Monetary equilibria in continuous time

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Abstract

In a stochastic cash-in-advance economy in continuous-time, with Ricardian fiscal policy competitive equilibria are indeterminate. As in discrete time, this indeterminacy is characterized by the initial price level and a nominal equivalent martingale measure over states of the world.

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Key words: cash-in-advance; continuous time; indeterminacy; equivalent martingale measure.

1 Introduction

A natural concern in any argument on monetary policy is whether monetary policy determines the path of prices at equilibrium. Whether it does may depend both on the structure of the economy and the conduct of monetary policy. Any indeterminacy that remains may be only nominal or also real.

Contemporary literature on monetary policy, theoretical as well as empirical, has employed extensively the framework of a cash-in-advance economy in Lucas and Stokey (1987). It is in a variant of this model that Woodford (1994),Woodford (1995) made the bold claim that policy, monetary and fiscal, does determine the path of prices as long as it is non-Ricardian: that is, as long as fiscal policy is not constrained by an intertemporal budget constraint, which he termed a fiscal theory of price determination. Buiter (2002) challenged the relevance, even propriety of the determinacy claim by pointing out that it relies on the violation of the public sector budget constraint albeit out of equilibrium.

Differently motivated, Dubey and Geanakoplos (2003a) and Dubey and Geanakoplos (2003b) arrived at similar determinacy results. In the presence of any small amount of outside money, in a Walrasian economy that is extended to encompass banks that issue balances while cash-in-advance constraints are operative, equilibrium prices are determinate. Which is even more striking, as it applies to economies with uncertainty and general, not necessarily recursive equilibrium processes.

In a contrary vain, Bloise, Drèze, and Polemarchakis (2005) demonstrated that, for a fixed path of rates of interest, there is a non-trivial multiplicity of equilibrium paths of prices of commodities. Determinacy requires that, subject to no-arbitrage and in addition to rates of interest, the prices of state-contingent revenues be somehow determined.

With non-Ricardian policy, in the fiscal theory or with outside money, determinacy obtains because, at equilibrium, the public budget must be balanced; with Ricardian policy, the public budget is balanced a fortiori.

Under uncertainty, the determinacy of equilibrium with a non-Ricardian specification relies on an additional (often tacit) assumption: that the monetary authority conduct open market operation between cash balances and, exclusively, short term nominal bonds. If open market operations are extended to involve assets with arbitrary distributions of payoffs, as they are in periods of quantitative easing, the determinacy result fails. This is the case both for the argument in Woodford (1994), Woodford (1995) and in Dubey and Geanakoplos (2003b). Cochrane (2001) made this point with reference to the maturity structure of the bonds exchanged in open market operations; McMahon, Peiris, and Polemarchakis (2015) demonstrated that quantitative easing compromises the control of monetary policy over the distribution of inflation across realizations of uncertainty.

It is important to point out that, with Ricardian policy, the structure of the set of equilibria with cash-in-advance constraints mirrors the structure of the set in economies that abstract from transactions, in particular the Walrasian model, where prices are quoted in abstract units of account. Ever since Arrow (1953) proposed what became the definitive sequence economy with complete financial markets, it has been recognized that equilibria display a high degree of nominal indeterminacy. In an insightful contribution, Cass (1985) showed that, when the asset market is incomplete, the indeterminacy is real; Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989) characterized the set of equilibria and, in particular, the dimensions of real and nominal multiplicity. The real indeterminacy with incomplete markets was interpreted as an argument for the relevance of monetary policy. objections to this interpretation, with reference to the absence of a money market in the argument of Cass simply miss the point: the structure of the set of equilibria is the same in an economy with a cash-in-advance constraint and a bona fide money market — that is, as long as the specification is Ricardian.

In an earlier paper, Nakajima and Polemarchakis (2005), we characterized the indeterminacy associated with Ricardian policy in a simple, stochastic, cash-in-advance economy. We showed that it equilibrium paths are described by the initial price level and a probability measure associated with state-contingent nominal bonds that we called an equivalent martingale measure. It follows that monetary policy determines an average, but not the distribution of inflation across realizations of uncertainty.

