

Perils of unconventional monetary policy¹

Michael McMahon² Udara Peiris³
Herakles Polemarchakis⁴

November 3, 2015
Current: December 29, 2016

¹This work was circulated earlier under the title “Perils of quantitative easing,” Discussion paper No.04, CRETA, Department of Economics, University of Warwick, November 2015. We would like to thank Ed Nelson for his very useful discussion of our work, John Geanakoplos for comments on outside money, and Ricardo Reis for helpful discussions on central bank balance sheet policy; also, (seminar) participants at Bank of England conference “Learning the Lessons from QE and other Unconventional Monetary Policies,” the Summer Workshop at the HKIMR, the SAET Conference, SWET, Core, Cowles and the University of Warwick for their comments and suggestions. Peiris’s work on this study has been supported by the Russian Academic Excellence Project ‘5-100.’

²University of Warwick, CEPR, CAGE (Warwick), CfM (LSE), and CAMA (ANU). Email: m.mcmahon@warwick.ac.uk

³ICEF, NRU Higher School of Economics, Russian Federation. Email: upeiris@hse.ru

⁴Department of Economics, University of Warwick. Email: h.polemarchakis@warwick.ac.uk

Abstract

Unconventional monetary policy, by relaxing restrictions on the composition of the balance sheet of the central bank, compromises control over the stochastic path of inflation; or, in open economies, over the stochastic path of exchange rates. If the composition of the balance sheet is unrestricted (either left to market demand or governed by a rule that depends on expected inflation rates) then a unique path of inflation cannot be implemented. This is the case under pure quantitative easing, where the target is the size of real money balances. In contrast, credit easing policies restrict the composition of the portfolio by targeting a specific expansion in the maturity profile of bonds bought, and so can implement a unique path of inflation. The composition of the portfolios traded by monetary-fiscal authorities determines premia in asset and currency markets.

Keywords: unconventional monetary policy; stochastic path of inflation.

JEL Codes: D50; E31; E52

Conventional monetary policy restricts assets on the balance sheet of the central bank to short-term Treasury Bills. Much analysis takes this as given and, as a result, the importance of restrictions on the central bank asset portfolio is typically overlooked. Unconventional monetary policy relaxes the restrictions on the central bank asset portfolio and allows for assets of varying maturity and risk profiles. In this paper, we explore what this potentially less-restricted portfolio means for the ability of the central bank to control the stochastic path of inflation.

We consider a stochastic cash-in-advance economy with flexible prices and a perfect, in particular complete, asset market, and we restrict attention to trades in securities of one-date maturity; trades in long-lived assets can duplicate such trades, as in [Kreps \(1982\)](#), and allow for a role for the maturity structure of debt, as in [Cochrane \(2001\)](#) or [Angeletos \(2002\)](#). Well-founded criticisms of the fiscal theory of the price level in [Buiter \(2002\)](#) and [Drèze and Polemarchakis \(2000\)](#) notwithstanding, we make the assumption of non-Ricardian seigniorage policy for the central bank; this, to remain in an environment that, under conventional monetary policy, yields a determinate price level. Our conclusion, that, surprisingly, has gone unnoticed, is that monetary policy, that sets a path of short-term, nominal interest rates, determines the path of expected or average inflation, but not the distribution of possible paths of inflation. The key result is that the stochastic path of inflation is determined by the adjustment of the portfolio of the monetary authority over time in response to market forces and expectations. Without adequate restrictions on the asset portfolio, indeterminacy is pervasive. In an open economy, indeterminacy extends to the stochastic path of exchange rates.

As uncertainty unfolds, market forces and expectations determine the value of the assets on the balance sheet. Variations in the value of the balance sheet drive the stochastic path of monetary injections or withdrawals that, in turn, determines the path of inflation; [Sims \(2016\)](#) explains this. Once portfolio allocations are selected, the distribution of inflation outcomes is unique. In fact, with full knowledge of the structure of the economic environment, the central bank could choose the portfolio weights to target a specific distribution of inflation. However, if the portfolio allocations are left unrestricted, the distribution of inflation is not pinned down.

Under a conventional (restricted) central bank asset portfolio, there is a unique stochastic distribution of inflation outcomes even though central banks typically target only expected inflation. That is, even though restrictions on the portfolio of assets are driven by risk aversion on behalf of the central bank, they imply that the central bank can keep the distribution of inflation anchored as it targets expected inflation and also restricts the port-

folio of assets held. However, when a central bank shifts to unconventional policies, as is often the case in response to hitting the zero lower bound, this changes the distribution of inflation driven by the change in the central bank portfolio. The change in the stochastic distribution of inflation depends on the change in portfolio policy. The composition of the portfolios traded by monetary-fiscal authorities determines premia in asset and currency markets.

While a common feature of unconventional policies is that these policies have expanded the central bank balance sheet, central banks have employed two distinct approaches to these policies which we discuss in detail below; namely, quantitative easing (QE) and credit easing (CE). QE focuses only on the expansion of the central bank liabilities and does not restrict the asset composition of the balance sheet. By contrast, CE targets a specific allocation of assets, much like conventional monetary policy that restricts open market operations to Treasury bills. Under CE, it is the explicit target for the composition of the balance sheet that allows the monetary authority to target the stochastic path of inflation: the target for the composition of the portfolio guarantees the necessary restrictions to obtain determinacy of the inflation distribution and limit the de-anchoring of the inflation distribution.

We also show that even policy rules that select the portfolio weights as a function of expected inflation (or any other nominal variable), do not overcome the indeterminacy problem. This may be surprising since policy rules may seem to imply restriction; as we show, they leave portfolio weights effectively unrestricted.

The indeterminacy of QE in our benchmark model is nominal; while the central bank loses the control of inflation, the indeterminacy does not affect the attainable equilibrium allocations. If the central bank were to switch to a money supply, rather than interest rate, policy, the indeterminacy would be real: it would affect real allocations. More importantly, the indeterminacy would be real if prices were sticky or the asset market is incomplete, as in [Bai and Schwarz \(2006\)](#).

There is a vast and important literature on indeterminacy of monetary equilibria: [Sargent and Wallace \(1975\)](#) pointed out the indeterminacy of the initial price level under interest rate policy; [Lucas and Stokey \(1987\)](#) derived the condition for the uniqueness of a recursive equilibrium with money supply policy; [Woodford \(1994\)](#) analysed the dynamic paths of equilibria associated with the indeterminacy of the initial price level under money supply policy. In this paper, we give the exact characterisation of recursive equilibria under quantitative easing with interest rate policy.

The fiscal theory of the price level in [Woodford \(1994\)](#), and [Dupor \(2000\)](#) in an open economy, take it for granted that a monetary authority trades exclusively in short-term, nominally risk-free bonds. This is also the case

in [Dubey and Geanakoplos \(2003\)](#), who argue that “outside money” suffices to eliminate the indeterminacy that prevails in economies with nominally denominated assets; an important claim, because, as noted by [Cass \(1984, 1985\)](#) and analysed in depth in [Balasko and Cass \(1989\)](#) and [Geanakoplos and Mas-Colell \(1989\)](#), when the asset market is incomplete, nominal indeterminacy has real effects. Here, we highlight the importance of the composition of the portfolio of the monetary authority for the determinacy of the path of prices, even, the determinacy of the price level.

The possible multiplicity of stochastic inflation paths at equilibrium was clear in [Bloise, Drèze, and Polemarchakis \(2005\)](#) and [Nakajima and Polemarchakis \(2005\)](#); but there, the specification was Ricardian, equilibria were indeterminate, and the point was to demonstrate that the indeterminacy can be parametrised by the price level and a nominal martingale measure. [Magill and Quinzii \(2014b\)](#) developed the argument that inflationary expectations can serve as an alternative parametrisation, which is more interesting. [Drèze and Polemarchakis \(2000\)](#) pointed out the need for “comprehensive monetary policy” that sets the stochastic path of the term structure of interest rates (or, equivalently, all state-contingent short-term rates) in order to determine the path of inflation. This theme was later developed in [Adao, Correia, and Teles \(2014\)](#), and [Magill and Quinzii \(2014a\)](#). Importantly, in this argument, the way out of indeterminacy involved targets or restrictions on the returns of assets. Our point here is that the composition of the balance sheet of the monetary authority matters as an instrument of immediate policy relevance.

We extend our main result to show that feedback rules, that set interest rates or the composition of the balance sheet as a function of future variables, are not sufficient to obtain a determinate inflation path. “Simple” inflation processes may only be compatible with conventional monetary policy. Importantly, our argument does not derive from an open-ended horizon or the stability of a steady state as is standard in the literature on the determinacy of equilibrium paths under a Taylor rule, following [Taylor \(1993\)](#); important contributions are [Woodford \(1999\)](#) and [Benhabib, Schmitt-Grohe, and Uribe \(2001\)](#).

In [Curdia and Woodford \(2011\)](#), the portfolio under unconventional policies has the same risk-profile as (or is collinear with) the portfolio under conventional policy; yet, unconventional policies may have real effects in the presence of segmented markets. In practice central banks have accommodated trade to include non-collinear assets (bonds of longer maturities or private sector liabilities). It is the trade in these assets that we focus on and our formulation is in the spirit of [Eggertsson and Woodford \(2003\)](#) where the central bank chooses a portfolio among a set of state-contingent assets.

In practice, unconventional policies are employed during times of crisis,

when interest rates may be constrained at the zero lower bound, and they target an increase of the size of the balance sheet. We do not consider quantitative aspects of unconventional monetary policy or reasons that motivate changes in the size of the balance sheet. We focus on the implications of changes in the composition of risky assets in the portfolio of the central bank for the stochastic path of inflation. As we show, interest rates being at the effective zero lower bound do not alter our main finding.

The portfolio balance channel operates when bonds of different maturities are not perfect substitutes and traders have maturity-specific bond demands. In this setting, the maturity structure of outstanding debt can affect term premia. Theoretical models describing the portfolio balance channel such as [Vayanos and Vila \(2009\)](#) and [Hamilton and Wu \(2012\)](#) neglect the consequences of variations in the composition of the monetary authority portfolio on the stochastic path of inflation. We show that as the composition of the portfolios of monetary-fiscal authorities determine the stochastic path of prices, they also determine the nominal stochastic discount factor. Independent of changes in expectations about the path of short-term interest rates, the correlation between the discount factor and asset prices, and nominal exchange rates, then generates risk premia and biases whose size and direction corresponds to the chosen portfolio composition.

Since the global financial crisis of 2007, there is a growing view that variations in capital flows should be managed, and that these variations may stem from the monetary policy of trading partners. Our results contribute to this view by highlighting that QE transmits indeterminacy across central bank portfolios and, consequently the path of exchange rates. If markets are incomplete within each country, then fluctuations in central bank portfolios resonate abroad by affecting, not only the nominal exchange rate, but also asset prices and risk premia. Furthermore, if central banks set interest rates according to a Taylor-type rule that accounts for changes in the nominal exchange rate, as in [Taylor \(2001\)](#), and trading partners conducted QE, then they would not be able to guarantee the desired outcomes can be implemented. In other words, QE by trading partners manifests itself as indeterminacy of both nominal and real risk-premia, globally, and more importantly, even in countries that conduct traditional monetary policy or CE.

