Identification of ambiguity [1]

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Abstract

Individuals behave differently when they know the objective probability of events and when they do not. The smooth ambiguity model accommodates both ambiguity (uncertainty) and risk. We consider an individual who trades financial assets to maximize a smooth ambiguity utility over two dates. For an incomplete, competitive asset market, we give sufficient conditions for consumption and asset demand functions generated by smooth ambiguity preferences to identify the ambiguity and risk indices as well as the ambiguity probability measure. An ambiguity free asset plays an important role for identification.

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Ambiguity preferences distinguish between uncertainty, where an individual cannot assign unambiguous probabilities to specific events, and risk, where such an assignment is possible[1]. Indeed, over the years following the critical contributions of Ellsberg (1961) in response to von Neumann and Morgenstern (1947) and Savage (1954), laboratory data have demonstrated that choices of individuals often do not conform to expected utility that does not distinguish between risk and uncertainty[2], and there has been continued interest in experimental tests that focus on this and related questions. Well known examples are Slovic and Tversky (1974), Einhorn and Hogarth (1986), Halevy (2007) and, with focus on ambiguity aversion, Trautmann and van der Kuillen (2015).

Even though the vast majority of studies of attitudes towards risk and uncertainty have considered lottery experiments, an alternative empirical approach considers an asset market rather than a lottery setting. Two important applications of this approach to ambiguity preferences are Bayer, Bose, Polisson, and Renou (2013) and Ahn, Choi, Gale, and Kariv (2014). In the former, the authors derive testable inequality conditions for the data to be consistent with ambiguity preferences. In the latter, ambiguity preference parameters are estimated under different parametric specifications using portfolio choice data from a laboratory experiment. Both papers were confined to the case of complete asset markets.

We assume that the consumption and asset demand functions are observable and generated by two-period smooth ambiguity preferences, and we derive sufficient conditions for the identification of an individual's distinct preferences over uncertainty and risk as well as the individual's subjective ambiguity probability beliefs. Importantly, the argument for identification applies to incomplete asset markets

A number of alternative models distinguish between uncertainty and risk: the seminal formulation of multiple priors and maxmin preferences of Gilboa and Schmeidler (1989), multiplier preferences of Anderson, Hansen, and Sargent (2003) , variational preferences of Maccheroni, Marinacci, and Rustichini (2006) and uncertainty averse preferences of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011). We focus on the model of smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005) or Seo (2009) for several reasons. As the authors note, the model (i) achieves a separation of ambiguity as characterized by their uncertainty beliefs and their aversion to uncertainty, and it (ii) generates smooth indifference curves, rather than

[1] Ghirardato (2004), p. 36.

[2] Camerer and Weber (1992) and Attanasi, Gollier, Montesano, and Pace (2014) and references cited there.

kinked indifference curves that may obfuscate the argument. In addition, the approach applies to first- and second-order distributions and, as a result, and can readily relate the analysis to the familiar expected utility case. Finally, the smooth ambiguity model has been used in important asset demand analysis, such as Gollier (2011). Mukerji and Tallon (2001) argue that competitive markets in which investors maximize ambiguity preferences display properties that are both empirically relevant and excluded by expected utility[3].

When conditional probability distributions are indexed by some unknown parameter, then the ambiguity state is the value of the parameter. Ju and Miao (2012) estimate a hidden Markov regime-switching process of US consumption and equity dividends, and identify two ambiguity states, a boom state and recession state, conditional on which equity returns have a lognormal distribution. In the robustness theory developed by Hansen and Sargent (2001) and Hansen (2007), where model misspecification is a concern, ambiguity states are possible models that give rise to different probability distribution. Chen, Ju, and Miao (2014) estimate two statistical models of stock returns in a portfolio allocation application.

The identification of fundamentals from observable market data can be posed, most simply, in the context of certainty. There, Mas-Colell (1977) demonstrates that the demand function identifies the preferences of the consumer. Importantly, the argument for identification is local: if prices are restricted to an open neighborhood, they identify fundamentals in an associated neighborhood. Evidently, the arguments extend to economies under pure risk, but with a complete system of markets in elementary securities. Identification becomes problematic, and more interesting, when the set of observations is restricted. Under pure risk, this arises when the asset market is incomplete and the payoffs to investors are restricted to a subspace of possible payoffs. Nevertheless, Green, Lau, and Polemarchakis (1979), Dybvig and Polemarchakis (1981) and Geanakoplos and Polemarchakis (1990) demonstrate that identification is possible as long as the utility function has an expected utility representation with a state-independent cardinal utility index, and the distribution of asset payoffs is known. Polemarchakis (1983) extends the argument to the joint identification of tastes and beliefs. The argument relies crucially on the presence of a risk free asset and, more importantly, does not allow risk due to future endowments[4].

[3]The smooth ambiguity model has not been without controversy – Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012).

[4]Kübler and Polemarchakis (2017) derive conditions that guarantee identification with no knowledge either of the cardinal utility index (attitudes towards risk) or of the distribution of future endowments or payoffs of assets. The argument applies even if the asset

Under the assumption that consumption and asset demands are indeed consistent with smooth ambiguity preferences, we derive sufficient conditions such that the uncertainty and risk indices as well as subjective ambiguity beliefs can be identified[5]. One key innovation in the extension of prior results under pure risk is the introduction of an ambiguity free asset, with payoff distributions that coincide across ambiguity states.

The consumption-portfolio indifference correspondence is an alternative to consumption and asset demand for identification.

1 Setup

There are two dates: date 0 and date 1, and uncertainty at date 1 is represented by states of the world. States of the world are $\omega \in \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is a finite set and has the following product structure: $\boldsymbol{\Omega} = \boldsymbol{A} \times \boldsymbol{S}$, where $a \in \boldsymbol{A}$ are ambiguity states, and $s \in \boldsymbol{S}$ are risk states. $\boldsymbol{\Omega}$ can be interpreted as a set of possible outcomes of two-stage lotteries. In this case, elements in $\boldsymbol{A}$ and $\boldsymbol{S}$ are, respectively, outcomes of first and second stage lotteries[6]. We assume it is risk states contingent on which asset payoff and consumption occur[7]. A probability measure on the set of states of the world, $\boldsymbol{\pi} \in \Delta(\boldsymbol{\Omega})$, can be expressed as $\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$, where $\boldsymbol{\mu} \in \Delta(\mathbf{A})$ is a probability measure over states of uncertainty, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is a family of conditional probability measures over states of risk, and $\pi_{as} = \mu_a \nu_{as}$.

A distribution of wealth across risk states at date 1 is

$$\boldsymbol{x} = (..., x_s, ...) \in \mathbb{R}^S_+.$$

A utility function over date 0 consumption and distributions of date 1 wealth is

$$U(x_0; \boldsymbol{x}) : \mathbb{R}^{S+1}_+ \to \mathbb{R}$$

that is twice differentiable, strictly monotonically increasing and strictly quasi-concave in x_0 and $\boldsymbol{x}$, and satisfies a boundary condition: the closure of the indifference curve through any strictly positive distribution is contained

market is incomplete and demand is observed only locally.