The intuition that underlies the indeterminacy in a discrete time specification is straight forward. At a date event, there are as many elementary security prices to be determined as immediate successors of the event. Monetary policy that sets the rate of interest fixes the sum of these prices or, equivalently the price of a short term nominal bond; all but one dimensions of indeterminacy remain. If the horizon of the economy is finite, the degrees of indeterminacy over all events aggregate to the number of terminal nodes minus 1, or the dimension of the space of probability measures over terminal nodes or states of the world. The argument extends to economies with an infinite horizon.

As described above, the intuition that underlies the indeterminacy of equilibrium prices over discrete time relies on the distinction at each instant of time between the ”present” and the ”immediate future.” The question then arises whether, in a continuous time specification, the indeterminacy persists.

Recently, for instance, continuous-time monetary models have been used to analyze the indeterminacy in the context of the Taylor rule Taylor (1993).[1] In this literature, attention is typically restricted in showing the indeterminacy of equilibrium and the existence of sunspot equilibria. Welfare properties of those equilibria have not been, and there has not been an attempt to characterize the entire set of equilibria.

In this paper, we extend the approach of Bloise, Drèze, and Polemarchakis (2005) and Nakajima and Polemarchakis (2005) to a continuous time. We show that, with Ricardian policy, as in discrete time, competitive equilibrium prices are indeterminate: there is a continuum of equilibria indexed by the initial price level and the equivalent martingale measure. We characterize the whole set of equilibria, which should be particularly useful in a normative analysis.

In order to illustrate our point simply, we consider a continuous-time version of the model of Lucas and Stokey (1987) and assume that monetary policy sets a stochastic process of the short-term nominal interest rate. In this case indeterminacy is purely nominal and the real allocation is uniquely determined. It should be possible to extend our approach to the case where indeterminacy is real, but this we leave for future research.

The rest of the paper is organized as follows. In Section 2 the model economy is described. In Section 3 the main result on indeterminacy is provided. Concluding remarks are in Section 4.

2 The Economy

The economy is a continuous-time version of the economy in Lucas and Stokey (1987); the continuous-time formulation builds on the previous work by Duffie and Zame (1989) and Duffie (2001, Chapter 10).

2.1 Primitives

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a finite time interval $[0, T]$. Here, Ω is a complete description of the exogenous uncertain environment from time 0 to time T, the σ-algebra $\mathcal{F}$ is the collection of events distinguishable at time T, and P is a probability measure over $(\Omega, \mathcal{F})$. An N-dimensional standard Brownian motion, Z, is defined on $(\Omega, \mathcal{F}, P)$. Let $\boldsymbol{F} = \{\mathcal{F}_t : t \in [0, T]\}$ be the standard filtration of sub-σ-algebras of $\mathcal{F}$ generated by Z. We assume $\mathcal{F}_T = \mathcal{F}$. The filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F}, P)$ is the basic model for information and beliefs. All probabilistic statements below

[1] Examples include, among others, Benhabib, Schmitt-Grohe, and Uribe (2001) and Benhabib, Schmitt-Grohe, and Uribe (2002).

should be understood to hold "almost surely" with respect to $(\Omega, \mathcal{F}, \boldsymbol{F}, P)$, unless otherwise indicated. Also, all the processes to appear are adapted to $\boldsymbol{F}$ unless otherwise specified.

There is a representative consumer. As in Lucas and Stokey (1987), there are two consumption goods available at each date: "cash goods," which are subject to a cash-in-advance constraint, and "credit good," which are not. We shall use $c_1(t)$ and $c_2(t)$, respectively, to denote the amount of consumption of cash goods and that of credit goods at date t. The consumption space is the set L of adapted processes $c = (c_1, c_2)$ in $\mathbb{R}^2$ satisfying $E\left(\int_0^T c_i(t)^2\, dt\right) < \infty$. The representative individual has the utility function:

$$U(c) = E\left[\int_0^T e^{-\rho t} u\big[c_1(t), c_2(t)\big]\, dt\right], \tag{1}$$

where $\rho > 0$ and the flow utility function u satisfies the following condition.