Unconventional monetary policy in practice

Before turning to the model, we first explore the practical adoption of unconventional monetary policy in major economies in order to give context to the model. There is general agreement on the objective of unconventional policies

as a mechanism to support credit and liquidity. However the implementation of policies vary as [Bernanke \(2009\)](#) explains:

“The Federal Reserve’s approach to supporting credit markets is conceptually distinct from quantitative easing (QE), the policy approach used by the Bank of Japan from 2001 to 2006. Our approach—which could be described as ‘credit easing’ (CE)—resembles quantitative easing in one respect: It involves an expansion of the central bank’s balance sheet. However, in a pure QE regime, the focus of policy is the quantity of bank reserves, which are liabilities of the central bank; the composition of loans and securities on the asset side of the central bank’s balance sheet is incidental.”

Heterogeneity of central bank implementation of unconventional policies which makes our theoretical distinction between the extreme cases of pure QE and pure CE particularly relevance. While no central bank has pursued pure QE or pure CE, some central banks have pursued policies much closer to CE and others policies much closer to QE. The key characteristic that makes the Fed’s policies closer to CE is that it sought to expand its balance sheet while committing to a specific asset composition in doing so.¹

As part of this, the Federal Reserve Bank of New York published how (in terms of portfolio weights) the total Large Scale Asset Purchases (LSAP) program purchases would be distributed across maturity sectors. Moreover, each two-week-long round of asset purchases would begin every-other Wednesday when the SOMA Desk would announce, along with the specific days on which it would be conducting the auctions, the maturity sectors in which it would be buying over the subsequent two weeks. In our model the indeterminacy is resolved by ex-ante restrictions on the composition of assets held on the central bank balance sheet.

The Fed’s approach contrasts somewhat with some other major central banks easing policies. During the first round of QE undertaken between 2001 and 2006, the Bank of Japan (BOJ) set new operational targets, called Current Account Balances, for monetary policy in terms of the central bank reserves held by financash-in-advance constraintl intermediaries . To achieve these targets, it made outright purchases of a long-term Japanese government bonds, stocks held by commercash-in-advance constraintl banks (from October 2002 to September 2003) and ABS (July 2003 to March 2006), but

¹The Federal Reserve reduced the target federal funds rate to effectively zero and also implemented a number of other programs and policies which led to significant changes to the Federal Reserve’s balance sheet; for example, [Bernanke \(2009\)](#), [Goodfriend \(2011\)](#), [Reis \(2009\)](#) and [Fawley and Christopher \(2013\)](#).

the specific portfolio of these assets was not the target of the central bank. Recent unconventional policies by the BOJ have targeted lending to banks rather than the outright purchase of assets from secondary markets though [Fawley and Christopher \(2013\)](#) argue that the BOJ was mainly concerned with generating reserves and provided limited restrictions on the range of assets. One may argue that the assets purchased by the BOJ were motivated by expectations of prices and premia. We show that even such a policy is insufficient to rule out indeterminacy. [Ugai \(2007\)](#) and [Maeda et al. \(2005\)](#) discuss the details of the BOJ experience with QE in more detail.

Recent unconventional policies by the ECB have also targeted lending to banks rather than the outright purchase of assets from secondary markets. Nonetheless, [Fawley and Christopher \(2013\)](#) argues that these programs may also be “considered pure QE in the sense that they targeted reserves and typically accepted a wide range of assets as collateral”. The Bank of England’s QE scheme was similar to the early BOJ scheme. The Monetary Policy Committee set an overall target for the amount of assets purchased through the Asset Purchase Facility (APF), the composition of assets was not the target. However, the Bank of England set some restrictions on the asset portfolio (medium- and long-term gilts) making the APF somewhat closer to the Fed, and CE, than the BOJ.²

1 The analytical argument

Monetary policy involves *quantitative easing* if open market operations extend to unrestricted portfolios of government bonds of different maturities or bonds issued by the private sector. It involves *credit easing* if open market operations extend beyond treasuries, but still target a specific composition for the balance sheet of the monetary-fiscal authority; as a limit case, monetary policy is *conventional* when open market operations are restricted to Treasury Bills, short term, nominally risk-free assets.

Fiscal policy is *Ricardian* if it is restricted to satisfy an intertemporal budget constraint or transversality condition; equivalently, if public debt vanishes for all possible, equilibrium or non-equilibrium, values of prices and interest

²Established in January 2009, the APF was first used as a tool of monetary policy in March 2009. Initially the APF would buy high-grade corporate bonds and government gilts with maturity 5-25 years, but more recently the APF only bought conventional gilts, though over a slightly extended maturity range starting at three years. In any given purchase operation, a broad range of assets, such as gilts with maturities 10-25 years, were up for purchase; a reverse auction determined the allocation and prices, remained determined by market conditions.

rates. It is *non-Ricardian*, if it is not restricted to satisfy an intertemporal budget constraint; in particular, outside money or initial liabilities of the public towards the private sector are not taxed back.

Quantitative easing generates indeterminacy indexed by a *nominal pricing measure* over states of the world. This measure determines the distribution of rates of inflation, up to a moment that is determined by the risk-free rate and non-arbitrage. Ricardian policy leaves the initial price level indeterminate as well. Determinacy and, by extension, monetary and financial stability, obtain under credit easing or monetary policy that is conventional. The indeterminacy is nominal only as long as prices are flexible, monetary policy sets nominal rates of interest, and the asset market is effectively complete; otherwise, there are, generically, real effects. It is worth pointing as our analysis considers a process of continual re-balancing of the monetary-fiscal authority balance sheet, our argument applies equally to the unwinding of quantitative easing as well as to the initiation of it. What is essential is the type of policy that determines the stochastic evolution of the balance sheet.

In Section 1.1 and 1.2 we characterize unconventional monetary policy under pure quantitative easing where the composition of the assets traded by the central bank is unrestricted. We show the indeterminacy inherent in a stochastic economy and link it to the mix of interest and non-interest bearing assets traded by the monetary-fiscal authority. In Section 1.3 we show that this is not a consequence of non-stationary equilibria or of exogenous interest rate paths, and we make explicit the role of the composition of the portfolio of the monetary-fiscal authority portfolio in the determination of stochastic inflation rates. In the presence of pure quantitative easing, interest-rate feedback rules are insufficient to obtain determinacy. We then show that pure credit easing policies (which set portfolio weights exogenously) obtains determinacy while policies that allow for feedback rules determining the composition of assets is insufficient to rule out indeterminacy. Finally, restricting attention to “simple” inflation processes may only be compatible with conventional monetary policy.

1.1 The short run

Activity extends over dates $t = 0, 1$, while a final third date, $t = 2$, serves for accounting purposes.³ Uncertainty over states of the world, $s \in \{1 \dots, s, \dots S\}$, is realised at $t = 1$; each state occurs with probability $f(\cdot)$. These states could be purely extrinsic or they could refer to some other “fundamentals,” such as monetary policy shocks across states. To maintain notational consis-

³In particular, debts are settled in this final date; there is no uncertainty after date 1.

tency with later sections, a date-event at date t is denoted s^t ; with s^0 being one element, and $s^1|s^0$ being one of S elements, as is $s^2|s^1$.

At dates 0 and 1, a representative individual supplies $l(s^t)$ units of labour to produce perishable output $y(s^t) = l(s^t)$ in exchange for competitive nominal wages $w(s^t)$, while consumption is $c(s^t)$ and the price level is $p(s^t)$. As real wages are 1, equilibrium nominal wages equal the price level at each-date event.

Utility is derived from consumption of goods and leisure and the intertemporal utility is separable and that flow utility function is continuously differentiable, strictly increasing, strictly concave and satisfies boundary conditions. The lifetime utility of an individual is

$$u(c(s^0), 1 - l(s^0)) + \beta \sum_{s^1} u(c(s^1), 1 - y(s^1))f(s^1).$$

An individual cannot use labour income to purchase goods but must instead use cash obtained from the asset market. The asset market opens (and closes) before the goods market, and any cash proceeds from the sale of output must be carried over to the next date.⁴

An individual enters date 0 with nominal wealth $\tau(s^0)$; the asset market is then open and cash $\hat{m}(s^0)$ and a complete set of contingent claims are traded. He purchases an amount $\theta(s^1|s^0)$ of the elementary security that pays one unit of currency in state s^1 and zero otherwise; the price of this contingent claim is $q(s^1|s^0)$. While the asset market is open, the individual faces the budget constraint

$$\hat{m}(s^0) + \sum_{s^1|s^0} q(s^1|s^0)\theta(s^1|s^0) \leq \tau(s^0).$$

The nominally risk-free one-date interest rate at date 0 is $r(s^0)$; the no arbitrage condition yields

$$\frac{1}{1 + r(s^0)} = \sum_{s^1} q(s^1|s^0).$$

The cash-in-advance constraint implies that the household must have sufficient cash ($\hat{m}(s^0)$) to cover its purchases:

$$p(s^0)c(s^0) \leq \hat{m}(s^0).$$

The cash brought by the individual into the next date is

$$m(s^0) = \hat{m}(s^0) - p(s^0)z(s^0),$$

⁴Our timing and monetary structure closely follows the cash-in-advance models in [Lucas and Stokey \(1987\)](#) and [Nakajima and Polemarchakis \(2005\)](#).

where we $z(s^0) = c(s^0) - l(s^0)$ is the net demand of the individual at date 0 (with a similar definition for $z(s^1|s^0)$).

It follows that the cash-in-advance constraint is

$$m(s^0) \geq p(s^0)y(s^0)$$

or ⁵

$$\frac{r(s^0)}{1+r(s^0)}m(s^0) \geq \frac{r(s^0)}{1+r(s^0)}p(s^0)y(s^0).$$

At the start of the date 1, nature determines the state and household wealth is

$$\tau(s^1|s^0) = m(s^0) + \theta(s^1|s^0).$$

It follows that, at date 0, the household faces the constraint

$$p(s^0)z(s^0) + m(s^0) + \sum_s q(s^1|s^0)\theta(s^1|s^0) \leq \tau(s^0).$$

At date 1, uncertainty has been fully resolved. Similar reasoning yields a single constraint,

$$p(s^1|s^0)z(s^1|s^0) + m(s^1|s^0) + \frac{1}{1+r(s^1|s^0)}\theta(s^2|s^1) \leq \tau(s^1|s^0),$$

and the cash-in-advance constraint

$$m(s^1|s^0) \geq p(s^1|s^0)y(s^1|s^0).$$

At date 2, the final date, all debts are repaid; wealth at the end of the date cannot be negative:

$$\tau(s^2|s^1) = m(s^1|s^0) + \theta(s^2|s^1) \geq 0.$$

Since individuals strictly prefer more to less, there will be no slack.

The lifetime budget constraint is

$$p(s^0)z(s^0) + m(s^0)\frac{r(s^0)}{1+r(s^0)} + \sum_s q(s^1|s^0)z(s^1|s^0) + \sum_s q(s^1|s^0)m(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)} \leq \tau(s^0).$$

⁵If $r(s^0) = 0$, then both sides are zero while if $r(s^0) > 0$, then utility-maximising households will not wish to hold excess cash between dates as doing so would forego a positive return.

