state probability measure $\boldsymbol{\mu}$.

Proof. We argue in a series of steps.

Step 1—identifying the risk index u.

Consider, in portfolio space $\mathbb{R}^J$, the plane $\Lambda_j = \{\boldsymbol{y} \in \mathbb{R}^J : y_i = 0, i \neq 1 \text{ or } j\}$. For any point $\overline{\boldsymbol{y}} = (\overline{y}_1, 0, ..., \overline{y}_j, ..., 0)$ in the plane Λ_j, from the implicit function theorem, in some neighborhood $\aleph_j$ of $(\overline{y}_1, 0, ..., \overline{y}_j, ..., 0)$, y_1 can be written as a unique twice continuously differentiable function $y_1 = f_j(y_j;\boldsymbol{\nu})$ such that

$$E_{\boldsymbol{\mu}}\phi(u^{-1}(E_{\boldsymbol{\nu}_a}u(f_j(y_j;\boldsymbol{\nu})\boldsymbol{r}_1 + y_j\boldsymbol{r}_j))) = \overline{U} \quad \text{(C.1)}$$

everywhere on $\aleph_j$. This is the parametric expression of an individual's indifference curve passing through $\overline{\boldsymbol{y}}$ in the plane Λ_j, and therefore function f_j is observable.

For each j $(j = 2, ..., J)$, totally differentiating equation (C.1) with respect to y_j gives

$$E_{\boldsymbol{\mu}}\phi^{'}(u^{-1}(Eu_{\boldsymbol{\nu}_a}(f_j(y_j;\boldsymbol{\nu})\boldsymbol{r}_1+y_j\boldsymbol{r}_j)))\frac{E_{\boldsymbol{\nu}_a}u^{'}(f_j(y_j;\boldsymbol{\nu})\boldsymbol{r}_1+y_j\boldsymbol{r}_j)(f_j^{'}(y_j;\boldsymbol{\nu})\boldsymbol{r}_1+\boldsymbol{r}_j)}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(f_j(y_j;\boldsymbol{\nu})\boldsymbol{r}_1+y_j\boldsymbol{r}_j)))}$$

$$= 0. \tag{C.2}$$

We restrict attention to plane Λ_2 and the corresponding function f_2. From the fact that the payoffs of asset 1 and 2 are invariant to ambiguity states, there exists a probability measure, $\tilde{\boldsymbol{\nu}} \in \Delta(\boldsymbol{S})$, and a matrix of asset payoffs over states of risk $\tilde{\boldsymbol{R}} = (\boldsymbol{1}_{\#\boldsymbol{S}}, \tilde{\boldsymbol{r}}_2)$, such that, the distribution of asset payoffs generated by $(\boldsymbol{\nu}_a, \boldsymbol{R}\boldsymbol{y})$, for any state of ambiguity, coincides with the distribution generated by $(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})$.

With $j = 2$, the above equation (C.2) becomes

$$f_2^{'}(\overline{y}_2;\tilde{\boldsymbol{\nu}}) = -\frac{E_{\tilde{\boldsymbol{\nu}}}u^{'}(f_2(\overline{y}_2;\tilde{\boldsymbol{\nu}})\boldsymbol{r}_1+\overline{y}_2\tilde{\boldsymbol{r}}_2)\tilde{\boldsymbol{r}}_2}{E_{\tilde{\boldsymbol{\nu}}}u^{'}(f_2(\overline{y}_2;\tilde{\boldsymbol{\nu}})\boldsymbol{r}_1+\overline{y}_2\tilde{\boldsymbol{r}}_2)\boldsymbol{r}_1}. \tag{C.3}$$