Assumption 1. The flow utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is smooth, strictly concave, and strictly increasing. Both goods are normal:

$$u_{11}u_2 - u_{12}u_1 < 0, \quad \text{and} \quad u_{22}u_1 - u_{12}u_2 < 0.$$

The Inada conditions are satisfied:

$$\lim_{c_1 \to 0} u_1 = \lim_{c_2 \to 0} u_2 = +\infty.$$

Goods are not storable. At each date $t \in [0, T]$, the representative individual receives endowment $e(t)$ of credit goods. She has an access to the technology that transforms credit goods into cash goods one-for-one. It follows that cash and credit goods have the same nominal price, $p(t)$, at each date. Each individual cannot consume the cash goods she produces; instead, she has to purchase them from other individuals with cash. Thus, consumption of cash goods, $c_1(t)$, is subject to the cash-in-advance constraint:

$$p(t)c_1(t) \leq m(t), \tag{2}$$

where $m(t)$ is the amount of nominal balances held by the individual at date t.[2] Following Duffie and Zame (1989), we make the following assumption on the endowment process e.

[2] Rebelo and Xie (1999) also use this type of cash-in-advance constraint in their deterministic continuous-time model.

Assumption 2. The endowment process $e \in L_+$ is an Ito process, bounded away from zero, where the stochastic differential representation

$$de(t) = \mu_e(t)\,dt + \sigma_e(t)\,dZ(t) \tag{3}$$

is such that $E\left(\int_0^T \sigma_e(t)\cdot\sigma_e(t)\,dt\right) < \infty$.

In addition to money, $N+1$ long-lived financial securities are traded at each date, which are indexed by $n = 0, 1, \ldots, N$. Security 0 is a nominal bond, which is locally riskless in nominal terms. The instantaneous nominal interest rate of the bond at date t is $r(t) \geq 0$. Securities $n = 1, \ldots, N$ are risky and represented by N-dimensional cumulative dividend process $D = (D^1, \ldots, D^N)$, measured in nominal units of account. For each security $n = 1, \ldots, N$, D^n is an Ito process such that $D^n(T)$ has finite variance. The stochastic differential representation of D is

$$dD(t) = \mu_D(t)\,dt + \sigma_D(t)\,dZ(t),$$

where μ_D and σ_D are, respectively, an $N \times 1$ vector process and an $N \times N$ matrix process. Following Duffie and Zame (1989), we assume the following spanning condition on the cumulative dividend process.

Assumption 3. The martingales $M^1, \ldots, M^N$ defined by

$$M^n(t) = E\big[D^n(T)\,|\,\mathcal{F}_t\big], \qquad t \in [0, T],$$

form a martingale generator.

If a stochastic differential representation of $M = (M^1, \ldots, M^N)$ is given by $dM(t) = \varphi(t)\,dZ(t)$, this assumption says that the matrix process φ has rank N almost everywhere.

2.2 Budget feasible plans

Each individual takes as given a price-level process $p \in L$, an interest-rate process, r, and an N-dimensional Ito security-price process $S = (S^1, \ldots, S^N)$. Here, we take S to be cum dividend. It follows that the gain process $G = (S + D)$ is also an Ito process, so that we can write

$$dG(t) = \mu_G(t)\,dt + \sigma_G(t)\,dZ(t),$$

for some μ_G and σ_G. A portfolio is an $(N+2)\times 1$ vector process, denoted generically by

$$\Big\{\alpha(t) = \Big(m(t), b(t), \theta(t) \equiv \big[\theta^1(t), \ldots, \theta^N(t)\big]\Big)\,;\, t \in [0, T]\Big\},$$