[5]In an an analogous vain, Polemarchakis and Selden (1984) identify the risk and time index in the context of Ordinal Certainty Equivalent (OCE) preferences

[6]Segal (1990) gives arguments for the potential superiority of the Anscombe and Aumann (1963) over the Savage (1954) setup in the analysis of ambiguity attitudes.

[7]If asset payoff and consumption are contingent on both ambiguity states and risk states, results here go through with some notational change. The assumption made here makes the argument stronger, and it is consistent with observations in practice.

in the strictly positive orthant or

$$(\bar{x}_0; \bar{\boldsymbol{x}}) \in \mathbb{R}^{S+1}_{++} \;\Rightarrow\; Cl\;\{(x_0; \boldsymbol{x}) : U(x_0; \boldsymbol{x}) = U(\bar{x}_0; \bar{\boldsymbol{x}})\} \subseteq \mathbb{R}^{S+1}_{++}.$$

In a static setting, Klibanoff, Marinacci, and Mukerji (2005) provide a set of axioms that are necessary and sufficient for the existence of a risk index, an ambiguity index, and a probability measure,

$$u : \mathbb{R}_{++} \to \mathbb{R}, \quad \tilde{\phi} : u(\mathbb{R}_{++}) \to \mathbb{R} \;\text{ and }\; \boldsymbol{\mu},$$

respectively, such that

$$U(\boldsymbol{x}) = E_{\boldsymbol{\mu}} \tilde{\phi}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big) \tag{1}$$

represents ambiguity preferences[8]. Alternatively, if

$$\phi = \widetilde{\phi} \circ u, \quad \phi : \mathbb{R}_{++} \to \mathbb{R},$$

then (1) takes the form

$$U(\boldsymbol{x}) = E_{\boldsymbol{\mu}} \phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\Big). \tag{2}$$

For the representation (1), a positive affine transformation of the risk index u does not change preferences if and only if a compensating transformation is applied to $\tilde{\phi}$. In contrast, the preferences corresponding to (2) are invariant to a positive affine transformation of the risk index[9]. Under the formulation (1), an individual is strictly ambiguity averse if $\tilde{\phi}$ is strictly concave (alternatively, in (2), ϕ is more concave than u), and ambiguity neutral if $\tilde{\phi}$ is linear (alternatively, in (2), ϕ is as concave as u). As argued in Selden and Wei (2014), for (2), an individual is strictly ambiguity averse if ϕ is strictly concave, and ambiguity neutral if ϕ is linear. This difference is a matter of interpretation, since clearly $\phi = \tilde{\phi} \circ u$ establishes the equivalence of the formulations[10].

[8] As defined earlier, index a in the representation refers to a typical ambiguity state, the probability measure μ is over ambiguity states, and $\boldsymbol{\nu}_a$ is the probability measure conditional on each ambiguity state a.

[9] This point is discussed in Klibanoff, Marinacci, and Mukerji (2005), p. 1858.

[10] As in Example 1 in Selden and Wei (2014), suppose we interpret $\widetilde{\phi}$ and u in (1), respectively, as the ambiguity and risk indices. Consider a specific lottery with no risk and only uncertainty. Then increasing the decision maker's risk aversion produces the counter intuitive result that the certainty equivalent of the lottery decreases. A considerably more intuitive conclusion is reached if, alternatively, we follow the suggestion of Selden and Wei (2014) to use the representation (2) and interpret ϕ and u, respectively, as the ambiguity and risk indices. Then, increasing the concavity of the risk index has no impact on ϕ and the certainty equivalent of the lottery, referenced above, does not change.

When considering a two-period model, we write equation (2) as

$$U(x_0;\boldsymbol{x}) = \phi(x_0) + \beta E_{\boldsymbol{\mu}}\phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(x_s)\big)\Big). \tag{3}$$

Baillon (2017) and Kajii and Xue (2016) use a two-period smooth ambiguity model based on equation (1):

$$U(x_0;\boldsymbol{x}) = \tilde{\phi}\big(u(x_0)\big) + \beta E_{\boldsymbol{\mu}}\tilde{\phi}\big(E_{\boldsymbol{\nu}_a}u(x_s)\big).$$

Clearly, if we define $\phi = \tilde{\phi}\circ u$, this functional form becomes equation (3)[11].

We assume that the objective probabilities $\boldsymbol{\nu}$ are observable. This is reasonable since we can allow for arbitrarily many conditional probabilities over risk states as in Klibanoff, Marinacci, and Mukerji (2005). In the experiments of Ellsberg (1961) or Ahn, Choi, Gale, and Kariv (2014), for each ambiguity state, that is, for each possible composition of color balls, the conditional probabilities $\boldsymbol{\nu}_a$ are objectively known to the subjects. In the asset pricing and portfolio allocation applications, the conditional distribution of stock returns are estimated by Ju and Miao (2012) and Chen, Ju, and Miao (2014) respectively. The assumption that objective conditional probability distributions $\boldsymbol{\nu}_a$ are known or observed is allowed for in the asset setting of Varian (1983) and the incomplete market demand tests in Kübler, Selden, and Wei (2020). Last but not least, as we arguec in *Remarck 3,* this is without loss of (much) generality.

Observability of the probability measure $\boldsymbol{\mu}$ is not required. In the two-stage lotteries framework of Anscombe and Aumann (1963), the probability measure $\boldsymbol{\mu}$, is subjective and, as a consequence, unlikely to be observable. However, if the domain of preferences consists of compound objective lotteries, then, observability of the probability measure $\boldsymbol{\mu}$ is not an unreasonable assumption. In financial economic applications, it is important to allow for μ that is not observable.

At date 0, the consumption good and financial assets are traded. Assets are $j \in \boldsymbol{J}$. Payoffs of asset j at date 1 defined across risk states are

$$\boldsymbol{r}_j = (..., r_{sj}, ...)',$$

[11] Klibanoff, Marinacci, and Mukerji (2009) propose a recursive smooth ambiguity model,

$$U(x_0;\boldsymbol{x}) = u(x_0) + \beta\tilde{\phi}^{-1}\Big(E_{\boldsymbol{\mu}}\tilde{\phi}\big(E_{\boldsymbol{\nu}_a}u(x_s)\big)\Big).$$

This functional form is not equivalent to our preference representation (3).

a column vector, conditional on risk state s. Payoffs of the set of assets are $\mathbf{R}_s = (..., r_{sj}, ...)$, a row vector, and the matrix of asset is

$$\boldsymbol{R} = (..., \boldsymbol{r}_j, ...) = (..., \boldsymbol{R}_s, ...)'$$

that has full column rank or, equivalently, payoffs of assets, $\{\boldsymbol{r}_j\}$, are linearly independent[12].