Further totally differentiating equation (C.3) with respect to y_2, we have

$$f_2^{''}(\overline{y}_2;\tilde{\boldsymbol{\nu}}) = -\frac{E_{\tilde{\boldsymbol{\nu}}}u^{''}(f_2(\overline{y}_2;\tilde{\boldsymbol{\nu}})\boldsymbol{r}_1+\overline{y}_2\tilde{\boldsymbol{r}}_2)(f_2^{'}(\overline{y}_2;\tilde{\boldsymbol{\nu}})\boldsymbol{r}_1+\tilde{\boldsymbol{r}}_2)^2}{E_{\tilde{\boldsymbol{\nu}}}u^{'}(f_2(\overline{y}_2;\tilde{\boldsymbol{\nu}})\boldsymbol{r}_1+\overline{y}_2\tilde{\boldsymbol{r}}_2)\boldsymbol{r}_1}. \tag{C.4}$$

At $(\overline{y}_1, \overline{y}_2, 0, ..., 0)$ with $\overline{y}_2 = 0$,

$$-\frac{u^{''}(\overline{y}_1)}{u^{'}(\overline{y}_1)} = \frac{f_2^{''}(0;\tilde{\boldsymbol{\nu}})}{E_{\tilde{\boldsymbol{\nu}}}(f_2^{'}(0;\tilde{\boldsymbol{\nu}})+\tilde{\boldsymbol{r}}_2)^2}. \tag{C.5}$$

Since the individual's indifference correspondence is observable, so are $f_2^{'}(\cdot)$ and $f_2^{''}(\cdot)$. Therefore, $-\frac{u^{''}(\overline{y}_1)}{u^{'}(\overline{y}_1)}$ is observable for all $\overline{y}_1 > 0$, and the risk index u will be identified on $\mathbb{R}_{++}$, up to a positive affine transformation.

Step 2—identifying the probability measure $\boldsymbol{\mu}$*.*

For each j $(j = 2, ..., J)$, if we restrict attention to the portfolio $\tilde{\boldsymbol{y}} = (x, 0, ..., 0)$, equation (C.2) gives

$$E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j = -f_j^{'}(0;\boldsymbol{\nu}),$$

which can be written in matrix form

$$[\mu_1, ..., \mu_A]\begin{bmatrix} E_{\boldsymbol{\nu}_1}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_1}\boldsymbol{r}_J \\ ... & E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j & ... \\ E_{\boldsymbol{\nu}_A}\boldsymbol{r}_2 & ... & E_{\boldsymbol{\nu}_A}\boldsymbol{r}_J \end{bmatrix} = [-f_2^{'}(0;\boldsymbol{\nu}), ..., -f_J^{'}(0;\boldsymbol{\nu})]. \tag{C.6}$$

The full row rank condition (2) implies that the probability measure $\boldsymbol{\mu}$ can be uniquely identified.

Step 3—identifying the ambiguity index ϕ.

For an ambiguous asset j (i.e., $j \neq 2$), further differentiating equation (C.2) with respect to y_j, and evaluating the resultant equation at the portfolio $(\overline{y}_1, 0, ..., \overline{y}_j, ..., 0)$ with $\overline{y}_j = 0$, we get

$$[E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j)^2]\frac{\phi^{''}(\overline{y}_1)}{\phi^{'}(\overline{y}_1)} =$$

$$[E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j)^2 - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_j - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j)^2]\frac{u^{''}(\overline{y}_1)}{u^{'}(\overline{y}_1)} + f_j^{''}(0;\boldsymbol{\nu}). \tag{C.7}$$

Since the conditional probability measures over risk states, $\boldsymbol{\nu}$, are known, and the probability measure over ambiguity states, $\boldsymbol{\mu}$, has been identified, all the moments $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j$, $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_j)^2$ and $E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j)^2$, can be computed. The full row rank condition implies that there exists at least one ambiguous asset j such that $(E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j)^2 \neq E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_j)^2$, i.e., the coefficient of $\frac{\phi^{''}(\overline{y}_1)}{\phi^{'}(\overline{y}_1)}$ does not vanish for all $\overline{y}_1 > 0$. Given the risk index u identified, equation (C.7) in turn identifies the ambiguity index ϕ on $\mathbb{R}_{++}$, up to a positive affine transformation. □