where $m(t)$, $b(t)$, and $\theta^n(t)$ are the amount of nominal balances, nominal bonds, and shares of the n-th security held at time t. Given p, r and G, we take the portfolio space A to be the vector space of $(N+2)$-dimensional processes $\alpha = (m, b, \theta)$ such that

$$E\left[\int_0^T \left(\frac{m(t)}{p(t)}\right)^2 dt\right] < \infty, \tag{4}$$

$$\int_0^T \left|r(t)b(t) + \theta(t)\cdot\mu_G(t)\right| dt < \infty \quad P\text{-a.s.}, \tag{5}$$

$$E\left(\int_0^T \theta(t)^\top \sigma_G(t)\sigma_G(t)^\top \theta(t)\, dt\right) < \infty. \tag{6}$$

Under these regularity conditions, the stochastic integral $\int \theta\, dG$ is well defined, and further that $\int \theta\, dG$ is a martingale whenever G is a martingale.[3]

Let w_0 be the nominal value of initial wealth of the representative agent. A budget-feasible plan is a pair $(c, \alpha) \in (L_+)^2 \times A$ such that, for any time t,

$$\begin{aligned} w(t) \le w_0 + \int_0^t \theta(s)\, dG(s) + \int_0^t r(s)b(s)\, ds \\ + \int_0^t p(s)\big[e(s) - c_1(s) - c_2(s)\big]\, ds - v(t), \end{aligned} \tag{7}$$

and

$$w(T) \ge 0, \tag{8}$$

Here, v denotes the cumulative lump-sum tax process which is specified in Section 2.3, and $w(t)$ is nominal wealth at t:

$$w(t) = m(t) + b(t) + \theta(t)\cdot S(t).$$

The representative consumer maximizes her lifetime utility (1) subject to the budget constraints (7)-(8) and the cash-in-advance constraint (2).

2.3 The public sector

A debt portfolio of the public sector is an $(N+2)$-dimensional process $(M, B, \Theta) \in A$ that satisfies

$$W(t) = W_0 + \int_0^t \Theta(s)\, dG(s) + \int_0^t r(s)B(s)\, ds - V(t), \tag{9}$$

[3]The latter property guarantees that the existence of a state price deflator precludes arbitrage (Duffie, 2001, Proposition 6C).

where $W(t)$ is the total liabilities of the public sector:

$$W(t) = M(t) + B(t) + \Theta(t) \cdot S(t),$$

W_0 is its initial value, $M(t)$ is the money supply, $B(t)$ is the bond supply, $\Theta(t)$ is the supply of the risky securities, and $V(t)$ is the cumulative lump-sum taxes with $V(0) = 0$. The public sector consists of the fiscal and monetary authorities, which conduct fiscal and monetary policy, respectively. For monetary policy, we focus on interest rate policy.

Monetary policy The monetary authority sets an adapted process of nominal interest rates, r, which is uniformly bounded: there exists a scalar $\overline{r} > 0$ such that $0 \leq r(\omega, t) \leq \overline{r}$, for almost all t and ω.

We assume that fiscal policy is "Ricardian" in the sense used in Woodford (1995) and Benhabib, Schmit-Grohé and Uribe (2001). The debt portfolio does not matter when fiscal policy is Ricardian (for example, Nakajima and Polemarchakis, 2005). So, for simplicity, we assume that the government does not trade risky securities: $\Theta(t) = 0$ for all t. The total liabilities of the public sector are $W(t) = B(t) + M(t)$, all t. Fiscal policy takes the following form:

Fiscal policy Given the initial liabilities W_0, the fiscal authority sets a cumulative tax process V as

$$V(t) = \tau(t) \left\{ W_0 + \int_0^t r(s) W(s)\, ds \right\} - \int_0^t r(s) M(s)\, ds, \tag{10}$$

where the repayment rate process, τ, has right-continuous and bounded variation sample paths with $0 < \tau(t) \leq 1$, all t, and $\tau(T) = 1$ almost surely.