A portfolio of assets is $\boldsymbol{y} = (..., y_j, ...)$ and it generates the distributions of wealth across risk states $\boldsymbol{x} = \boldsymbol{Ry}$. The set of portfolios that generate strictly positive $\boldsymbol{x}$ is non-empty,

$$\boldsymbol{Y} = \{\boldsymbol{y} : \boldsymbol{Ry} \gg \mathbf{0}\} \neq \emptyset,$$

that is open. The domain of asset prices not allowing for arbitrage is

$$\boldsymbol{P} = \{\boldsymbol{p} : \boldsymbol{Ry} > 0 \Rightarrow \boldsymbol{py} > 0\} = \{\boldsymbol{p} = \boldsymbol{qR}, \boldsymbol{q} \gg \mathbf{0}\}.$$

Given the consumption price p_0, asset price vector $\boldsymbol{p} = (p_1, ..., p_J)$ and conditional probability measures $\boldsymbol{\nu}$, the optimization problem of the individual is

$$\max_{x_0 > 0, \boldsymbol{y} \in \boldsymbol{Y}} U(x_0, \boldsymbol{Ry}), \quad s.t. \;\; p_0 \cdot x_0 + \boldsymbol{p} \cdot \boldsymbol{y} \leq 1. \tag{4}$$

A solution to the optimization problem, $x_0(p_0, \boldsymbol{p})$ and $\boldsymbol{y}(p_0, \boldsymbol{p})$, exists, satisfies $x_0(p_0, \boldsymbol{p}) > 0$ and $\boldsymbol{Ry}(p_0, \boldsymbol{p}) \gg 0$, and is unique. It defines the demand function for consumption and assets,

$$x_0 : (p_0, \boldsymbol{p}) \to \mathbb{R}_+, \quad \text{and} \quad \boldsymbol{y} : (p_0, \boldsymbol{p}) \to \boldsymbol{Y}.$$

Importantly, the demand function is invertible.

2 Identification

We address the following question: Suppose that data based on consumption and asset demand functions is consistent with the existence of two-period smooth ambiguity preferences (3). Can the underlying ambiguity and risk indices as well as the ambiguity probability measure be identified?

The demand for consumption and assets satisfies the necessary and sufficient first order conditions for the optimization problem (4),

$$\begin{gathered} \boldsymbol{D}U(x_0, \boldsymbol{Ry}) = \lambda(p_0, \boldsymbol{p}), \;\; \lambda > 0, \\ p_0 x_0 + \boldsymbol{py} = 1. \end{gathered} \tag{5}$$

[12]For notational ease, we assume asset payoffs do not change across observations. However, all our results hold if asset payoffs vary, but are observable.

These conditions generate the family of marginal rates of substitution of consumption and assets[13],

$$m_j : (\mathbb{R}_{++}, \boldsymbol{Y}) \to (0, \infty),$$

defined by

$$m_{j0}(x_0, \boldsymbol{y}) =$$

$$\frac{\frac{\partial U(x_0, \boldsymbol{Ry})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{Ry})}{\partial x_0}} = \frac{\beta E_{\boldsymbol{\mu}} \phi'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}))) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{Ry}) \boldsymbol{r}_j}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))}}{\phi'(x_0)} > 0 \tag{6}$$

and

$$m_{jk}(x_0, \boldsymbol{y}) =$$

$$\frac{\frac{\partial U(x_0, \boldsymbol{Ry})}{\partial y_j}}{\frac{\partial U(x_0, \boldsymbol{Ry})}{\partial y_k}} = \frac{E_{\boldsymbol{\mu}} \phi'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}))) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{Ry}) \boldsymbol{r}_j}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))}}{E_{\boldsymbol{\mu}} \phi'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}))) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{Ry}) \boldsymbol{r}_k}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))}} > 0, \tag{7}$$

where $\boldsymbol{\mu}$ is the probability measure over ambiguity states, and $\boldsymbol{\nu}_a$ is the probability measure conditional on each ambiguity state associated with the distribution of returns for each asset.

Before presenting the identification result, we give two simple examples in which the identification of $\boldsymbol{\mu}$ fails even if the risk index u, the ambiguity index ϕ and the discount factor β are known. Note that we are concerned with the more demanding case: the joint identification of u, ϕ, β and $\boldsymbol{\mu}$.

Example 1. *Suppose $\phi(x) = x^{\frac{1}{3}}$, $u(x) = \ln x$, $\beta = 1$, there is only one ambiguous asset, and its payoff satisfies $\sum_s \pi_{1s} \ln r_s = \sum_s \pi_{2s} \ln r_s$.*

The investor solves the maximization problem

$$\max_{x_0, y} \ (x_0)^{\frac{1}{3}} + \sum_a \mu_a \left(e^{\sum_s \pi_{as} \ln(r_s y)}\right)^{\frac{1}{3}},$$

$$s.t. \ p_0 x_0 + py = 1.$$

The first order condition of (6) characterizes demand for the asset,

$$\sum_a \mu_a \left(e^{\sum_s \pi_{as} \ln r_s + \ln y}\right)^{\frac{1}{3}} = \frac{py}{(x_0)^{\frac{2}{3}} p_0 x_0}.$$

[13] Since we assume demand is invertible, we can take as given either the demand for consumption and assets or, alternatively, the associated marginal rates of substitution.

Since $\sum_s \pi_{1s} \ln r_s = \sum_s \pi_{2s} \ln r_s$*,* μ_1 *and* μ_2 *cannot be separately identified.*

NB *Note that the functional form of the ambiguity index is immaterial.*

Example 2. *Suppose* $\phi(x) = x^{\frac{1}{3}}$*,* $u(x) = \ln x$*,* $\beta = 1$*, there are two ambiguous assets with* $r_{11} = r_{21} = r_{12} = r_{22}$*,* $\pi_{11} + \pi_{12} = \pi_{21} + \pi_{22}$*,* $\pi_{1s} = \pi_{2s}$ *for* $s = 3, ..., S$*.*

The investor solves the maximization problem

$$\max_{x_0, y_1, y_2} \ (x_0)^{\frac{1}{3}} + \sum_a \mu_a \left(e^{\sum_s \pi_{as} \ln(r_{1s} y_1 + r_{2s} y_2)}\right)^{\frac{1}{3}},$$

$$s.t. \ p_0 x_0 + p_1 y_1 + p_2 y_2 = 1.$$

First order conditions of (6) characterize demand for the asset,

$$\sum_a \mu_a e^{(\pi_{a1}+\pi_{a2}) \ln r_{11}(y_1+y_2) + \sum_{s=3}^S \pi_{as} \ln(r_{1s}y_1 + r_{2s}y_2)} \left((\pi_{a1} + \pi_{a2}) \frac{1}{y_1+y_2} + \sum_{s=3}^S \pi_{as} \frac{r_{1s}}{r_{1s}y_1 + r_{2s}y_2}\right) \left(e^{(\pi_{a1}+\pi_{a2}) \ln r_{11}(y_1+y_2) + \sum_{s=3}^S \pi_{as} \ln(r_{1s}y_1 + r_{2s}y_2)}\right)^{-\frac{2}{3}} = \frac{p_1}{(x_0)^{\frac{2}{3}} p_0 x_0},$$

and

$$\sum_a \mu_a e^{(\pi_{a1}+\pi_{a2}) \ln r_{11}(y_1+y_2) + \sum_{s=3}^S \pi_{as} \ln(r_{1s}y_1 + r_{2s}y_2)} \left((\pi_{a1} + \pi_{a2}) \frac{1}{y_1+y_2} + \sum_{s=3}^S \pi_{as} \frac{r_{2s}}{r_{1s}y_1 + r_{2s}y_2}\right) \left(e^{(\pi_{a1}+\pi_{a2}) \ln r_{11}(y_1+y_2) + \sum_{s=3}^S \pi_{as} \ln(r_{1s}y_1 + r_{2s}y_2)}\right)^{-\frac{2}{3}} = \frac{p_2}{(x_0)^{\frac{2}{3}} p_0 x_0}.$$