The optimization problem of an individual is to choose $c(s^0)$, $c(s^1|s^0)$, $y(s^0)$ and $y(s^1|s^0)$ subject to the intertemporal budget constraint and the cash-in-advance constraints; first order conditions are

$$\frac{\frac{\partial u[c(s^0), 1-l(s^0)]}{\partial c(s^0)}}{\frac{\partial u[c(s^0), 1-l(s^0)]}{\partial l(s^0)}} = 1 + r(s^0),$$

$$\frac{\frac{\partial u[c(s^1|s^0), 1-l(s^1|s^0)]}{\partial c(s^1|s^0)}}{\frac{\partial u[c(s^1|s^0), 1-l(s^1|s^0)]}{\partial l(s^1|s^0)}} = 1 + r(s^1|s^0),$$

$$\frac{\frac{f(s^1|s^0)\partial u[c(s^1|s^0), 1-l(s^1|s^0)]}{\partial c(s^1|s^0)}}{\frac{\partial u[c(s^0), 1-l(s^0)]}{\partial c(s^0)}} = \tilde{q}(s^1|s^0) = \frac{q(s^1|s^0)p(s^1|s^0)}{p(s^0)}.$$

The consolidated monetary-fiscal authority sets the nominal interest rate $r(s^0)$ and $r(s^1|s^0)$, manages its asset portfolio by choosing the securities which it trades, $\delta(s^1|s^0)$. The monetary fiscal authority enters with liabilities $T(s^0)$. The monetary fiscal authority prints money $M(s^0)$ which it introduces to the economy either via open market operations in the asset market. This yields

$$M(s^0) + \sum_s q(s^1|s^0)B(s^1|s^0) = T(s^0),$$

$$T(s^1|s^0) = M(s^0) + B(s^1|s^0),$$

$$M(s^1|s^0) + \frac{1}{1+r(s^1|s^0)}B(s^2|s^1) = T(s^1|s^0),$$

$$T(s^2|s^1) = M(s^1|s^0) + B(s^2|s^1).$$

We write the composition of the monetary fiscal authority portfolio in each security, $B(s^1|s^0)$, as a security specific share, $\delta(s^1|s^0)$, times a measure of the total size of the holdings, D_1 :

$$B(s^1|s^0) = \delta(s^1|s^0)D_1.$$

The present-value budget constraint is

$$M(s^0)\frac{r(s^0)}{1+r(s^0)} + \sum_{s^1|s^0} q(s^1|s^0)M(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)} = T(s^0).$$

If initial liabilities, $T(s^0)$, vary with prices to satisfy the present-value budget constraint, then the fiscal-policy regime is Ricardian, otherwise it is non-Ricardian. The distinction between the two regimes results in either the

initial price level being indeterminate or not. As our focus is on the distribution of second date prices, the fiscal-policy regime is not important for our results.

As described earlier, conventional monetary policy involves only trades of risk-free securities: $\delta(s^1|s^0) = 1/s$. QE leaves portfolio shares unrestricted, or, equivalently, they depend on expected inflation rates. CE restricts the portfolio shares, or, equivalently, they do not depend on expected inflation rates; it does not trade only risk-free securities.

In addition to individual optimization, equilibrium requires market clearing or

$$\begin{aligned} c(s^0) &= y(s^0), & c(s^1|s^0) &= y(s^1|s^0), \\ m(s^0) &= M(s^0), & m(s^1|s^0) &= M(s^1|s^0), \\ \tau(s^0) &= T(s^0), & \tau(s^1|s^0) &= T(s^1|s^0), \\ \tau(s^2|s^1) &= T(s^2|s^1). \end{aligned}$$

From market clearing and the first order conditions for optimization, we obtain the allocation; and, given the allocation, the real price of the state-contingent bond, $\tilde{q}(s^1|s^0)$. From the present value budget constraint and cash-in-advance constraints we obtain an expression for the real value of initial nominal wealth. Under a non-Ricardian fiscal policy, this then determines the initial price level. Without further restrictions, the indexed bond price cannot be uniquely decomposed and the distribution of inflation rates is left undetermined.

Let a tilde over a variable denote the real value of the variable. The second date household budget constraint and market clearing gives us in equilibrium

$$\begin{aligned} p(s^1|s^0)z(s^1|s^0) + m(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)} &= \\ \theta(s^1|s^0) + m(s^0) &= \delta(s^1|s^0)D_1 + m(s^0), \\ 1 + \pi(s^1|s^0) &= \frac{\delta(s^1|s^0)\tilde{D}_1 + \tilde{m}(s^0)}{\tilde{m}(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)}}. \end{aligned}$$

If $\delta(s^1|s^0)$ does not depend on the expected rate of inflation, we have S relationships between the expected inflation rates and the real scale of debt and the real value of money balances.⁶

⁶On the other hand, suppose that the portfolio weights depend on the expected inflation rate and are of the form $\delta(s^1|s^0) = [1 + \pi(s^1|s^0)]/[\sum_{s^1|s^0} 1 + \pi(s^1|s^0)] \times constant_s$ such

The no-arbitrage condition is used to express security prices $q(s^1|s^0)$ as depending on the nominal equivalent Martingale measure $\nu(s^1|s^0)$ in the form

$$q(s^1|s^0) = \frac{\nu(s^1|s^0)}{1+r(s^0)},$$

$$\nu(s^1|s^0) = \tilde{q}(s^1|s^0) \frac{1+r(s^0)}{1+\pi(s^1|s^0)}.$$

The sum of the martingale probabilities being 1. If interest rates are positive, real money balances are $\tilde{m}(s^0) = y(s^0)$ and $\tilde{m}(s^1|s^0) = y(s^1|s^0)$ and we obtain one equation to determine the scale of debt.

With the real scale of debt in hand, we obtain the distribution of inflation rates as a function of the portfolio weights. If policy is conventional or one of credit easing, then the portfolio weights are *restricted* and policy uniquely determines the path of inflation. On the other hand, under quantitative easing, the portfolio weights are *unrestricted* (either free or depends on expected inflation), and a unique path of inflation cannot be implemented by policy.

In [Eggertsson and Woodford \(2003\)](#), when real money balances are not saturated, the central bank portfolio uniquely determines the path of prices. This is because the composition of the portfolio depends on current state variables and is *restricted*. If QE focuses exclusively on expansion of real money balances, then the assumption made in their paper is a strong one. A relaxation of this assumption leaves the portfolio *unrestricted* even when the zero lower bound is not binding, the central bank loses control of the path of inflation.

The Effect of the Zero Lower Bound

Although our derivations above used positive interest rates, this was only for analytical convenience and our results remain robust to scenarios where the economy is either temporarily or permanently at the zero lower bound. One may also be concerned that our analysis precludes the expansion of real money balances through unconventional policy. When interest rates are positive, the path of real money balances are determined solely from the path of the real interest rate. In contrast, when interest rates are zero, the composition of assets held by the monetary-fiscal authority determines the path of real money balances.

that $\sum_{s^1|s^0} \delta(s^1|s^0) = 1$. In this case, the state-contingent inflation rates cancel out and what remains is the sum. It will be clear in what follows that this precludes uniquely determining the state-contingent inflation rates.

Temporary Zero Lower Bound

Suppose that the economy is temporarily at the zero lower bound; at date 0 interest rates are 0, but in the second date the economy exits the zero bound and interest rates are positive. If date 0 interest rates were 0, then real money balances may be greater than real income at date 0; we lose one equation.

The real monetary fiscal-authority present-value budget constraint when date 0 interest rates are 0 is

$$\sum_s \tilde{q}(s^1|s^0) \tilde{M}(s^1|s^0) \frac{r(s^1|s^0)}{1+r(s^1|s^0)} = \frac{W(s^0)}{p(s^0)},$$

and, under a non-Ricardian policy, it gives us the initial price level.

The real date 0 budget constraint in equilibrium gives

$$\tilde{M}(s^0) + \tilde{D}_1 \sum_s \frac{\tilde{q}(s^1|s^0)}{1+\pi(s^1|s^0)} \delta(s^1|s^0) = \tilde{w}(s^0).$$

Note that

$$\frac{\tilde{q}(s^1|s^0) \delta(s^1|s^0)}{1+\pi(s^1|s^0)} = \tilde{q}(s^1|s^0) \delta(s^1|s^0) \frac{z(s^1|s^0) + \tilde{M}(s^1|s^0) \frac{r(s^1|s^0)}{1+r(s^1|s^0)}}{\delta(s^1|s^0) \tilde{D}_1 + \tilde{M}(s^0)},$$

and, if interest rates are positive, real money balances are $\tilde{m}(s^0) = y(s^0)$ and $\tilde{m}(s^1|s^0) = y(s^1|s^0)$. Furthermore $[W(s^0)/p(s^0)] = \tilde{w}(s^0)$ and $\tilde{M}(s^1|s^0) = \tilde{m}(s^1|s^0)$. In equilibrium,

$$\begin{aligned} \tilde{m}(s^0) + \tilde{D}_1 \sum_s \tilde{q}(s^1|s^0) \delta(s^1|s^0) \frac{\tilde{m}(s^1|s^0) \frac{r(s^1|s^0)}{1+r(s^1|s^0)}}{\delta(s^1|s^0) \tilde{D}_1 + \tilde{m}(s^0)} = \\ \sum_s \tilde{q}(s^1|s^0) \tilde{m}(s^1|s^0) \frac{r(s^1|s^0)}{1+r(s^1|s^0)}, \end{aligned}$$

and, together with the no-arbitrage condition, the real scale of debt can be solved uniquely if monetary policy is restricted. Importantly, the real value of date 0 money balances depends on the portfolio shares; balance policy determines both the size of real money balances and the path of inflation. Note that the Ricardian/non-Ricardian fiscal policy distinction is not driving the results; we only require that fiscal policy is chosen to be compatible with equilibrium.

Permanent Zero Lower Bound

If interest rates are zero in both dates, we require a more fully articulated fiscal policy though the Ricardian/non-Ricardian policy distinction remains unimportant. Let the monetary-fiscal authority levy indexed transfers at all date-events, in addition to the initial nominal liabilities.

The equilibrium monetary-fiscal authority real budget constraints now become, with all interest rates zero,

$$\begin{aligned} \tilde{m}(s^0) + \tilde{D}_1 \sum_{s^1|s^0} \tilde{q}(s^1|s^0) \delta(s^1|s^0) + g(s^0) = \\ \frac{T(s^0)}{p(s^0)} (1 + \pi(s^1|s^0)) g(s^1|s^0) = \tilde{D}_1 \delta(s^1|s^0) + \tilde{m}(s^0), \end{aligned}$$

and the real present-value budget constraint is

$$g(s^0) + \sum_{s^1|s^0} \tilde{q}(s^1|s^0) g(s^1|s^0) = \frac{T(s^0)}{p(s^0)},$$

where g is the real value of the indexed nominal transfer. The household optimality conditions remain the same. The real scale of debt and the real value of date 0 money balances is solved from

$$\begin{aligned} \tilde{m}(s^0) + \tilde{D}_1 \sum_s \tilde{q}(s^1|s^0) \delta(s^1|s^0) = \\ \sum_s \tilde{q}(s^1|s^0) g(s^1|s^0) \sum_{s^1|s^0} \tilde{q}(s^1|s^0) g(s^1|s^0) \frac{1+r(s^0)}{\tilde{D}_1 \delta(s^1|s^0) + \tilde{m}(s^0)} = 1, \end{aligned}$$

where the second equation is the no-arbitrage condition. Hence, in contrast to conventional policy or credit easing, if the portfolio weights are unrestricted under QE (either free or depends on expected inflation rates) then the path of inflation cannot be uniquely determined from policy.