Given this form of fiscal policy, the total liabilities of the public sector, W, evolves as

$$W(t) = \big[1 - \tau(t)\big] \left\{ W_0 + \int_0^t r(s) W(s)\, ds \right\}. \tag{11}$$

It follows that $W(T) = 1$ almost surely, regardless of other exogenous and endogenous variables.

3 Equilibria

To guarantee the existence of an equilibrium, we make an additional assumption on the flow utility function.

Assumption 4. The flow utility function u satisfies

$$\lim_{c\to 0} \frac{u_1(c, e-c)}{u_2(c, e-c)} = +\infty,$$

for each $e > 0$.

Since individuals are identical, market clearing conditions are

$$c_1(t) + c_2(t) = e(t), \quad \theta(t) = \Theta(t) = 0, \quad b(t) = B(t), \quad m(t) = M(t).$$

Also consistency requires that $w_0 = W_0$ and $v(t) = V(t)$.

Given initial nominal wealth, $w_0 = W_0$, interest rate policy, r, financial policy, $\Theta = 0$, and fiscal policy, τ, a competitive equilibrium consists of an allocation, $\{c_1, c_2\}$, a portfolio of individuals, $\{m, b, \theta\}$, debt of the monetary-fiscal authority, $\{M, B\}$, a transfer process, V, a spot-market price process, p, and a stock price process, S, such that

1. given W_0 and $\{r, M\}$, fiscal-financial policy $\{\tau, \Theta\}$ determines transfers V and debt $\{B, W\}$;
2. the monetary authority accommodates the money demand, $M = m$;
3. given interest rates, r, spot-market prices, p, stock prices, S, and transfers, $v = V$, the individual's problem is solved by c_1, c_2, m, b, and θ;
4. all markets clear.

3.1 Equivalent martingale measures

We will show that our economy has indeterminacy regarding the initial price level and a "nominal" equivalent martingale measure. A nominal equivalent martingale measure is an equivalent martingale measure associated with nominal asset prices.

It is instructive to transform the budget constraints (7)-(8) into a single, lifetime budget constraint. Define κ by

$$\kappa(t) = -\sigma_G(t)^{-1}\big[\mu_G(t) - r(t)S(t)\big]. \quad (12)$$

We shall see that Assumption 3 guarantees that $\sigma_G(\omega, t)$ is indeed invertible for almost all ω and t. Suppose for the moment that the process κ is uniformly bounded, that is, there exists a $\overline{K} < \infty$ such that $|\kappa(\omega, t)| \leq \overline{K}$ for almost all ω and t. Later, we shall only consider equilibria which satisfy this condition.

Define η by

$$\eta(t) = \exp\left\{\int_0^t \kappa(s) dZ(s) - \frac{1}{2}\int_0^t \kappa(s)\cdot\kappa(s)\, ds\right\}, \tag{13}$$

which is well defined because κ is uniformly bounded. Note that η is the Ito process with stochastic differential representation:

$$d\eta(t) = \eta(t)\kappa(t)\, dZ(t).$$

Since κ is uniformly bounded, η is a martingale and $\eta(T)$ has finite variance. Hence we can define the probability measure Q equivalent to P by

$$Q(F) \equiv \int_F \eta(\omega, T)\, P(d\omega), \qquad \forall F \in \mathcal{F}. \tag{14}$$

This is referred to as a (nominal) equivalent martingale measure. A standard Brownian motion under Q is given by (Cox and Huang, 1991, Proposition 2.1):

$$Z^*(t) = Z(t) - \int_0^t \kappa(s)\, ds, \qquad t \in [0, T].$$

Now let $R(t)$ be the cumulative interest rate process:

$$R(t) = \exp\left(\int_0^t r(s)\, ds\right).$$

Using Ito's lemma, the discounted wealth process $w(t)/R(t)$ becomes

$$\begin{aligned}
&\frac{w(t)}{R(t)} - w_0 \\
&\quad \leq \int_0^t R(s)^{-1}\Big\{p(s)\big[e(s) - c_1(s) - c_2(s)\big] - r(s)m(s)\Big\}\, ds \\
&\qquad + \int_0^t R(s)^{-1}\Big\{\theta(s)\cdot\mu_G(s) - \theta(s)\cdot r(s)S(s)\Big\}\, ds \\
&\qquad + \int_0^t R(s)^{-1}\theta(s)^\top\sigma_G(s)\, dZ(s) - R(t)^{-1}v(t), \\
&\quad = \int_0^t R(s)^{-1}\Big\{p(s)\big[e(s) - c_1(s) - c_2(s)\big] - r(s)m(s)\Big\}\, ds \\
&\qquad + \int_0^t R(s)^{-1}\theta(s)^\top\sigma_G(s)\, dZ^*(s) - R(t)^{-1}v(t).
\end{aligned}$$

Under our assumption, $\int(\theta^\top \sigma_G/R)\,dZ^*$ is a martingale under Q. It follows that

$$E^*\left[\int_0^T R(t)^{-1}\Big\{p(t)\big[c_1(t)+c_2(t)-e(t)\big]+r(t)m(t)\Big\}\,dt+\frac{v(T)}{R(T)}\right]\leq w_0,$$

where E^* is the expectation operator associated with Q, and we have used the terminal wealth condition (8). The above inequality is equivalent to

$$\begin{aligned} E\left[\int_0^T q(t)\Big\{p(t)\big[c_1(t)+c_2(t)\big]+r(t)m(t)\Big\}\,dt\right] \\ \leq w_0+E\left[\int_0^T q(t)p(t)e(t)\,dt-q(T)v(T)\right], \end{aligned} \tag{15}$$

where $q(t)$ is the (nominal) state price deflator:

$$q(t)=\frac{\eta(t)}{R(t)}.$$

Equation (15) shows that nominal interest rate r serves as tax on nominal balances m. We now combine this and the cash-in-advance constraint (2). Since the cash-in-advance constraint binds whenever $r(t)>0$, we have

$$r(t)m(t)=r(t)p(t)c(t).$$

Inserting this equation into the lifetime budget constraint (15) yields

$$\begin{aligned} E\left[\int_0^T q(t)\Big\{p(t)\big[1+r(t)\big]c_1(t)+p(t)c_2(t)\Big\}\,dt\right] \\ \leq w_0+E\left[\int_0^T q(t)p(t)e(t)\,dt-q(T)v(T)\right]. \end{aligned} \tag{16}$$

Thus, any budget feasible plan meeting the cash-in-advance constraint satisfies (16). The above equation shows that the nominal interest rate, r, works as a distortionary tax on cash goods. This is, of course, the argument for the Friedman rule.

Similarly, the fiscal policy rule (10) implies that the public sector satisfies the lifetime budget constraint:

$$E\left[\int_0^T q(t)r(t)M(t)\,dt+q(T)V(T)\right]=W_0, \tag{17}$$

which says that the present discounted value of taxes and seigniorage equals the initial debt.

Consider the problem that maximizes (1) subject to the single budget constraint (16). Let λ be a scalar Lagrange multiplier on the lifetime budget constraint (16). By the Saddle Point Theorem (Duffie, 2001, Appendix B), a solution to this utility maximization problem is characterized by the first order conditions:

$$e^{-\rho t}u_1\big[c_1(t), c_2(t)\big] = \lambda q(t)\big[1 + r(t)\big]p(t), \tag{18}$$

$$e^{-\rho t}u_2\big[c_1(t), c_2(t)\big] = \lambda q(t)p(t), \tag{19}$$

and the complementary-slackness condition:

$$E\left[\int_0^T q(t)\Big\{p(t)\big[1 + r(t)\big]c_1(t) + p(t)c_2(t)\Big\}\,dt\right] = w_0 + E\left[\int_0^T q(t)p(t)e(t)\,dt - q(T)v(T)\right]. \tag{20}$$

Here, the existence of a solution is straightforward, because the complementary-slackness condition (20) follows from fiscal policy of the public sector.