Since $\pi_{11} + \pi_{12} = \pi_{21} + \pi_{22}$*,* $\pi_{1s} = \pi_{2s}$ *for* $s = 3, ..., S$*,* μ_1 *and* μ_2 *cannot be separately identified.*

NB *Note that the functional forms of the ambiguity or the risk index is immaterial. It is the distribution of asset payoffs that drives the argument.*

To obtain positive identification result, we introduce one risk free asset and one ambiguity free asset.

Definition 1. *An asset is* risk free *if it generates the same payoff across risk states.*

Note that in Example 2, even if one asset is risk free, the identification of $\boldsymbol{\mu}$ is still not possible.

Definition 2. *An asset is* ambiguity free *if, conditional on each ambiguity state, it generates the same distribution of returns.*

Example 3. *There are 3 risk states and 2 ambiguity states. The probability distributions conditional on ambiguity states are* $(\frac{1}{2}, 0, \frac{1}{2})$ *and* $(\frac{1}{2}, \frac{1}{2}, 0)$. *An asset that pays* $(1, 1, 1)$ *across risk states is risk free. An asset that pays* $(1, a, a)$ *across risk states is ambiguity free, even if* $a \neq 1$ *and the asset is not risk free.*

The introduction of an ambiguity free asset makes identification possible because a portfolio that involves only the risk free asset and the ambiguity free asset allows for a clear distinction between risk aversion and ambiguity aversion. Since the existence of an ambiguity free asset plays a crucial role in the identification argument, it deserves attention[14]. Evidently, for arbitrary asset payoffs and conditional probabilities, an ambiguity free asset need not exist. Being ambiguity free is a joint restriction on asset payoffs $\boldsymbol{r} = (r_1, ..., r_s, ..., r_S)$[15] and the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ where $\boldsymbol{\nu}_a = (\nu_{a1}, ..., \nu_{as}, ..., \nu_{aS})$. One extreme case is a risk free asset, with $r_s = r_{s'}$ for all s and s', that is ambiguity free independently of conditional probabilities[16]. At the other extreme, an asset with $r_s \neq r_{s'}$ for any s and s' cannot be ambiguity free for any conditional probabilities. To understand ambiguity free asset returns, we partition the set of risk states $S = \{1, ..., s, ..., S\}$ into N disjoint subsets $\{S^n\}_{n=1}^{N}$, such that

$$S^n = \{s, s' \in S : r_s = r_{s'}\}.$$

That is, S^n is a subset of risk states on which the payoffs of the asset coincide. For the three risk states in Example 3, we partition them into two subsets: $S^1 = \{s = 1\}$, $S^2 = \{s = 2, s = 3\}$.

Remark 1. *An asset with payoff* $r = (r_1, ..., r_s, ..., r_S)$ *is ambiguity free under conditional probability distributions* $\{\nu_a\}_{a=1}^{A}$ *if and only if* $\sum_{s \in S^n} \nu_{as} = \sum_{s \in S^n} \nu_{a's}$ *for all* n, a *and* a'.

[14] An ambiguity free asset appears in Klibanoff, Marinacci, and Mukerji (2005) (p.1876), where the effect of ambiguity and risk attitudes on portfolio choice is examined numerically. Note, however, that, different from our assumption, asset payoffs in their example depend on both ambiguity and risk states, which is atypical in the financial economics literature.

[15] For the analysis of an ambiguity free asset, we consider a single asset and omit its index.

[16] In the remainder of this paper when we refer to an ambiguity free asset, we will mean it is ambiguity free, but, risky, even though we do not emphasize the latter property.

The argument is immediate.

To state the identification theorem, define the $(n-1)$ dimensional unit sphere $\Lambda^{n-1} = \{\alpha \in R^n : \sum_{i=1}^n \alpha_i^2 = 1\}$.

Definition 3. *Functions* $\{f_i\}_{i=1}^n$ *with* $f_i : \mathfrak{D} \subset \mathbb{R}^m \to \mathbb{R}$ *are* linearly independent *if there does not exist* $\alpha \in \Lambda^{n-1}$, *such that* $\sum_{i=1}^n \alpha_i f_i(x) = 0$ *for all* $x \in \mathfrak{D}$.

For identification, we use a result from Kübler and Polemarchakis (2017).

Lemma 1. *If functions* $f_1,\ldots,f_n$ *are linearly independent on a set* $\mathfrak{D}$, *then there must exist finitely many points* $x_1,\ldots,x_N \in \mathfrak{D}$, *such that, for no* $\alpha \in \Lambda^{n-1}$, *is* $\sum_{i=1}^n \alpha_i f_i(x_j) = 0$ *for all* $j = 1, ..., N$.

We assume that

1. the smooth ambiguity utility (3) satisfies the condition that $\phi\left(u^{-1}(\cdot)\right)$ is strictly concave on $\mathbb{R}$, with the indices u and ϕ being twice differentiable, strictly increasing, and strictly concave on $\mathbb{R}_{++}$,
2. there is an asset, $j = 1$ that is risk free: $\boldsymbol{r}_1 = 1$ across states of the world,
3. there is an asset, $j = 2$ that is ambiguity free: its payoff distribution is invariant to the states of ambiguity, and
4. the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \Delta(\boldsymbol{S})$ is known.

Theorem 1. *If the functions*

$$\{f_a(\boldsymbol{y}) = \phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y}))) \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))}\}_{a=1}^A$$

are linearly independent on $\boldsymbol{Y}$, *then, the demand for consumption assets identifies the risk index* u *and the ambiguity index* ϕ *on* $\mathbb{R}_{++}$, *each up to a positive affine transformation, the discount factor* β, *as well as the ambiguity state probability measure* $\boldsymbol{\mu}$.