Date 0 real money balances is given by

$$\sum_{s^1|s^0} \frac{\tilde{q}(s^1|s^0) g(s^1|s^0)}{\sum_s \tilde{q}(s^1|s^0) g(s^1|s^0) - \tilde{m}(s^0)} \delta(s^1|s^0) + \tilde{m}(s^0) = \frac{1}{1+r(s^0)}.$$

Loosely speaking, date 0 real money balances are increasing the correlation between the portfolio shares and the present value of state-contingent real indexed transfers. To be explicit, our results remain robust to a world where unconventional policies simultaneously expand the size of the balance sheet and alter the composition of assets held in the portfolio.

Our results on the uniqueness of prices at the zero lower bound contrasts with [Eggertsson and Woodford \(2003\)](#) as fiscal transfers in their model are purely nominal and the presence of indexed transfers there would result in a link between the composition of the portfolio and inflation in their setting.

1.2 A stochastic dynamic economy

For ease of exposition, we now examine infinite horizon equilibria to show that our results do not depend on a finite horizon or on not restricting attention to stationary equilibria. As our results do not depend on interest rates, for analytical convenience we assume interest rates are positive throughout this section.

Time, t , is discrete, and it extends into the infinite future: $t = 1, \dots$. Events, s^t , at each date are finitely many. An immediate successor of a date-event is $s^{t+1}|s^t$, and, inductively, a successor is $s^{t+k}|s^t$. Conditional on s^t , probabilities of successors are $f(s^{t+1}|s^t)$ and, inductively, $f(s^{t+k}|s^t) = f(s^{t+k}|s^{t+k-1})f(s^{t+k-1}|s^t)$.

At a date-event, a perishable input, labor, $l(s^t)$, is employed to produce a perishable output, consumption, $y(s^t)$, according to a linear technology:

$$y(s^t) = a(s^t)l(s^t), \quad a(s^t) > 0.$$

A representative individual is endowed with 1 unit of leisure at every date-event. He supplies labor and demands the consumption good, and he derives utility according to the cardinal utility index $u(c(s^t), 1 - l(s^t))$ that satisfies standard monotonicity, curvature and boundary conditions. The preferences of the individual over consumption-employment paths commencing at s^t are described by the separable, von Neumann-Morgenstern intertemporal utility function

$$u(c(s^t), 1 - l(s^t)) + \mathbb{E}_{s^t} \sum_{k>0} \beta^k u(c(s^{t+k}|s^t), 1 - l(s^{t+k}|s^t)),$$

where $0 < \beta < 1$. Balances, $m(s^t)$, provide liquidity services. Elementary securities, $\theta(s^{t+1}|s^t)$, serve to transfer wealth to and from immediate successor date-events. The price level is $p(s^t)$, and the wage rate is $w(s^t) = a(s^t)p(s^t)$, as profit maximization requires. The nominal, risk-free interest rate is $r(s^t)$.

At each date-event, the asset market opens after the uncertainty, s^t , has realized, and, as a consequence, purchases and sales in the markets for labor and the consumption good are subject to standard cash-in-advance constraints; the effective cash-in-advance constraint is⁷

$$a(s^t)p(s^t)l(s^t) \leq m(s^t).$$

Prices of elementary securities are

$$q(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)}{1 + r(s^t)},$$

⁷Nakajima and Polemarchakis (2005) provide an explicit derivation.

with $\nu(\cdot|s^t)$ a “nominal pricing measure” or transition probabilities, which guarantees the non-arbitrage relation

$$\sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) = \frac{1}{1+r(s^t)}.$$

Inductively,

$$\nu(s^{t+k}|s^t) = \nu(s^{t+k}|s^{t+k-1})\nu(s^{t+k-1}|s^t), \quad k > 1,$$

and the implicit price of revenue at successor date-events is

$$q(s^{t+k}|s^t) = \frac{\nu(s^{t+k}|s^{t+k-1})}{1+r(s^{t+k-1}|s^t)}q(s^{t+k-1}|s^t), \quad k > 1.$$

The individual has initial wealth $\tau(s^1) = \omega$. Initial wealth constitutes a claim against the monetary-fiscal authority; alternatively, it can be interpreted as outside money. It is exogenous in a non-Ricardian specification. In a Ricardian specification, it is set endogenously so as to satisfy the transversality condition imposed on monetary-fiscal policy.

The flow budget constraint is

$$p(s^t)z(s^t) + m(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)\theta(s^{t+1}|s^t) \leq \tau(s^t),$$

where $z(s^t) = c(s^t) - a(s^t)l(s^t)$ is the effective excess demand for consumption.

Wealth at successor date-events is

$$\tau(s^{t+1}|s^t) = \theta(s^{t+1}|s^t) + m(s^t),$$

and, after elimination of the trade in assets, the flow budget constraint reduces to

$$p(s^t)z(s^t) + \frac{r(s^t)}{1+r(s^t)}a(s^t)p(s^t)l(s^t) + \sum_{s_{t+1}} q(s_{t+1}|s^t)\tau(s_{t+1}|s^t) \leq \tau(s^t).$$

Debt limit constraints are

$$-\tau(s^t) \leq \sum_{k>0} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) \frac{1}{1+r(s^t)} a(s^{t+k})p(s^{t+k}).$$

Alternatively, $\tilde{m}(s^t) = (1/p(s^t))m(s^t)$ are real balances, $\tilde{\tau}(s^t) = (1/p(s^t))\tau(s^t)$ is real wealth, $\pi(s^{t+1}|s^t) = (p(s^{t+1})/p(s^t)) - 1$ is the rate of inflation, and

$$\tilde{q}(s^{t+1}|s^t) = q(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t)) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1+r(s^t)}$$

are prices of indexed elementary securities.

Real wealth at successor date-events is

$$\tilde{\tau}(s^{t+1}|s^t) = \left(\frac{\theta(s^{t+1}|s^t) + m(s^t)}{p(s^t)} \right) \frac{1}{1 + \pi(s^{t+1}|s^t)},$$

and the flow budget constraint reduces to

$$z(s^t) + \frac{r(s^t)}{1 + r(s^t)} a(s^t) l(s^t) + \sum_{s^{t+1}} \tilde{q}(s^{t+1}|s^t) \tilde{\tau}(s^{t+1}|s^t) \leq \tilde{\tau}(s^t).$$

First order conditions for an optimum are

$$\frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial l(s^t)} \left(\frac{a(s^t)}{1+r(s^t)} \right)^{-1},$$

$$\beta f(s^{t+1}|s^t) \frac{\partial u(c(s^{t+1}), 1-l(s^{t+1}))}{\partial c(s^{t+1})} \tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)},$$

and the transversality condition is

$$\lim_{k \rightarrow \infty} \sum_{s^{t+k}|s^t} \tilde{q}(s^{t+k}|s^t) \tilde{\tau}(s^{t+k}|s^t) = 0.$$

The monetary-fiscal authority sets rates of interest and accommodates the demand for balances. It supplies balances, $M(s^t)$, and trades in elementary securities subject to a flow budget constraint that, after elimination of the trade in assets, reduces to

$$T(s^t) \leq \frac{r(s^t)}{1 + r(s^t)} M(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) T(s^{t+1}|s^t),$$

where $T(s^t)$ and, similarly, $T(s^{t+1}|s^t)$ are obligations towards the private sector; initial obligations are $\Omega = T(s^1)$. Ricardian policy imposes on the monetary-fiscal authority the transversality condition

$$\lim_{k \rightarrow \infty} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) T(s^{t+k}|s^t) = 0$$

or, equivalently, as prices vary, it sets the initial claims of the private sector as

$$\Omega = \frac{r(s^1)}{1 + r(s^1)} M(s^1) + \sum_{t>0} \sum_{s^t|s^1} \frac{r(s^t|s^1)}{1 + r(s^t|s^1)} q(s^t|s^1) M(s^t|s^1).$$

For equilibrium, it is necessary and sufficient that the excess demand for output vanishes:

$$z(s^t) = c(s^t) - a(s^t)l(s^t) = 0.$$

From the first order conditions for an optimum, this determines the path of employment and consumption:

$$\frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial l(s^t)} \left(\frac{a(s^t)}{1 + r(s^t)} \right)^{-1},$$

and, in turn, the prices of indexed elementary securities:

$$\beta f(s^{t+1}|s^t) \frac{\partial u(c(s^{t+1}), 1 - l(s^{t+1}))}{\partial c(s^{t+1})} \tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)}.$$

The initial price level serves to guarantee that, at equilibrium, the transversality condition of the monetary-fiscal authority holds. If monetary-fiscal policy is Ricardian, the price level remains indeterminate. If it is non-Ricardian, in that initial claims are given, then the equilibrium path of nominal asset prices determines the present-discounted value of unindexed transfers and so the initial price level.

More importantly, without further restrictions, as is the case under QE, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s^t)$, and a nominal pricing measure, $\nu(\cdot|s^t)$, remains indeterminate: if the nominal pricing measure, $\nu(\cdot|s^t)$, is specified arbitrarily, the inflation process, $\pi(\cdot|s^t)$, adjusts to implement the equilibrium; that is, to satisfy

$$\tilde{q}(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}.$$

The determinacy in [Woodford \(1994\)](#) highlights the importance of the present value of the monetary-fiscal authority budget constraint in the determination of the price level. We examine here, and what is often overlooked, is that the stochastic evolution of government wealth is essential for the determination of the stochastic path of prices. That [Woodford \(1994\)](#) restricts attention to conventional policy, in which case the portfolio of the monetary-fiscal authority is composed solely of Treasury bills, obscures this second point. Our results are not dependent on the infinite horizon of the economy, or the open-endedness of the policies that we describe. In previous versions of this paper we showed that our results remain valid in a finite horizon economy.

It may be confusing that we abstract from fiscal transfers after the initial date; we only do so because their implications are straightforward and do not

affect the argument. It is worth pointing out, however, that the dichotomy between the nominal pricing measure and the initial price level that obtains when transfers are indexed, no longer holds when transfers are not indexed: the nominal pricing measure, indeterminate under quantitative easing, affects the aggregate volume of claims against the monetary-fiscal authority and, as a consequence, the initial price level as well.

Under QE, the nature of the interest-elasticity of money demand does not determine the stationary equilibrium path, though it determines the stability of the path. Following [Sims \(1994\)](#), the introduction of a portfolio of securities (rather than the single risk-free bond that was considered) under a policy of QE leaves the difference equations, that otherwise determine the unique path of money, to depend on the state-contingent return on the portfolio of the monetary-fiscal authority and the portfolio-to-money supply ratio. A given (stationary) distribution of portfolio returns, that satisfies the no-arbitrage condition given by the fixed short-term nominal interest rate, then corresponds to a stationary distribution of portfolios, even for the fiscal policy rules that were considered there. Put simply, adequate consideration of the interest-elasticity of money demand guarantees a stationary distribution, but not a unique one.

1.3 A stationary economy

In this section we examine the nature of stationary equilibria, abstracting from consideration of the stability of stationary equilibria and, therefore, also from the interest-elasticity of money demand. We show that the argument extends to stationary economies and stationary equilibria or steady states.

The resolution of uncertainty follows a stationary stochastic process. Elementary states of the world are s , finitely many, and transition probabilities are $f(s'|s)$.