Based on the first-order conditions (18)-(19) and the lifetime budget constraints (15)-(17), we would expect that only intertemporal relative prices, $q(t)p(t)/p(0)$, are determined in equilibrium. Thus, equilibria would be indexed by $p(0)$ and κ, because $q(t) = \eta(t)/R(t)$ and $\eta(t)$ is defined by (13).

3.2 Indeterminacy

The next proposition summarizes the indeterminacy result.

Proposition 1. Given initial nominal wealth, $w_0 = W_0$, interest-rate policy, r and fiscal policy, τ and $\Theta = 0$, if Assumptions 1-4 are satisfied, then

1. a competitive equilibrium exists;
2. the equilibrium allocation $\{c_1, c_2\}$ is unique;
3. the initial price, $p(0)$, and the nominal equivalent martingale measure, Q, are indeterminate: for any $p(0) > 0$ and for any uniformly bounded process κ, any prices and portfolio $\{p, M, B, W\}$ satisfying

$$p(t) = \frac{p(0)R(t)}{\eta(t)}\frac{e^{-\rho t}u_2\big[c_1(t), c_2(t)\big]}{u_2\big[c_1(0), c_2(0)\big]},$$

$$M(t) \geq p(t)c_1(t), \qquad \text{(equality if } r(t) > 0\text{)},$$

(11), and $B(t) = W(t) - M(t)$, support the allocation $\{c_1, c_2\}$, where η is defined by κ as in (13).

Proof: Let $p(0) > 0$ be an arbitrary positive scalar and κ be any uniformly bounded process. Define η and Q as in (13) and (14), respectively. A standard Brownian motion under Q, Z^Q, is

$$dZ_t^Q = dZ_t - \kappa(t)\,dt.$$

Given η and R, let the cum-dividend asset price process, S, be

$$S(t) = \frac{R(t)}{\eta(t)} E\left[\int_t^T \frac{\eta(s)}{R(s)}\,dD(s)\,\Big|\,\mathcal{F}_t\right], \qquad t \in [0,T). \tag{21}$$

We first see that given κ, if S is defined this way, equation (12) does hold. For this, consider the deflated processes:

$$S^R(t) = \frac{S(t)}{R(t)}, \quad D^R(t) = \int_0^t \frac{1}{R(s)}\,dD(s), \quad G^R(t) = S^R(t) + D^R(t).$$

Then G^R is a martingale under Q:

$$G^R(t) = E^Q\big[G^R(T)\,\big|\,\mathcal{F}_t\big] = E^Q\big[D^R(T)\,\big|\,\mathcal{F}_t\big].$$

Let $dG = \mu_G(t)\,dt + \sigma_G(t)\,dZ(t)$ be the stochastic differential representation of G. Since G^R is a martingale under Q, we can write

$$\begin{aligned} dG^R(t) &= \sigma_G(t)\,dZ^Q(t) \\ &= -\sigma_G(t)\kappa(t)\,dt + \sigma_G(t)\,dZ(t) \\ &= \big[\mu_G(t) - r(t)S(t)\big]\,dt + \sigma_G(t)\,dZ(t). \end{aligned}$$

Thus, it is indeed true that for an arbitrary κ if S is defined as in (21), then μ_G, σ_G, S, and r satisfy

$$\kappa(t) = -\sigma_G(t)^{-1}\big[\mu_G(t) - r(t)S(t)\big],$$

where the invertibility of $\sigma(t)$ is guaranteed by Assumption 3.