Proof. We argue in a series of steps. *Step 1—identification of the discount factor* β. From the marginal rate of substitution between consumption and risk free asset in equation (6),

$$\phi^{'}(x_0) = \beta E_{\boldsymbol{\mu}}\phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{Ry})\boldsymbol{r}_1}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))}\frac{1}{m_{10}(x_0,\boldsymbol{y})}. \quad (8)$$

At $x_0 = \overline{x}_0$ and portfolio $\overline{y} = (\overline{x}_0, 0, ..., 0)$

$$\beta = m_{10}(\overline{x}_0, \overline{\boldsymbol{y}}). \quad (9)$$

Step 2—identification of the ambiguity index ϕ. From equation (8), if normalizing $\phi^{'}(\overline{x}_0) = 1$,

$$\phi^{'}(x_0) = \frac{m_{10}(\overline{x}_0, \boldsymbol{y})}{m_{10}(x_0, \boldsymbol{y})}, \quad (10)$$

which identifies the ambiguity index ϕ on $\mathbb{R}_{++}$ up to a positive affine transformation. *Step 3—identification of the risk index u.* We restrict attention to the portfolios $\boldsymbol{y} = (y_1, y_2, 0, ..., 0)$, and we let $\tilde{\boldsymbol{y}} = (y_1, y_2)$ be the associated truncated portfolio. Since the distribution of payoffs for assets 1 and 2 is invariant across states of ambiguity, there exist a probability measure, $\tilde{\boldsymbol{\nu}} \in \Delta(\boldsymbol{S})$, and a matrix of payoffs of assets over states of risk $\tilde{\boldsymbol{R}} = (\boldsymbol{1}_{\#\boldsymbol{S}}, \tilde{\boldsymbol{r}}_2)$,[17] such that, the distribution of payoffs of assets generated by $(\boldsymbol{\nu}_a, \boldsymbol{Ry})$, for any state of ambiguity, coincides with the distribution generated by $(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})$. As a consequence, from equation (7),

$$m_{12}(\tilde{\boldsymbol{y}}; \tilde{\boldsymbol{\nu}}) = \frac{E_{\tilde{\boldsymbol{\nu}}}u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})}{E_{\tilde{\boldsymbol{\nu}}}u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})\tilde{\boldsymbol{r}}_2} > 0. \quad (11)$$

Identification of the cardinal risk index u on $\mathbb{R}_{++}$, then follows as under pure risk in Dybvig and Polemarchakis (1981). *Step 4—identification of the probability measure $\boldsymbol{\mu}$.* From the marginal rate of substitution between consumption and ambiguous asset j in equation (6),

$$\beta E_{\boldsymbol{\mu}}\phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{Ry})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))} = m_{j0}(x_0, \boldsymbol{y})\phi^{'}(x_0). \quad (12)$$

We define

$$f_a(\boldsymbol{y}) = \phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{Ry})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})))}.$$

[17] $\boldsymbol{1}_{\#\boldsymbol{S}}$ is the vector of 1's of dimension $\#\boldsymbol{S}$, the cardinality of $\boldsymbol{S}$.

Equation (12) can be rewritten as

$$\sum_{a=1}^{A} \beta \mu_a f_a(\boldsymbol{y}) = m_{j0}(x_0, \boldsymbol{y}) \phi^{'}(x_0). \tag{13}$$

By Lemma 1, if the functions $\{f_a\}_{a=1}^{A}$ are linearly independent, we can find a positive integer N and points $\boldsymbol{y}_1, ..., \boldsymbol{y}_N$, such that the system of equations

$$\sum_{a=1}^{A} \beta \alpha_a f_a(\boldsymbol{y}_i) = 0, \quad i = 1, ..., N$$

has no solution with $\alpha \neq 0$. Since the first-order conditions (13) holds on the open set $\boldsymbol{Y}$, we can find $\{(x_{0i}, \boldsymbol{p}_i)\}_{i=1}^{N}$, such that

$$\sum_{a=1}^{A} \beta \mu_a f_a(\boldsymbol{y_i}) = m_{j0}(x_{0i}, \boldsymbol{y_i}) \phi^{'}(x_{0i}). \tag{14}$$

This is a linear system in $\{\mu_a\}_{a=1}^{A}$, and it must have a unique solution, which identifies the probability measure $\boldsymbol{\mu}$. □

Remark 2. *The functions $\{f_a(\boldsymbol{y})\}_{a=1}^{A}$ involve ϕ and u. Given both ϕ and u have been identified, and the conditional distribution of the asset payoffs is known, the linear independence of $\{f_a(\boldsymbol{y})\}_{a=1}^{A}$ can be directly checked. Actually, since both ϕ and u are identified from consumption and asset demand, the linear independence of functions $\{f_a(\boldsymbol{y})\}_{a=1}^{A}$ can be equivalently defined in terms of observable consumption and asset demand and conditional probability distribution.*

Remark 3. *The identification in Theorem 1 assumes observation of the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$. This is not essantial since we can allow for arbitrarily many conditional probabilities. Once $\boldsymbol{\mu}$ is identified, the subset of $\{\boldsymbol{\nu}_a\}_{a=1}^{A}$ which is subjectively relevant is also identified: it is the subset that consists of the conditional probability distributions assigned nonzero probability by $\boldsymbol{\mu}$.*

Remark 4. *Although the continuum case is excluded, the number of ambiguity states can be arbitrary, and can be more or less than the number of risk states. For the identification, only one ambiguous asset is needed, and the markets can be very incomplete.*

Remark 5. *As under pure risk in Dybvig and Polemarchakis (1981), knowing the second moment of the distribution of asset payoffs that is invariant across states of ambiguity permits identification of the risk index u, as well as identification of the asset payoffs independent of the states of ambiguity.*

Remark 6. *The proposition requires the existence of a risk free asset. As under pure risk in Green, Lau, and Polemarchakis (1979), we can show that, without a risk free asset, the marginal rate of substitution between two ambiguity free assets identifies the risk index u so long as the underlying risk index u is analytic at $x = 0$. Once the risk index u is identified, the identification of the ambiguity index and the ambiguity probability measure follows the same argument as in the theorem.*

It should be noted that Theorem 1 provides conditions for there to be a unique β, u, ϕ and μ. Also, it provides algorithms for the recovery of β, u and ϕ, but not for μ. To recover μ, we need to rely on linear independence in Theorem 1 is satisfied. Linear independence in Theorem 1 is satisfied if functions are differentiably linearly independent. To determine differentiable linear independence is an easier task. Define a differential operator

$$\Delta_k = (\frac{\partial}{\partial x_1})^{j_1} \cdots (\frac{\partial}{\partial x_m})^{j_m}, \quad j_1 + ... + j_m \leq k.$$

Definition 4. *The functions $f_1, ..., f_n$ are* differentiably linearly independent *on $\mathfrak{D}$ if there is some $k \geq n-1$ and some $\bar{x} \in \mathfrak{D}$, such that each f_i is at least C^k at $\bar{x}$ and such that there are differential operators $k_1, ...k_n$, with $k_i \leq k$, for all $i = 1, ...n$, such that the matrix*

$$\boldsymbol{W} = \begin{pmatrix} \Delta_{k_1}(f_1) & \cdots & \Delta_{k_1}(f_i) & \cdots & \Delta_{k_1}(f_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{k_j}(f_1) & \cdots & \Delta_{k_j}(f_i) & \cdots & \Delta_{k_j}(f_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{k_n}(f_1) & \cdots & \Delta_{k_n}(f_i) & \cdots & \Delta_{k_n}(f_n) \end{pmatrix}$$

is nonsingular.