Rates of interest, $r(s^1|s^0)$, determine the path of consumption, $c(s^1|s^0)$, and employment, $l(s^1|s^0)$, at equilibrium, which, in turn, determine the prices of indexed elementary securities:

$$\beta f(s'|s) \frac{\partial u(c(s'), 1 - l(s'))}{\partial c(s')} \tilde{q}(s'|s)^{-1} = \frac{\partial u(c(s^1|s^0), 1 - l(s^1|s^0))}{\partial c(s^1|s^0)}$$

or

$$\tilde{Q} = \beta Du(s^1|s^0)^{-1} F Du(s').$$

Here,

$$Du(s^1|s^0) = \text{diag} \left(\dots, \frac{\partial u(c(s^1|s^0), 1 - l(s^1|s^0))}{\partial c(s^1|s^0)}, \dots \right)$$

is the diagonal matrix of marginal utilities of consumption, and

$$F = (f(s'|s)) \quad \text{and} \quad \tilde{Q} = (\tilde{q}(s'|s))$$

are, respectively, the matrices of transition probabilities and of prices of indexed elementary securities.

With

$$\tilde{y} = (\dots \frac{r(s^1|s^0)}{1+r(s^1|s^0)} a(s^1|s^0) l(s^1|s^0) \dots)'$$

the vector of net, real expenditures on balances at equilibrium, real claims against the fiscal-monetary authority at the steady state,

$$\tilde{\tau} = (\dots \tilde{\tau}(s^1|s^0), \dots)'$$

are determined by the equation

$$\tilde{y} + \tilde{Q}\tilde{\tau} = \tilde{\tau} \quad \text{or} \quad \tilde{\tau} = (I - \tilde{Q})^{-1}\tilde{y};$$

since $0 < \beta < 1$,

$$(I - \tilde{Q})^{-1} = \sum_{k=0}^{\infty} \tilde{Q}^k = \sum_{k=0}^{\infty} \beta^k Du(s^1|s^0)^{-1} F^k Du(s'),$$

and, since F is a Markov transition matrix, while $\tilde{y} \gg 0$, the real claims against the monetary-fiscal authority at the steady state are strictly positive:

$$\tilde{\tau} \gg 0.$$

Non-Ricardian monetary-fiscal policy determines the initial price level by setting exogenously the level of initial nominal claims; otherwise, the price level remains indeterminate.

More importantly, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s)$, and a nominal pricing measure, $\nu(\cdot|s)$, remain indeterminate:

$$\tilde{Q} = R^{-1}N \otimes \Pi,$$

where \otimes denotes the Hadamard product. Here,

$$R = \text{diag}(\dots, (1 + r(s^1|s^0)), \dots)$$

is the diagonal matrix of interest factors, and

$$N = (\nu(s'|s)) \quad \text{and} \quad \Pi = ((1 + \pi(s'|s)))$$

are, respectively, the Markov transition matrix of “nominal pricing transition probabilities” and the matrix of inflation factors.

Alternative specifications of the stochastic process of inflation serve to characterize the set of equilibria and to highlight the role of the balance sheet policy of the monetary-fiscal authority.

The role of the balance sheet policy is the focus of the analysis here; it was not dealt with in [Drèze and Polemarchakis \(2000\)](#) or [Bloise, Drèze, and Polemarchakis \(2005\)](#).

QE: In the absence of restrictions on the balance sheet of the monetary fiscal authority, which is the case under QE, the set of steady state equilibria is indexed by the nominal pricing transition probabilities, $\nu(\cdot|s)$, that can be set arbitrarily, while the inflation factors, $\pi(\cdot|s)$, adjust to implement the equilibrium; alternatively, the inflation factors are set arbitrarily, up to a scale effect, and the nominal pricing transition probabilities adjust to implement the equilibrium.

The argument is as follows: with

$$(1 + \pi(s'|s)) = h(s^1|s^0)\gamma(s'|s),$$

an arbitrary (for the moment) decomposition of the inflation process into a term (the scale effect) that depends only on the current state and a term of (relative) inflation factors and is Markovian, the equilibrium condition takes the form

$$\tilde{Q} = R^{-1}N \otimes H\Gamma;$$

here, H be the diagonal matrix of the $h(s^1|s^0)$ and Γ the matrix of the $\gamma(s'|s)$.

Given Γ , there are H, N that guarantee equilibrium; the argument is straightforward:

$$\tilde{Q} = R^{-1}N \otimes \Pi \Rightarrow \tilde{Q} = R^{-1}N \otimes H\Gamma \Rightarrow \tilde{Q} \oslash \Gamma = (R^{-1}H)N,$$

the last step, since H is a diagonal matrix.⁸

Since N is Markovian if and only if it is non-negative and $N\mathbf{1}_S = \mathbf{1}_S$,

$$(\tilde{Q} \oslash \Gamma)\mathbf{1}_S = (R^{-1}H)\mathbf{1}_S,$$

which allows us to solve for $h(s^1|s^0)$.

With H, Γ in hand, we can solve for N , that shall indeed, be Markovian.

If N is given, there are H, Γ (or, equivalently, Π) that guarantee equilibrium.

⁸ \oslash denotes Hadamard division.

Taylor rules: We now show that the indeterminacy obtained is not ruled out by interest-feedback rules. Any process can be written uniquely as

$$(1 + \pi(s'|s)) = h(s^1|s^0)\gamma(s'|s), \quad \gamma(s'|s) = \frac{\delta(s'|s)}{f(s'|s)}, \quad \sum_{s'} \delta(s'|s) = 1,$$

in which case,

$$h(s^1|s^0) = E_{s'}(1 + \pi(s'|s));$$

With $r(s^1|s^0)$ not set exogenously, but as a function of $h(s^1|s^0)$, this is a [Taylor \(1993\)](#) rule, and indeterminacy persists. In other words, policy that specifies the path of nominal interest rates as a function of expected inflation, does not pin down the stochastic path of inflation.⁹

Evidently, with $r(s^1|s^0)$ not set exogenously, but as a function of $h(s^1|s^0)$, equilibrium requires solution of the equation

$$\frac{h(s^1|s^0)}{1 + r(h(s^1|s^0))} = \sum_{s' \in S} \frac{f(s'|s)}{\gamma(s'|s)} \beta \frac{\frac{\partial u(c(h(s'), 1-l(h(s')))}{\partial c(h(s'))}}{\frac{\partial u(c(h(s^1|s^0), 1-l(h(s^1|s^0)))}{\partial c(s^1|s^0)}}},$$

where the allocation, as a function of $h(s^1|s^0)$, is solved from the individual optimality conditions. If a solution to this system of equations exists and is unique, for example if the function/rule is linear, then the solution still depends on the (arbitrarily chosen) Γ .

CE: Alternatively,

$$(1 + \pi(s'|s)) = h(s^1|s^0)\gamma(s'|s), \quad \gamma(s'|s) = \frac{\delta(s'|s)}{\tilde{\tau}(s')}, \quad \sum_{s'} \delta(s'|s) = 1,$$

in which case, $\delta(s'|s)$ are portfolio weights that determine the composition of assets in the balance sheet of the monetary-fiscal authority.

Monetary-fiscal policy conducted as CE sets the composition of the balance sheet; that is, it sets explicit positive portfolio weights, $\delta(s'|s) > 0$; claims against the monetary-fiscal authority in real terms, $\tilde{\tau}(s^1|s^0)$, are determined, at the steady-state, by fundamentals, and, as a consequence, under CE, the matrix Γ is determined.

Since

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H\mathbf{1}_S = (R\tilde{Q} \oslash \Gamma)\mathbf{1}_S,$$

⁹That the Taylor rule does not depend on realized rates of inflation is appropriate for (stochastic) steady-state equilibria.

the Markov transition matrix, N , is well defined ($h \gg 0$) and determinate; it follows that the equilibrium is determinate as well.

Under conventional monetary-fiscal policy, the portfolio of the monetary-fiscal authority consists of Treasury bills, nominally risk-free bonds of short maturity. Here, this corresponds to one-date nominally risk-free bonds: $\delta(s'|s) = 1/S$.

Determinacy obtains for arbitrary, but, importantly, portfolio weights in the balance sheet of the monetary-fiscal authority that only depend on fundamentals and/or realized variables at s . If the portfolio weights are chosen by policy to depend on endogenous nominal variables at s' , such as the expected stochastic rate of inflation, then indeterminacy obtains. To be explicit, consider, for example, that the portfolio weights depended on expectations of the future nominal value of wealth: $\delta(s'|s) = [\tilde{\tau}(s')(1 + \pi(s'|s))]/[\sum_{s'} \tilde{\tau}(s')(1 + \pi(s'|s))] \Rightarrow h(s^1|s^0) = \sum_{s'} \tilde{\tau}(s')(1 + \pi(s'|s))$. In a model where there are long-dated securities, setting portfolio weights as a function of expected nominal asset prices would have the same outcome. This contrasts with [Magill and Quinzii \(2014b\)](#) and [Adao, Correia, and Teles \(2014\)](#), where explicit targets for asset prices, independent of equilibrium, pin down portfolio weights.

Simple inflation processes: It is instructive to consider whether restricting the inflation process to depend endogenously only on either the current or future state is compatible with an equilibrium policy choice. Suppose that the inflation process, which is endogenous, is restricted to take the form

$$(1 + \pi(s'|s)) = h(s^1|s^0)b(s'),$$

where $b(s^1|s^0) > 0$ is positive function of the fundamentals of the economy determined at the steady state and, as a consequence,

$$N \otimes \Pi = HNB.$$

Here,

$$b = (\dots, b(s^1|s^0), \dots), \quad \text{and} \quad h = (\dots, h(s^1|s^0), \dots),$$

and B and H are the assocash-in-advance constraintted diagonal matrices.

Then,

$$\tilde{Q} = R^{-1}N \otimes \Pi \quad \Leftrightarrow \quad R\tilde{Q}B^{-1} = HN,$$

which determines the inflation process as well as nominal pricing probabilities, since

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad N = \left(\text{diag}(R\tilde{Q}B^{-1}\mathbf{1}_S) \right)^{-1} R\tilde{Q}B^{-1},$$

a Markov transition matrix, as required.

This is indeed the case under conventional monetary policy.

Real wealth at successor date-events is

$$\tilde{\tau}(s') = \left(\frac{\theta(s'|s) + m(s^1|s^0)}{p(s)} \right) \frac{1}{1 + \pi(s'|s)},$$

and conventional monetary policy requires that

$$\theta(s'|s) = \theta(s^1|s^0)$$

or

$$(1 + \pi(s'|s)) = \underbrace{\left(\frac{\theta(s^1|s^0) + m(s^1|s^0)}{p(s)} \right)}_{h(s^1|s^0)} \underbrace{\frac{1}{\tilde{\tau}(s')}}_{b(s')}.$$

Suppose, instead, that inflation is restricted to depend endogenously only on the future state,

$$(1 + \pi(s'|s)) = h(s')b(s^1|s^0),$$

and, as a consequence,

$$N \otimes \Pi = BNH.$$

In this case,

$$\tilde{Q} = R^{-1}N \otimes \Pi \quad \Leftrightarrow \quad B^{-1}R\tilde{Q} = NH,$$

and

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad N = B^{-1}R\tilde{Q} \left(\text{diag}((B^{-1}R\tilde{Q})^{-1}\mathbf{1}_S) \right)$$

that need not be positive. In other words equilibrium inflation may be restricted to depend endogenously only on the current state but not only on the future one. However such a restriction precludes analysis of the effects of unconventional monetary policy on changes in the composition of the balance sheet of the monetary-fiscal authority and their subsequent determination of the stochastic path of inflation.