For each t, let $c_1(t)$ be a solution to

$$\frac{u_1\big[c_1(t), e(t) - c_1(t)\big]}{u_2\big[c_1(t), e(t) - c_1(t)\big]} = 1 + r(t).$$

It uniquely exists under our assumptions. Let $c_2(t) = e(t) - c_1(t)$. Both c_1 and c_2 are bounded away from zero under our assumptions. Let $p(t)$ be

$$p(t) = \frac{p(0)}{q(t)} \frac{e^{-\rho t} u_2\big[c_1(t), c_2(t)\big]}{u_2\big[c_1(0), c_2(0)\big]}.$$

Note that since c_1 and c_2 are bounded away from zero, $q(t)p(t)$ is a bounded process, so that the integral in (16) is well defined.

Equilibrium portfolio is constructed as follows. Let $\theta(t) = 0$, all t; $w(t) = W(t)$ be the solution to (11); $m(t) = M(t) = p(t)c_1(t)$ whenever $r(t) > 0$. When $r(t) = 0$, let $m(t) = M(t)$ be any number above $p(t)c_1(t)$. Let $b(t) = B(t) = W(t) - M(t)$. It is clear that $\{c_1, c_2, m, b, \theta\}$ constructed this way is indeed a budget feasible plan that satisfies (7)-(8). Thus, it is a utility maximizing plan.

□

4 Conclusion

Here, the indeterminacy nominal. It would be real if monetary policy set a path of money supplies; if it followed a Taylor rule; if prices were sticky; or if markets were incomplete.

References

K. J. Arrow. Le rôle des valeurs boursières pour la repartition la meilleure des risques. *Économétrie, Colloques Internationaux du CNRS*, 11:41–47, 1953. Translated as The role of securities in the optimal allocatin of risk bearing. Review of Economic Studies, 31: 91-96, 1964.

Y. Balasko and D. Cass. The structure of financial equilibrium i: exogenous yields and unrestricted participation. *Econometrica*, 57:135–162, 1989.

J. Benhabib, S. Schmitt-Grohe, and M. Uribe. Monetary policy and multiple equilibria. *American Economic Review*, 91:167–186, 2001.

J. Benhabib, S. Schmitt-Grohe, and M. Uribe. Avoiding liquidity traps. *Journal of Political Economy*, 110:535–563, 2002.

G. Bloise, J. H. Drèze, and H. Polemarchakis. Monetary equilibria over and infinite horizon. *Economic Theory*, 25:51–74, 2005. URL http://www.polemarchakis.org/a67-mei.pdf.

W. H. Buiter. The fiscal theory of the price level: a critique. *The Economic Journal*, 112:459–480, 2002.

D. Cass. On the 'number' of equilibrium allocations with financial assets. Unpublished manuscript, 1985.

J. H. Cochrane. Long-term debt and optimal policy in the fiscal theory of the price level. *Econometrica*, 69:69–116, 2001.

P. Dubey and J. Geanakoplos. From nash to walras via shapley and shubik. *Journal of Mathematical Economics*, 39, 2003a.

P. Dubey and J.D. Geanakoplos. Inside-outside money, gains to trade and is-lm. *Economic Theory*, 21:347–397, 2003b.

J. D. Geanakoplos and A. Mas-Colell. Real indeterminacy with financial assets. *Journal of Economic Theory*, 47:22–38, 1989.

R. E. Lucas and N. L. Stokey. Money and rates of interest in a cash-in-advance economy. *Econometrica*, 55:491–513, 1987.

M. McMahon, M. U. Peiris, and H. Polemarchakis. Perils of unconventional monetary policy. Unpublished manuscript, 2015. URL http://www.polemarchakis.org/u04-pum.pdf.

T. Nakajima and H. Polemarchakis. Money and prices under uncertainty. *Review of Economic Studies*, 72:223–246, 2005. URL http://www.polemarchakis.org/a66-mpu.pdf.

J. B. Taylor. Diecretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy*, 39:195–214, 1993.

M. Woodford. Monetariy policy and price level determinacy in a cash-in-advance economy. *Economic Theory*, 4:345–380, 1994.

M. Woodford. Price-level determination without control of a monetary aggregate. *Carnegie-Rochester Conference Series on Public Policy*, 43:1–46, 1995.