It is easy to see that, if $f_1, ..., f_n$ are differentiably linearly independent on $\mathfrak{D}$, they are linearly independent since differentiable linear independence implies that there cannot be an open neighborhood of $\bar{x}$ and some $\alpha \in \Lambda^{n-1}$ such that $\sum_{i=1}^{n} \alpha_i f_i(x) = 0$ for all x in the neighborhood. Although the converse is generally not true, linear independence and differentiably linear independence are equivalent if functions are analytic. In this case, we can take $\Delta_{k_i} = \Delta_{i-1}$ and the $\boldsymbol{W}$ matrix becomes the Wronskian matrix.

We show that, if the functions

$$\{f_a(\boldsymbol{y}) = \phi^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}))) \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))}\}_{a=1}^{A}$$

are differentiably linearly independent, then the probability measure $\boldsymbol{\mu}$ can be computed explicitly.

We fix one ambiguous asset j (there could be more ambiguous assets available), and let

$$w_a(\boldsymbol{y}) = u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})), \quad e_a(\boldsymbol{y}) = \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u^{'}(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})))},$$

$$w_a^{(n)}(\boldsymbol{y}) = \frac{\partial^n w_a(\boldsymbol{y})}{\partial y_j^n}, \qquad e_a^{(n)}(\boldsymbol{y}) = \frac{\partial^n e_a(\boldsymbol{y})}{\partial y_j^n},$$

$$u^{(n)}(x) = \frac{d^n u(x)}{dx^n}, \qquad \phi^{(n)}(w) = \frac{d^n \phi(w)}{dw^n},$$

$$m_{j1}^{(n)}(x_0, \boldsymbol{y}) = \frac{\partial^n m_{j1}(x_0, \boldsymbol{y})}{\partial y_j^n}.$$

For any n and ambiguity state a, let

$$f_a^n(\boldsymbol{y}) = \frac{\partial^n f_a(\boldsymbol{y})}{\partial y_j^n} = \frac{\partial^n \phi^{'}(w_a(\boldsymbol{y}))e_a(\boldsymbol{y})}{\partial y_j^n}$$

$$= \sum_{m=0}^n \binom{n}{m} \sum_{\Delta(mk)} \frac{m!}{k_1! \dots k_m!} \phi^{(k+1)}(w_a(\boldsymbol{y}))(\frac{w_a^{(1)}(\boldsymbol{y})}{1!})^{k_1} \dots \tag{15}$$

$$(\frac{w_a^{(m)}(\boldsymbol{y})}{m!})^{k_m} e_a^{(n-m)}(\boldsymbol{y}),$$

where $\sum_{\Delta(mk)}$ is the sum over all $k_1, ..., k_m$ for which $k_1+2k_2+\ldots+mk_m = m$, and $k = k_1 + k_2 + \ldots + k_m$. When $n = 0$, we define $\binom{0}{0} = 1$, and $0! = 1$. When $m = 0$, we define

$$\sum_{\Delta(mk)} \frac{m!}{k_1! \dots k_m!} \phi^{(k+1)}(w_a(\boldsymbol{y}))(\frac{w_a^{(1)}(\boldsymbol{y})}{1!})^{k_1} \dots (\frac{w_a^{(m)}(\boldsymbol{y})}{m!})^{k_m} e_a^{(n-m)}(\boldsymbol{y})$$

in equation (15) to be $\phi^{(1)}(w_a(\boldsymbol{y}))e_a^{(n)}(\boldsymbol{y})$.

Proposition 1. *Given β, u and ϕ, if the functions $\{f_a(\boldsymbol{y})\}_{a=1}^A$ are differentiably linearly independent on $\boldsymbol{Y}$, then, the probability measure $\boldsymbol{\mu}$ can be explicitly computed.*

Proof. Let j be an ambiguous asset. Under our notation, equation (6) can be rewritten as

$$E_{\boldsymbol{\mu}} \phi^{'}(w_a(\boldsymbol{y}))e_a(\boldsymbol{y}) = \frac{1}{\beta} m_{j0}(x_0, \boldsymbol{y})\phi^{'}(x_0). \tag{16}$$

To take successive differentiation of equation (16) with respect to y_j, we apply the Leibniz formula[18]to the product of two functions $\phi^{'}(w_a(\boldsymbol{y}))$ and $e_a(\boldsymbol{y})$ in the left hand side of equation (16), for any n, we get

$$E_{\boldsymbol{\mu}}\frac{\partial^n\phi^{'}(w_a(\boldsymbol{y}))e_a(\boldsymbol{y})}{\partial y_j^n} = E_{\boldsymbol{\mu}}\sum_{m=0}^{n}\binom{n}{m}\frac{\partial^m\phi^{'}(w_a(\boldsymbol{y}))}{\partial y_j^m}\frac{\partial^{n-m}e_a(\boldsymbol{y})}{\partial y_j^{n-m}}).$$

Notice that $\phi^{'}(w_a(\boldsymbol{y}))$ is a composition function, by the Faa Di Bruno formula for the mth derivative of the composition function, we have

$$\frac{\partial^m\phi^{'}(w_a(\boldsymbol{y}))}{\partial y_j^m} = \sum_{\Delta(mk)}\frac{m!}{k_1!\cdots k_m!}\phi^{(k+1)}(w_a(\boldsymbol{y}))$$
$$(\frac{w_a^{(1)}(\boldsymbol{y})}{1!})^{k_1}\cdots(\frac{w_a^{(m)}(\boldsymbol{y})}{m!})^{k_m},$$

where $\sum_{\Delta(mk)}$ is sum over all $k_1, ..., k_m$ for which $k_1 + 2k_2 + \cdots + mk_m = m$, and $k = k_1 + k_2 + \cdots + k_m$. Therefore, after nth derivative, the left hand side of equation (16) is

$$E_{\boldsymbol{\mu}}\sum_{m=0}^{n}\binom{n}{m}\sum_{\Delta(mk)}\frac{m!}{k_1!\cdots k_m!}$$
$$\phi^{(k+1)}(w_a(\boldsymbol{y}))(\frac{w_a^{(1)}(\boldsymbol{y})}{1!})^{k_1}\cdots(\frac{w_a^{(m)}(\boldsymbol{y})}{m!})^{k_m}e_a^{(n-m)}(\boldsymbol{y}). \tag{17}$$

For the right hand side of equation (16), after successive differentiation, for any n, we get

$$\frac{1}{\beta}m_{j0}^{(n)}(x_0,\boldsymbol{y})\phi^{'}(x_0).$$

Then, after the $(N-1)$th differentiation, these N equations can be written in marix form, with $(\mu_1, ..., \mu_A)$ unknown :

$$\begin{pmatrix} f_1^0 & \cdots & f_a^0 & \cdots & f_A^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^n & \cdots & f_a^n & \cdots & f_A^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{N-1} & \cdots & f_a^{N-1} & \cdots & f_A^{N-1} \end{pmatrix}_{N\times A}\begin{pmatrix}\mu_1\\ \vdots \\ \mu_a \\ \vdots \\ \mu_A\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta}m_{j0}^{(0)}(x_0,\boldsymbol{y})\phi^{'}(x_0)\\ \vdots \\ \frac{1}{\beta}m_{j0}^{(n)}(x_0,\boldsymbol{y})\phi^{'}(x_0) \\ \vdots \\ \frac{1}{\beta}m_{j0}^{(N-1)}(x_0,\boldsymbol{y})\phi^{'}(x_0)\end{pmatrix}$$