2 A large open economy

We now relax the assumption that the economy is closed. Much of the model remains the same, but we can show the effect of unconventional monetary policy on

There are two countries in the world, home and foreign, each inhabited by a representative agent. Foreign variables, both macro and those relating to foreign agents, will be denoted with an asterisk (*). The transactions of agents in the home and foreign country will be denoted with a subscript “h” and “f,” respectively. It suffices to specify explicitly mostly only the constraints and variables relevant for the home agent and country; the foreign agent and country are equivalent.

At a date-event, a perishable non-tradable input, labor, $l(s^t)$, is employed to produce a perishable domestic tradable output, consumption, $y(s^t)$, according to a linear technology. The representative home individual is endowed with 1 unit of leisure at every date-event. He supplies non-tradable labor and demands the tradeable consumption good, and he derives utility according to the cardinal utility index $u(c(s^t), 1 - l(s^t))$ that satisfies standard monotonicity, curvature and boundary conditions. The preferences of the individual over consumption-employment paths commencing at s^t are described by the separable, von Neumann-Morgenstern, intertemporal utility function. Balances, $m_h(s^t)$ and $m_f(s^t)$ provide liquidity services in the home and foreign country respectively. Elementary securities, $\theta(s^{t+1}|s^t)$, serve to transfer wealth to and from immediate successor date-events. The price level is $p(s^t)$, and the wage rate is $w(s^t) = a(s_t)p(s^t)$, as profit maximisation requires. The nominal, risk-free interest rate is $r(s^t)$. As the goods produced in each country are perfect substitutes, the law-of-one-price holds and determines the exchange rate $e^*(s^t) = p(s^t)/p^*(s^t)$.

At each date-event, the asset (and currency) market opens after the uncertainty, s^t , has realized, and, as a consequence, purchases and sales in the markets for labor and the consumption good are subject to standard cash-in-advance constraints; the effective cash-in-advance constraint is¹⁰

$$a(s^t)p(s^t)l(s^t) \leq m_h(s^t), \quad 0 \leq m_f^*(s^t).$$

Prices of elementary securities in the domestic country are

$$q(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)}{1 + r(s^t)},$$

with $\nu(\cdot|s^t)$ the domestic “nominal pricing measure.” Note that the nominal prices of elementary securities and the “nominal pricing measure” are unique to the currency in which they are denominated. As there are a complete set of state-contingent bonds in each currency, the prices of securities which deliver currency in the same state are related by the following no-arbitrage

¹⁰Nakajima and Polemarchakis (2005) provide an explicit derivation.

condition

$$\frac{\nu^*(s^{t+1}|s^t)}{\nu(s^{t+1}|s^t)} \left\{ \frac{1+r(s^t)}{1+r^*(s^t)} \right\} = \frac{e^*(s^{t+1}|s^t)}{e^*(s^t)}.$$

In other words, the path of nominal exchange rates depends on the ratio of the “nominal pricing measure” across countries and implies the uncovered interest parity condition

$$\frac{1+r(s^t)}{1+r^*(s^t)} e^*(s^t) = \sum_{s^{t+1}|s^t} \nu(s^{t+1}|s^t) e^*(s^{t+1}|s^t).$$

This gives the risk-neutral expected exchange rate. As markets are complete, variations in the nominal equivalent martingale measure in each country only have nominal effects on the implicit premium in the exchange rate¹¹. If there were nominal rigidities or frictions which prevented the law of one price from holding, then the covariance between the nominal equivalent martingale measure and nominal exchange rate, and hence the premium in expected exchange rates, would imply different allocations of (real) resources.

The individual has initial nominal wealth $\tau_h(s^t)$ and $\tau_f(s^t)$ in each country. Initial wealth constitutes a claim against the respective monetary-fiscal authority; alternatively, it can be interpreted as outside money. It is exogenous in a non-Ricardian specification. In a Ricardian specification, it is set endogenously so as to satisfy the transversality condition imposed on monetary-fiscal policy.

The flow budget constraint is¹²

$$\begin{aligned} & p(s^t)c_h(s^t) + e^*(s^t)p^*(s^t)c_f(s^t) + m(s^t) \\ & + \sum_{s^{t+1}|s^t} \{q(s^{t+1}|s^t)\theta_h(s^{t+1}|s^t) + e^*(s^t)q^*(s^{t+1}|s^t)\theta_f(s^{t+1}|s^t)\} \\ & \leq p(s^t)a(s^t)l(s^t) + \tau_h(s^t) + e^*(s^t)\tau_f(s^t). \end{aligned}$$

Debt limit constraints are

$$\tau_h(s^t) + e^*(s^t)\tau_f(s^t) \geq - \sum_{k>0} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) \frac{1}{1+r(s^t)} a(s^{t+k})$$

or, equivalently,

$$\lim_{k \rightarrow \infty} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) \{ \tau_h(s^{t+k}|s^t) + e^*(s^{t+k}|s^t)\tau_f(s^{t+k}|s^t) \} \geq 0.$$

¹¹The difference between the risk neutral and objective expected exchange rate.

¹²Foreign money balances are dominated by foreign bonds and are zero in equilibrium, while the effective cash-in-advance constraint guarantees that domestic money balances are positive.

Wealth at successor date-events is

$$\tau_h(s^{t+1}|s^t) = \theta(s^{t+1}|s^t) + m(s^t) \quad \text{and} \quad \tau_f(s^{t+1}|s^t) = \theta_f(s^{t+1}|s^t),$$

and, after elimination of the trade in assets and using the law-of-one-price, the flow budget constraint reduces to

$$p(s^t)z(s^t) + \frac{r(s^t)}{1+r(s^t)}a(s^t)p(s^t)l(s^t) + \sum_{s_{t+1}} q(s_{t+1}|s^t)\tau(s_{t+1}|s^t) \leq \tau(s^t),$$

where $z(s^t) = c(s^t) - a(s^t)l(s^t)$ is the effective excess demand for consumption, $c(s^t)$ is the sum of consumption at home and abroad and $\tau(s^t) = \tau_h(s^t) + e^*(s^t)\tau_f(s^t)$.

Alternatively, $\tilde{m}(s^t) = (1/p(s^t))m(s^t)$ are real balances, $\tilde{\tau}(s^t) = (1/p(s^t))\tau(s^t)$ is real wealth, $\pi(s^{t+1}|s^t) = (p(s^{t+1})/p(s^t)) - 1$ is the rate of inflation, and

$$\tilde{q}(s^{t+1}|s^t) = q(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t)) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}$$

are prices of indexed elementary securities¹³.

Real wealth at successor date-events is

$$\tilde{\tau}(s^{t+1}|s^t) = \left(\frac{\theta(s^{t+1}|s^t) + m(s^t) + e^*(s^{t+1}|s^t)\theta(s^{t+1}|s^t)}{p(s^t)} \right) \frac{1}{1 + \pi(s^{t+1}|s^t)},$$

and the flow budget constraint reduces to

$$z(s^t) + \frac{r(s^t)}{1+r(s^t)}a(s^t)l(s^t) + \sum_{s_{t+1}} \tilde{q}(s_{t+1}|s^t)\tilde{\tau}(s_{t+1}|s^t) \leq \tilde{\tau}(s^t).$$

First order conditions for an optimum are

$$\frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial l(s^t)} \left(\frac{a(s^t)}{1+r(s^t)} \right)^{-1},$$

$$\beta f(s^{t+1}|s^t) \frac{\partial u(c(s^{t+1}), 1-l(s^{t+1}))}{\partial c(s^{t+1})} \tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)},$$

and the transversality condition is

$$\lim_{k \rightarrow \infty} \sum_{s^{t+k}|s^t} \tilde{q}(s^{t+k}|s^t)\tilde{\tau}(s^{t+k}|s^t) = 0.$$

¹³From the no-arbitrage condition for assets, this is also the same in the foreign country.

The monetary-fiscal authority in each country sets domestic one date rates of interest and accommodates the demand for domestic balances. It supplies domestic balances, $M(s^t)$, and trades in elementary securities subject to a flow budget constraint as in the closed economy case. For equilibrium, it is necessary and sufficient that the excess demand for output vanishes:

$$z(s^t) + z^*(s^t) = c(s^t) + c^*(s^t) - a(s^t)l(s^t) - a^*(s^t)l^*(s^t) = 0,$$

which determines the path of employment and consumption for each household:

$$\frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial l(s^t)} \left(\frac{a(s^t)}{1 + r(s^t)} \right)^{-1};$$

in turn, this determines the prices of indexed elementary securities:

$$\beta f(s^{t+1}|s^t) \frac{\partial u(c(s^{t+1}), 1 - l(s^{t+1}))}{\partial c(s^{t+1})} \tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)}.$$

The initial price level serves to guarantee that, at equilibrium, the transversality condition of the monetary-fiscal authority holds. If monetary-fiscal policy is Ricardian, the price level remains indeterminate.

More importantly, without further restrictions, as is the case under QE, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s^t)$, and a nominal pricing measure, $\nu(\cdot|s^t)$, remains indeterminate: if the nominal pricing measure, $\nu(\cdot|s^t)$, is specified arbitrarily, the inflation process, $\pi(\cdot|s^t)$, adjusts to implement the equilibrium; that is, to satisfy

$$\tilde{q}(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}.$$

Furthermore, the path of the nominal exchange rate remains indeterminate. Arbitrary nominal pricing measures in each country determine the stochastic future exchange rate to satisfy

$$\frac{\nu^*(s^{t+1}|s^t)}{\nu(s^{t+1}|s^t)} \left\{ \frac{1 + r(s^t)}{1 + r^*(s^t)} \right\} e^*(s^t) = e^*(s^{t+1}|s^t).$$

2.1 A stationary economy

The argument extends to stationary economies and stationary equilibria or steady states.

The resolution of uncertainty follows a stationary stochastic process. Elementary states of the world are s , finitely many, and transition probabilities are $f(s'|s)$.

Rates of interest, $(r(s^1|s^0), r^*(s^1|s^0))$ determine the path of consumption, $(c(s^1|s^0), c^*(s^1|s^0))$ and employment, $(l(s^1|s^0), l^*(s^1|s^0))$ at equilibrium, which, in turn, determine the prices of indexed elementary securities:

$$\beta f(s'|s) \frac{\partial u(c(s'), 1 - l(s'))}{\partial c(s')} \tilde{q}(s'|s)^{-1} = \frac{\partial u(c(s^1|s^0), 1 - l(s^1|s^0))}{\partial c(s^1|s^0)}$$

or

$$\tilde{Q} = \beta Du(s^1|s^0)^{-1} F Du(s').$$

Note that the prices of indexed elementary securities is independent of the country. The nominal elementary securities, and hence martingale measures, across countries differ in their stochastic rates of inflation (and consequently the no-arbitrage condition).

Here,

$$Du(s^1|s^0) = \text{diag}(\dots, \frac{\partial u(c(s^1|s^0), 1 - l(s^1|s^0))}{\partial c(s^1|s^0)}, \dots)$$

is the diagonal matrix of marginal utilities of consumption, and

$$F = (f(s'|s)) \quad \text{and} \quad \tilde{Q} = (\tilde{q}(s'|s))$$

are, respectively, the matrices of transition probabilities and of prices of indexed elementary securities.