[18]The formula of Leibniz for the nth derivative of the product of two functions is given by $\frac{d^n f(x)g(x)}{dx^n} = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)$, where $\binom{0}{0} = 1$. The formula of Faa di Bruno for the nth derivative of the composition of two functions is given by $\frac{d^n f(g(x))}{dx^n} = \sum_{\Delta(nk)}\frac{n!}{k_1!\cdots k_n!}f^{(k)}(g(x))(\frac{g^{(1)}(x)}{1!})^{k_1}\cdots(\frac{g^{(n)}(x)}{n!})^{k_n}$,where $\sum_{\Delta(nk)}$ is sum over all $k_1, ..., k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$, and $k = k_1 + k_2 + \cdots + k_n$, and $0! = 1$. For references, see Roman (1980).

Given that the functions u and ϕ have been identified, and the conditional probability distributions $\{\boldsymbol{\nu}_a\}_{a=1}^{M}$ are known, the left hand side matrix is computable. The fact that the functions $\{f_a(\boldsymbol{y})\}_{a=1}^{A}$ are differentiably linearly independence implies that the left hand side matrix has rank A at some portfolio $\boldsymbol{y}$, then the probability measure μ can be computed. □

We give an example to illustrate the argument in the proposition.

Example 4. *Suppose $\phi(x) = x^{\frac{1}{4}}$, $u(x) = x^{\frac{1}{2}}$, $\beta = 1$. There are two ambiguity states with conditional probability $\nu_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ and $\nu_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12})$. And there are three assets with payoff: $r_1 = (1, 1, 1, 1)$, $r_2 = (1, 2, 0, 0)$, and $r_3 = (0, 0, 0, 1)$. That is, the first asset is risk free and the second asset is ambiguity free. Then the optimization problem is*

$$\max_{x_0,\boldsymbol{y}} \; x_0^{\frac{1}{4}} + \{\mu_1[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1)^{\frac{1}{2}} + \frac{1}{6}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{2}}$$
$$+ \mu_2[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{4}(y_1)^{\frac{1}{2}} + \frac{1}{12}(y_1 + y_3)^{\frac{1}{2}}]^{\frac{1}{2}}\},$$

$$s.t. \; p_0x_0 + p_1y_1 + p_2y_2 + p_3y_3 = 1.$$

The corresponding equation (16) with $j = 3$ is

$$\frac{\mu_1}{2}[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1)^{\frac{1}{2}} + \frac{1}{6}(y_1 + y_3)^{\frac{1}{2}}]^{-\frac{1}{2}}[\frac{1}{12}(y_1 + y_3)^{-\frac{1}{2}}]+$$
$$\frac{\mu_2}{2}[\frac{1}{2}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{6}(y_1 + 2y_2)^{\frac{1}{2}} + \frac{1}{4}(y_1)^{\frac{1}{2}} + \frac{1}{12}(y_1 + y_3)^{\frac{1}{2}}]^{-\frac{1}{2}}[\frac{1}{24}(y_1 + y_3)^{-\frac{1}{2}}]$$
$$= \frac{p_3}{p_0}(x_0, \boldsymbol{y})\frac{1}{4}(x_0)^{-\frac{3}{4}}.$$

Define

$$f_1(y) = \frac{1}{2}[\frac{1}{2}(y_1+y_2)^{\frac{1}{2}}+\frac{1}{6}(y_1+2y_2)^{\frac{1}{2}}+\frac{1}{6}(y_1)^{\frac{1}{2}}+\frac{1}{6}(y_1+y_3)^{\frac{1}{2}}]^{-\frac{1}{2}}[\frac{1}{12}(y_1+y_3)^{-\frac{1}{2}}],$$

and

$$f_2(y) = \frac{1}{2}[\frac{1}{2}(y_1+y_2)^{\frac{1}{2}}+\frac{1}{6}(y_1+2y_2)^{\frac{1}{2}}+\frac{1}{4}(y_1)^{\frac{1}{2}}+\frac{1}{12}(y_1+y_3)^{\frac{1}{2}}]^{-\frac{1}{2}}[\frac{1}{24}(y_1+y_3)^{-\frac{1}{2}}].$$

Then, at $y = (1, 0, 0)$, the matrix

$$\begin{bmatrix} f_1(y) & f_2(y) \\ f'_1(y) & f_2'(y) \end{bmatrix}$$

has full rank. Therefore, $f_1(y)$ and $f_2(y)$ are differentiably linearly independent and $\boldsymbol{\mu}$ can be identified and computed.

Remark 7. *An alternative approach to the identification of risk and ambiguity indices is to assume knowledge of the individual's consumption-portfolio indifference correspondence*

$$\begin{aligned} I(\bar{x}_0, \bar{\boldsymbol{y}}) = \{(x_0, \boldsymbol{y}) \in \mathbb{R}^{J+1} : &\phi(x_0) + \beta E_{\boldsymbol{\mu}} \phi(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{x}))) \\ &= \phi(\bar{x}_0) + \beta E_{\boldsymbol{\mu}} \phi(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\bar{\boldsymbol{y}})))\}. \end{aligned}$$

The argument is not surprising: the indifference correspondence suffices to trace out consumption and asset demands, while consumption and asset demands can be obtained through integration of the indifference correspondence. We omit the details.

Remark 8. *[Perils of identification]The identification argument assumes the existence of underlying preferences or a utility function that satisfy particular properties, and ascertains their uniqueness given the demand function. It would lead to erroneous conclusions if the assumption on the underlying preferences fails. An alternative approach, initiated by Afriat (1967), starts with finite observations of demand and price, and shows that if these observations satisfy the Strong Axiom of Revealed Preference, then there exists a preference relation rationalizing the data, and a piece-wise linear concave utility function that can be explicitly constructed. Needless to say, the preference relation or utility function is not unique. Mas-Colell (1978) addresses the issue and shows that the preference relations determined by finite observations in Afriat (1967) converge to the unique true preference if observations from a continuous demand function that is income Lipschitzian and satisfies the Strong Axiom of Revealed Preference and a boundary condition become dense. For a finite set of observations, Varian (1983) provides conditions necessary and sufficient for portfolio choices to be generated by expected utility maximization with a known distribution of asset payoffs in incomplete markets. Kübler, Selden, and Wei (2020) work in a two-period model with uncertainty, they derive demand restrictions necessary and sufficient for portfolio choices and certainty intertemporal consumption to have been generated by the maximization of Kreps-Porteus-Selden (KPS) preferences, and they provide conditions for the recovery time and risk preference utilities. For finite data, they derive a set of linear inequalities that are necessary and sufficient for observations to be consistent with the maximization of KPS preferences. In recent work, Kübler and Polemarchakis (2017) prove the convergence of preferences and beliefs constructed in Varian (1983) or Echenique and Saito (2015) to a unique profile as the number of observations becomes dense. It is straight forward to extend the revealed preference characterization in Varian (1983) and the convergence argument in Kübler and Polemarchakis (2017) to the ambiguity setting.*

It is important to note, nevertheless, that convergence bridges the gap between the recoverability and the identification of ambiguity preferences, and it answers the question whether demand is indeed generated by ambiguity preferences. Identification refers to the uniqueness of unobservable characteristics. Recoverability refers to a method by which these characteristics can be known.