For the home household,

$$\tilde{m} = (\dots \frac{r(s^1|s^0)}{1 + r(s^1|s^0)} a(s^1|s^0) l(s^1|s^0) \dots)$$

is the vector of net, real balances at equilibrium,

$$\tilde{z} = (\dots z(s^1|s^0) \dots)$$

is the vector of excess demands and the real wealth at the steady state is given by

$$\tilde{\tau} = (\dots \tau(s^1|s^0), \dots).$$

$\tilde{\tau}$ is determined by by the equations

$$\tilde{z} + \tilde{m} + \tilde{Q}\tilde{\tau} = \tilde{\tau} \quad \text{or} \quad \tilde{\tau} = (I - \tilde{Q})^{-1} [\tilde{z} + \tilde{m}].$$

$$\tilde{z}^* + \tilde{m}^* + \tilde{Q}\tilde{\tau}^* = \tilde{\tau}^* \quad \text{or} \quad \tilde{\tau}^* = (I - \tilde{Q})^{-1} [\tilde{z}^* + \tilde{m}^*].$$

The real wealth of the monetary-fiscal authorities in the home country, \tilde{T} , is determined by

$$\tilde{M} + \tilde{Q}\tilde{T} = \tilde{T} \quad \text{or} \quad \tilde{T} = (I - \tilde{Q})^{-1}\tilde{M},$$

where $\tilde{M} = (\dots \frac{r(s^1|s^0)}{1+r(s^1|s^0)}a(s^1|s^0)l(s^1|s^0)\dots)$ and, since F is a Markov transition matrix, while $\tilde{M} \gg 0$, the real claims against the monetary-fiscal authority at the steady state are strictly positive:

$$\tilde{T} \gg 0.$$

Note that the real claims against the monetary-fiscal authorities can only be jointly determined, $\tilde{T} + \tilde{T}^* = \tilde{\tau} + \tilde{\tau}^*$.

As we have solved the entire real economy without nominal variables, the initial price level in each country remains indeterminate. More importantly, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s)$, and a nominal pricing measure, $\nu(\cdot|s)$, remain indeterminate in each country. For the home country:

$$\tilde{Q} = R^{-1}N \otimes \Pi.$$

Here,

$$R = \text{diag}(\dots, (1 + r(s^1|s^0)), \dots)$$

is the diagonal matrix of interest factors, and

$$N = (\nu(s'|s)), \quad \Pi = ((1 + \pi(s'|s))), \quad \text{and} \quad E = (e^*(s'|s)/e^*(s^1|s^0))$$

are, respectively, the matrices of “nominal pricing transition probabilities”, inflation factors and exchange rate factors. The stochastic growth rates of nominal exchange rates are given by¹⁴

$$E = \Pi \oslash \Pi^*$$

In the absence of restrictions on the balance sheet of the monetary fiscal authority, which is the case under QE, the set of steady state equilibria is indexed by the nominal pricing transition probabilities, $\nu(\cdot|s)$, that can be set arbitrarily; the inflation factors, $\pi(\cdot|s)$, and exchange rate factors, $e(s'|s)/e(s^1|s^0)$, then adjust to implement the equilibrium.

In each country, the composition of the balance sheet of the monetary-fiscal authority can be described by portfolio weights, $\delta(s'|s)$ (that is, $0 < \delta(s'|s) \leq 1$, and $\sum_{s'} \delta(s'|s) = 1$), and scale factors $h(s^1|s^0)$, such that

$$h(s^1|s^0)\delta(s'|s) = \tilde{T}(s')(1 + \pi(s'|s));$$

¹⁴Entry-by-entry multiplication is \otimes , while \oslash is entry-by-entry division.

this is the case since the inflation factor is the rate of exchange of output between a date-event and an immediate successor.

The equilibrium condition, then reduces to

$$\tilde{Q} = R^{-1}N \otimes H\Gamma = R^{-1}HN \otimes \Gamma.$$

Here,

$$H = \text{diag}(\dots, h(s^1|s^0), \dots)$$

is the diagonal matrix of scale factors, and

$$\Gamma = \begin{pmatrix} \delta(s'|s) \\ \tilde{T}(s') \end{pmatrix}$$

is the matrix of portfolio weights relative to the payoff of the balance sheet.

Monetary-fiscal policy conducted as CE sets the composition of the balance sheet; that is, it sets positive portfolio weights, $\delta(s'|s) > 0$; claims against the monetary-fiscal authority in real terms, $\tilde{T}(s^1|s^0)$, are determined, at the steady-state, by fundamentals, and, as a consequence, under CE, the matrix Γ is determined.

Since

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H = (R\tilde{Q} \oslash \Gamma)\mathbf{1}_S,$$

the Markov transition matrix, N , is well defined ($h \gg 0$) and determinate; it follows that the equilibrium is determinate as well.

Under conventional monetary-fiscal policy, the portfolio of the monetary-fiscal authority consists of treasury bills, nominally risk-free bonds of short maturity. Here, this corresponds to one-date nominally risk-free bonds: $\delta(s'|s) = 1/S$. \square

Concerning the indeterminacy that obtains in the open economy, further remarks are in order:

1. Our results under unconventional quantitative policies remain valid if the law of one price failed to hold, as in [Corsetti and Pesenti \(2005\)](#), or if there were pricing rigidities. However, in these cases the indeterminacy may have real effects.
2. The indeterminacy under QE obtained is not a consequence of deviations from steady-state equilibria and will not be eliminated by an interest rate feed-back rule, such as a ‘‘Taylor rule’’. This will be discussed at the end of the following section to avoid repetition. The non-stationary equilibria results presented above allow for extreme paths inflation and exchange rates. As the Fisher equation only guarantees

an expected rate of inflation, it is entirely possible that there are paths of ever increasing inflation and a path of ever decreasing inflation (deflation), and consequently large stochastic changes in nominal exchange rates, and is reminiscent of the literature on speculative hyperinflation such as [Obstfeld and Rogoff \(1983\)](#).

3. Our requirement that the present-value budget constraints of the monetary-fiscal authority in each country be satisfied individually is not innocuous. Equilibrium only requires that the individual household budget constraints are satisfied, and as a consequence, only the joint budget constraint of the two government budget constraints will be satisfied. In that case the non-Ricardian assumption only guarantees that the present value of the monetary liabilities of both central banks, weighted by the exchange rate, equals the initial nominal wealth, also weighted by the exchange rate. As a consequence, neither the price levels in each country nor the exchange rate is determinate. This is the point of [Dupor \(2000\)](#). Here, the non-Ricardian assumption in each country results in the price-level in each country to be uniquely determined. The subsequent indeterminacy is then restricted to the indeterminacy of the stochastic path of inflation, and is convenient to identify the role that QE plays in generating this indeterminacy.
4. A managed exchange rate, satisfying uncovered interest parity, will either transmit or eliminate the indeterminacy. If, for arguments sake, the home country conducts traditional monetary policy (and has a determinate path of inflation), then the foreign country may partake in quantitative easing and provided that they also target a path of the exchange rate, then the law-of-one-price guarantees that foreign prices are also determinate. If, however, the home country also conducts quantitative easing, then management of the path of the exchange rate leaves the rates of inflation in each country indeterminate. This is because the law-of-one-price only determines the ratio of prices across countries to equal the nominal exchange rate, but the (stochastic) levels are left free.
5. Our argument allows monetary-fiscal authorities to arbitrarily select the composition of initial assets, and independently of the initial quantity of money and price level, which are determined by the initial fiscal liabilities. Our argument is valid when the monetary-fiscal authority attempts to affect the initial quantity of money by purchasing assets with newly printed money: this would be analagous to increasing the

outstanding liabilities that need to be returned through seignorage profits

6. The argument holds for the policies of unwinding of quantitative easing that are dependent on realized rates of inflation. This will be made more explicit in the following section, but intuitively, the monetary-fiscal authority here are faced with a new portfolio every date, due to the one-date contracts we focus on. This implies that the degrees of indeterminacy are $S - 1$ in each country and state. Hence, even if the initial portfolio composition is fixed, the consequent evolution of the portfolio (ie the unwinding phase) is not and indeterminacy will result.

It is worth pointing out that the indeterminacy we obtain is not a consequence of the stochastic nature of our economy per se, but rather that, given the uncertainty, the non-collinearity of assets traded by the monetary-fiscal authority. In a related note, [McMahon et al. \(2012\)](#), we examine the consequences of the recent European Central Bank (ECB) policy on purchasing the debt of member countries (Outright Monetary Transactions, or OMT). If the bonds purchased by the ECB are not expected to default, which such a policy is in fact designed to support, then the bonds of the member countries are collinear and there is no requirement to provide ex-ante restrictions on the composition of assets held by the ECB. If however such a policy cannot prevent default, then the bonds are no longer collinear and short-term interest rates may no longer be sufficient to determine the path of Eurozone inflation.

Unconventional Monetary Policy and Premia

In cash-in-advance specifications, liquidity costs generate a wedge between cash and credit goods, and consequently affect marginal utilities and equilibrium prices. This generates a positive correlation between the (real) stochastic discount factor and expected nominal interest rates, and, as a consequence, a real risk premium that causes the term structure of interest rates to lie above levels predicted by the pure expectation hypothesis. In a closed economy, [Espinoza et al. \(2009\)](#) show that the risk-premia generated by the non-neutrality of monetary policy exist in addition to the ones derived from the stochastic distribution of endowments as presented in [Lucas \(1978\)](#) and [Breedon \(1979\)](#). They provide a potential explanation for the Term Premium Puzzle¹⁵. In an open economy, the argument extends, whereby the path of

¹⁵There is a large literature on the difficulties of the uncovered interest parity holding empirically. The forward premium anomaly, as documented by [Fama \(1984\)](#), [Hodrick](#)

nominal interest rates in each country can affect real risk-premia on the path of nominal exchange rates as in [Peiris and Tsomocos \(2015\)](#)¹⁶. This is in contrast to equilibrium models where monetary policy is neutral, as in [Lucas \(1982\)](#), where, as risk premia are constant, interest rate differentials move one-for-one with the expected change in the exchange rate. We extend this literature by showing how the composition of the monetary-fiscal authority balance sheet, in addition to policy setting the path of interest rates or money supplies, affects premia in the bond and currency markets. The premia that we obtain is purely nominal though our results extend to economies with incomplete markets and price rigidities, in which case the premia would also be real.

Term Premia

Our analysis utilises the stationary equilibrium results obtained in the previous section.

Consider the price, in the home country, of a two-date nominally risk-free bond, at state s :

$$q_2(s^1|s^0) = \sum_{s'} q(s'|s) \sum_{s''|s'} q(s''|s') = \frac{1}{1 + r(s^1|s^0)} \sum_{s'|s} \frac{\nu(s'|s)}{1 + r(s'|s)}$$

In other words, the forward rate gives the risk-neutral expectation of the future one-date interest rates:

$$q_2(s^1|s^0)(1 + r(s^1|s^0)) = \sum_{s'|s} \frac{\nu(s'|s)}{1 + r(s'|s)}.$$

(1987), and [Backus et al. \(1995\)](#) among others, states that when a currency's interest rate is high, that currency is expected to appreciate in-advance constraint. Roughly speaking, the expected change in the exchange rate is constant and interest differentials move approximately one-for-one with risk premia.

¹⁶In that paper, there are two countries each inhabited by a representative agent and who must use domestic money for domestic trades, such as in the present paper. A cash-in-advance structure means that nominal interest rates affect the wedge between the marginal utilities of income and expenditure. Furthermore, markets are incomplete and agents may default upon their nominal obligations. Monetary policy, by altering the wedge, affects the volume of real trade, and hence marginal utilities, default probabilities and implied risk neutral probabilities. Consequently, there is a covariance between nominal exchange rates, and real and nominal premia which affects the difference between the risk-neutral and objective expectation of future exchange rates. In the present paper, this difference is generated purely by altering the composition of assets traded by the monetary-fiscal authorities in each country.