References

S. Afriat. The construction of a utility function from demand data. *International Economic Review*, 8:67–77, 1967.

D. Ahn, S. Choi, D. Gale, and S. Kariv. Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics*, 5:195–223, 2014.

E. Anderson, L. P. Hansen, and T. Sargent. A quartet of semi-groups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association*, 1:68–123, 2003.

F. Anscombe and R. J. Aumann. A definition of subjective probability. *Annals of Mathematical Statistics*, 34:199–205, 1963.

G. Attanasi, C. Gollier, A. Montesano, and N. Pace. Eliciting ambiguity aversion in unknown and in compound lotteries: a smooth ambiguity model experimental study. *Theory and Decision*, 77:485–530, 2014.

A. Baillon. Prudence with respect to ambiguity. *Economic Journal*, 127: 1731–1755, 2017.

R. C. Bayer, S. Bose, M. Polisson, and L. Renou. Ambiguity revealed. Unpublished manuscript, 2013.

C. Camerer and M. Weber. Recent developments in modeling preferences: uncertainty and ambiguity. *Journal of Risk and Uncertainty*, 5:325–370, 1992.

S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Uncertainty averse preferences. *Journal of Economic Theory*, 146:1275–1330, 2011.

H. Chen, N. Ju, and J. Miao. Dynamic asset allocation with ambiguous return predictability. *Review of Economic Dynamics*, 17:799–823, 2014.

P. Dybvig and H. Polemarchakis. Recovering cardinal utility. *Review of Economic Studies*, 48:159–166, 1981. URL http://www.polemarchakis.org/a14-rcu.pdf.

F. Echenique and K. Saito. Savage in the market. *Econometrica*, 83:1467–1495, 2015.

H. Einhorn and R. Hogarth. Decision making under ambiguity. *Journal of Business*, 59 (Supplement):225–S250, 1986.

D. Ellsberg. Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics*, 75:643–669, 1961.

L. Epstein. A paradox for the "smooth ambiguity" model of preference. *Econometrica*, 78:2085–2099, 2010.

J. D. Geanakoplos and H. Polemarchakis. Observability and optimality. *Journal of Mathematical Economics*, 19:153–165, 1990. URL http://www.polemarchakis.org/a39-oop.pdf.

P. Ghirardato. Defining ambiguity and ambiguity attitude. In I. Gilboa, editor, *Uncertainty in Economic Theory*, pages 36–45. Routledge, 2004.

I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.

C. Gollier. Portfolio choice and asset prices: the comparative statics of ambiguity aversion. *Review of Economic Studies*, 78:141–154, 2011.

J. R. Green, L. J. Lau, and H. M. Polemarchakis. On the recoverability of the von Neumann-Morgenstern utility function from asset demands. In J. R. Green and J. A. Scheinkman, editors, *Equilibrium, Growth and Trade: Essays in Honor of L. McKenzie*, pages 151–161. Academic Press, 1979. URL http://www.polemarchakis.org/a12-inm.pdf.

Y. Halevy. Ellsberg revisited: an experimental study. *Econometrica*, 75: 503–536, 2007.

L.P. Hansen. Beliefs, doubts and learning: the valuation of macroeconomic risk. *American Economic Review*, 97:1–30, 2007.

L.P. Hansen and T.J. Sargent. Robust control and model uncertainty. *American Economic Review*, 91:60–66, 2001.

N. Ju and J. Miao. Ambiguity, learning and asset returns. *Econometrica*, 80: 559–591, 2012.

A. Kajii and J. Xue. Precautionary saving with changing income ambiguity. Unpublished manuscript, 2016.

P. Klibanoff, M. Marinacci, and S. Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73:1849–1892, 2005.

P. Klibanoff, M. Marinacci, and S. Mukerji. Recursive smooth ambiguity preferences. *Journal of Economic Theory*, 144:930–976, 2009.

P. Klibanoff, P. Marinacci, and M. Mukerji. On the smooth ambiguity model: a reply. *Econometrica*, 80:1303–1321, 2012.

F. Kübler and H. Polemarchakis. The identification of beliefs from asset demands. *Econometrica*, 85:1219–1238, 2017. URL http://www.polemarchakis.org/a86-idb.pdf.

F. Kübler, L. Selden, and X. Wei. Incomplete market demand tests for Kreps-Porteus-Selden preferences. *Journal of Economic Theory*, 185:1–33, 2020.

F. Maccheroni, M. Marinacci, and A. Rustichini. Dynamic variational preferences. *Journal of Economic Theory*, 128:4–44, 2006.

A. Mas-Colell. On the recoverability of consumers' preferences from demand behavior. *Econometrica*, 45:1409–1430, 1977.

A. Mas-Colell. On revealed preference analysis. *Review of Economic Studies*, 45:121–131, 1978.

S. Mukerji and J.-M. Tallon. Ambiguity aversion and incompleteness of financial markets. *Review of Economic Studies*, 68:883–904, 2001.

H. Polemarchakis. Observable probabilistic beliefs. *Journal of Mathematical Economics*, 11:65–75, 1983. URL http://www.polemarchakis.org/a20-opb.pdf.

H. Polemarchakis and L. Selden. On the recoverability of risk and time preferences from consumption and asset demand. *European Economic Review*, 25:115–133, 1984. URL http://www.polemarchakis.org/a26-rca.pdf.

S. Roman. The formula of Faa di Bruno. *The American Mathematical Monthly*, 87:805–809, 1980.

L. J. Savage. *The Foundations of Statistics.* John Wiley and Sons, 1954.

U. Segal. Two-stage lotteries without the reduction axiom. *Econometrica*, 58:349–377, 1990.

L. Selden and X. Wei. Effects of ambiguity aversion on savings: an interpretive note. mimeo, 2014.

K. Seo. Ambiguity and second-order belief. *Econometrica*, 77:1575–1605, 2009.

P. Slovic and A. Tversky. Who accepts Savage's axiom? *Behavioral Science*, 19:368–373, 1974.

S. T. Trautmann and G. van der Kuillen. Ambiguity attitudes. In *The Wiley Blackwell Handbook of Judgment and Decision Making*, volume 1, pages 89–116. Blackwell, 2015.

H. R. Varian. Nonparametric tests of models of investor behavior. *Journal of Financial and Quantitative Analysis*, 18:269–278, 1983.

J. von Neumann and O. Morgenstern. *Games and Economic Behavior.* Princeton University Press, 1947.