The term premia are then described by

$$\sum_{s'|s} \frac{\nu(s'|s) - f(s'|s)}{1 + r(s'|s)}.$$

The stationary distribution of the term premia is

$$N \otimes R - F \otimes R = [((R\tilde{Q} \otimes \Gamma)\mathbf{1}_S)^{-1}R\tilde{Q} \otimes \Gamma - F] \otimes R.$$

Recall that Γ is the matrix of portfolio weights relative to the payoff of the balance sheet of the monetary-fiscal authority. Hence, given the fundamentals of the economy, and a given path of one-date interest rates, the term premia depends on the composition of the monetary-fiscal authority balance sheet. More precisely, a correlation is generated between the nominal martingale measure and nominal interest rates which results in risk-neutral pricing being systematically biased (from subjective pricing alone).

Currency Premia

Recall the Uncovered Interest Parity equation in state s

$$e^*(s^1|s^0) \frac{1 + r(s^1|s^0)}{1 + r^*(s^1|s^0)} = \sum_{s'|s} \nu(s'|s) e^*(s'|s),$$

where $\sum_{s'|s} \nu(s'|s) e^*(s'|s)$ is the risk neutral expectation of exchange rates. The realised distribution of exchange rates implies an (objective) expectation of $\sum_{s'|s} f(s'|s) e^*(s'|s)$. The difference between these two will be the currency premium. The stationary distribution of the premium is:

$$\begin{aligned} N \otimes E - F \otimes E &= N \otimes \Pi \otimes \Pi^* - F \otimes \Pi \otimes \Pi^* = \\ R\tilde{Q} \otimes \Pi^* - F \otimes \Pi \otimes \Pi^* &= (R\tilde{Q} - F \otimes \Pi) \otimes \Pi^* = \\ &= (R\tilde{Q} - F \otimes H\Gamma) \otimes (H^*\Gamma^*) = \\ (R\tilde{Q} - F \otimes (R\tilde{Q} \otimes \Gamma)\mathbf{1}_S\Gamma) &\otimes ((R^*\tilde{Q}^* \otimes \Gamma^*)\mathbf{1}_S\Gamma^*). \end{aligned}$$

This is entirely in terms of real variables and nominal interest rates and portfolio weights set by policy. Furthermore there is a clear separation between home and foreign variables and policy parameters. It follows then that stationary portfolio weights chosen in each country correspond to varying premia in the currency markets. The sign and magnitude of the premium can be chosen arbitrarily by appropriate choices of nominal interest rates and

portfolio weights. Note that varying the nominal interest rates results in the premium having a real (risk) component while varying the portfolio weights affects the stationary distribution of inflation and exchange rates which is purely nominal. From the equation, it is clear that the joint distribution of interest rates and inflation across countries matters in addition to the mean and variance of the inflation process in each country.

We have considered only interest rate targeting; the results do extend to policies that target the paths of money supplies. In that case, although the path of money is given by policy, fluctuations in demands for assets affect the path of interest rates and changes in the composition of monetary-fiscal authority portfolio has real effects. That is, in a money growth targeting regime, the path of real risk-premia depends on the composition of assets held by the monetary-fiscal authority.¹⁷

References

- B. Adao, I. Correia, and P. Teles. Short and long interest rate targets. *Journal of Monetary Economics*, 66:95–107, 2014.
- F. Alvarez, Andrew Atkeson, and P. J. Kehoe. Time-varying risk, interest rates, and exchange rates in general equilibrium. *The Review of Economic Studies*, 76:851–878, 2009.
- G. M. Angeletos. Fiscal policy with noncontingent debt and the optimal maturity structure. *The Quarterly Journal of Economics*, 117:1105–1131, 2002.
- D. K. Backus, S. Foresi, and C. I. Telmer. Interpreting the forward premium anomaly. *The Canadian Journal of Economics / Revue canadienne d'Économique*, 28:S108–S119, 1995.
- J. H. Bai and I. Schwarz. Monetary equilibria in a cash-in-advance economy with incomplete financial markets. *Journal of Mathematical Economics*, 42, 2006.

¹⁷Nakajima and Polemarchakis (2005) show this in a closed economy. Alvarez et al. (2009) consider an open economy similar to ours, but with segmented participation in the asset market; in the absence of a credit good, monetary policy is otherwise neutral. If the monetary-fiscal authority portfolio is left unrestricted, introducing a credit good may mean that the correlation between interest rates and risk premia depend both on the path of money and the evolution of the composition of the monetary-fiscal authority portfolio.

- Y. Balasko and D. Cass. The structure of financial equilibrium i: exogenous yields and unrestricted participation. *Econometrica*, 57:135–162, 1989.
- J. Benhabib, S. Schmitt-Grohe, and M. Uribe. Monetary policy and multiple equilibria. *American Economic Review*, 91:167–186, 2001.
- B. Bernanke. The crisis and the policy response. The Stamp Lecture, The London School of Economics and Political Science, 2009.
- G. Bloise, J. H. Drèze, and H. Polemarchakis. Monetary equilibria over and infinite horizon. *Economic Theory*, 25:51–74, 2005. URL <http://www.polemarchakis.org/a67-mei.pdf>.
- D. T. Breeden. An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics*, 7:265–296, 1979. ISSN 0304-405X.
- W. H. Buiter. The fiscal theory of the price level: a critique. *The Economic Journal*, 112:459–480, 2002.
- D. Cass. Competitive equilibria with incomplete financial markets. Working Paper No. 84-09, CARESS, University of Pennsylvania, 1984.
- D. Cass. On the ‘number’ of equilibrium allocations with financial assets. Working Paper No. 85-00, CARESS, University of Pennsylvania, 1985.
- J. H. Cochrane. Long-term debt and optimal policy in the fiscal theory of the price level. *Econometrica*, 69:69–116, 2001.
- G. Corsetti and P. Pesenti. International dimensions of optimal monetary policy. *Journal of Monetary Economics*, 52(2):281 – 305, 2005.
- V. Curdia and M. Woodford. The central-bank balance sheet as an instrument of monetary policy. *Journal of Monetary Economics*, 58:54–79, 2011.
- J. H. Drèze and H. Polemarchakis. Monetary equilibria. In G. Debreu, W. Neufeind, and W. Trockel, editors, *Economic Essays: A Festschrift in honor of W. Hildenbrand*, pages 83–108. Springer, 2000. URL <http://www.polemarchakis.org/o15-meq.pdf>.
- P. Dubey and J. D. Geanakoplos. Monetary equilibrium with missing markets. *Journal of Mathematical Economics*, 39:585–618, 2003.
- B. Dupor. Exchange rates and the fiscal theory of the price level. *Journal of Monetary Economics*, 45:613–630, 2000.

- G. B. Eggertsson and M. Woodford. The zero bound on interest rates and optimal monetary policy. *Brookings Papers on Economic Activity*, 2003: 139–211, 2003.
- R. A. Espinoza, C. A. E. Goodhart, and D. P. Tsomocos. State prices, liquidity, and default. *Economic Theory*, 39:177–194, 2009.
- E. F. Fama. Forward and spot exchange rates. *Journal of Monetary Economics*, 14:319–338, 1984.
- B. W. Fawley and N. J. Christopher. Four stories of quantitative easing. *Federal Reserve Bank of St. Louis Review*, 95:51–58, 2013.
- J. D. Geanakoplos and A. Mas-Colell. Real indeterminacy with financial assets. *Journal of Economic Theory*, 47:22–38, 1989.
- M. Goodfriend. Central banking in the credit turmoil: an assessment of federal reserve practice. *Journal of Monetary Economics*, 58:1–12, 2011. mimeo.
- J. D. Hamilton and J. C. Wu. The effectiveness of alternative monetary policy tools in a zero lower bound environment. *Journal of Money, Credit, and Banking*, 44(s1):3–46, 2012.
- R. J. Hodrick. *The Empirical Evidence on the Efficiency of Forward and Futures Foreign Exchange Markets*. Harwood Academic Publishers, 1987.
- D. M. Kreps. Multiperiod securities and the efficient allocation of risk: a comment on the black-scholes option pricing model. In J. McCall, editor, *The Economics of Uncertainty and Information*, pages 203–232. University of Chicago Press, 1982.
- R. E. Lucas. Asset prices in an exchange economy. *Econometrica*, 46:1429–1445, 1978.
- R. E. Lucas. Interest rates and currency prices in a two-country world. *Journal of Monetary Economics*, 10:335–359, 1982.
- R. E. Lucas and N. L. Stokey. Money and rates of interest in a cash-in-advance economy. *Econometrica*, 55:491–513, 1987.
- E. Maeda, B. Fujiwara, A. Mineshima, and K. Taniguchi. Japan’s open market operations under the quantitative easing policy. Working paper, Bank of Japan, 2005.

- M. Magill and M. Quinzii. Anchoring expectations of inflation. *Journal of Mathematical Economics*, 50:86–105, 2014a.
- M. Magill and M. Quinzii. Term structure and forward guidance as instruments of monetary policy. *Economic Theory*, 56:1–32, 2014b.
- M. McMahon, M.U. Peiris, and H. Polemarchakis. Outright monetary transaction sterilised? *Vox*, 2012. URL <http://www.voxeu.org/article/outright-monetary-transactions-sterilised>.
- T. Nakajima and H. Polemarchakis. Money and prices under uncertainty. *Review of Economic Studies*, 72:223–246, 2005. URL <http://www.polemarchakis.org/a66-mpu.pdf>.
- M. Obstfeld and K. Rogoff. Speculative hyperinflations in maximizing models: can we rule them out? *Journal of Political Economy*, 91:675–687, 1983.
- M. Udara Peiris and Dimitrios P. Tsomocos. International monetary equilibrium with default. *Journal of Mathematical Economics*, 56(0):47–57, 1 2015.
- R. Reis. Interpreting the unconventional u.s. monetary policy of 2007-09. *Brookings Papers on Economic Activity*, 40(2, Fall):119–182, 2009.
- T. Sargent and N. Wallace. Rational expectations, the optimal monetary instrument and the optimal money supply rule. *Journal of Political Economy*, 83:241–254, 1975.
- C. A. Sims. A simple model for the study of the determination of the price level and the interaction of monetary and fiscal policy. *Economic Theory*, 4:381–000, 1994.
- C. A. Sims. Fiscal policy, monetary policy and central bank independence. unpublished manuscript, 2016.
- J. B. Taylor. Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy*, 39:195–214, 1993.
- John B. Taylor. The role of the exchange rate in monetary-policy rules. *The American Economic Review*, 91(2):263–267, 2001.
- H. Ugai. Effects of the quantitative easing policy: a survey of empirical analyses. *Monetary and Economic Studies*, 25:1–48, 2007.

- D. Vayanos and J.-L. Vila. A preferred-habitat model of the term structure of interest rates. *National Bureau of Economic Research Working Paper Series*, No. 15487, November 2009 2009.
- M. Woodford. Monetary policy and price level determinacy in a cash-in-advance economy. *Economic Theory*, 4:345–380, 1994.
- M. Woodford. Price-level determination under interest rate rules. Unpublished manuscript, 1999.