Bayesian dialogs [1]

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Abstract

Eventual consensus is the only property of a *bayesian dialog,* the prototype of a *rational dialog.*

At each stage, one of two interlocutors states his beliefs formed after the revision prompted by the beliefs stated by the other at the previous stage. The dialog terminates, possibly at infinity, when nothing is left to be said. A third party, with access only to the transcript of a dialog, cannot distinguish a bayesian dialog from an arbitrary sequence of alternating utterances that terminates with agreement.

Equivalently, two rational individuals who learn form each other and will eventually agree, can hold different and divergent beliefs for any number of rounds of communication prior to consensus.

Key words: dialog, rationality, agreement.

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A *bayesian dialog* is a sequential exchange of beliefs, and it is the prototype of a *rational dialog*. At each stage, one of two interlocutors states his beliefs formed after the revision prompted by the beliefs stated by the other at the previous stage. The dialog terminates when nothing is left to be said.

Dialogs first appear in Sumerian literary tablets, composed before the second millennium bc, much before the earliest Greek and Hebrew literary works according to Kramer (1963), who adds: "*The disputations and dialogues, eleven in number, ... are the forerunners and prototypes of similarly literary compositions current all over the ancient world as far as India on the east and probably Greece on the west, ... and they provided the literary and stylistic framework for even such profound philosophic works as Plato's dialogues.*" Hösle (2012) refers to the discussion between Uddâlaka Âruni and his son Śvetaketu in the seventeenth chapter of the *Chandoygya-Upanishad* and to the *Book of Job,* and adds: "*[T]he philosophical dialogue was not invented by the Greeks. ... [But,] despite the many dialogues in other cultures, we can maintain that the Greeks succeeded in elevating the genre to new level. ... Important metaphysical speculations were also made in India – but not the comprehensive attempt to ground in reason, and solely in reason, the norms that guide our conduct.*".

Following Plato, the dialog has been a literary genre of choice for the elucidation of arguments and the transmission of knowledge. In practice, structured dialogs have been employed extensively to pool the information of experts and, more recently, and ambitiously, in the search for consensus, even the resolution of conflict and peacemaking. Dalkey (1969) documented and assessed the Delphi Method, introduced in the beginning of the cold war, while Nielsen, Brandenburger, Geanakoplos, McKelvey, and Page (1990) provided a formal analysis of multi-person communication and consensus.

Aumann (1976) defined *common knowledge* and proved that consensus is a necessary condition for common knowledge for finite partitions. Geanakoplos and Polemarchakis (1982) proved that finite bayesian dialogs terminate in consensus and common knowledge. Nielsen (1984) generalized both arguments by allowing for knowledge structures given by sigma algebras rather than finite partitions and proving that bayesian dialogs converge to consensus and common knowledge. Bacharach (1979) coined the term and looked at bayesian dialogs when information is normally distributed.

Here, a third party, with access only to the transcript of a dialog, cannot distinguish a bayesian dialog from an arbitrary sequence of alternating utterances: the only property of a rational dialog is eventual consensus.

The argument extends to the special case of a *didactic dialog*, in which an expert is better informed than his interlocutor. The expert never changes his opinion, but the interlocutor follows an arbitrary path to agreement. The dialogs of Plato are, typically, didactic. *Parmenides* is an exception. In the didactic dialogs, Socrates, whose personality dominates and with whom the author, Plato, identifies, knows the truth, to which he guides his interlocutor. Indeed, in the course of the dialog, the knowledgeable expert, Socrates, may lead the interlocutor, temporarily, to error. Levine (1998), writing on the dialog between Socrates and *Protagoras*, comments: "*By offering a more plausible interpretation, and then making this appear useless and 'vulgar'.*" Unlike Plato, Hume (1779) does not take sides in the dialog between the sceptic, Philo, and Cleanthes who defends the argument by design – at least not explicitly.

Turing (1950) and, in a simpler form, Newman, Turing, Jefferson, and Braithwaite (1952) posed the question whether automatic calculating machines can be said to think: "*The idea of the test is that the machine has to pretend to be a man, by answering questions put to it, and it will only pass if the pretense is reasonably convincing ....* " The argument here is relevant.

A bayesian dialog is an adjustment path to common knowledge. And, though common knowledge necessarily displays agreement, a refutable property, the adjustment path is arbitrary [1]. Which bears an analogy to general competitive analysis: Equilibrium prices and quantities are not arbitrary, in Brown and Matzkin (1996), and, furthermore, in Chiappori, Ekeland, Kubler, and Polemarchakis (2004), they identify fundamentals. Nevertheless, as follows from Debreu (1974), the Walrasian tâtonnement that leads to equilibrium, if it does, is arbitrary.

It should not be overlooked that rational individuals who learn form each other and will eventually agree, can hold different and divergent beliefs for any number of rounds of communication prior to consensus.

[1] Evidently, common knowledge displays agreement for common prior beliefs. Di Tillio, Lehrer, and Samet (2020) argued that, in an infinite dialog, convergence to common knowledge and agreement at a rate they characterize can serve as a test of common priors.

1 The analytical argument

States of the world are $\omega \in \Omega$, measurable sets are elements of a $\sigma-$ algebra, $\mathcal{F}$, and beliefs are probability measures on the measurable space $(\Omega, \mathcal{F})$.

For simplicity, we partition of the set of states of the world into information cells or sets in a rectangular array $Q = (Q_{m.n})_{n=1,\dots N}^{m=1,\dots M}$ that is either finite, $MN < \infty$, or countably infinite, $MN = \infty$.

Two individuals, $i = 1, 2$, receive information according to their private information partitions. Information sets of individual 1 are rows of the array of information cells, $R = \{R_m = \cup_{n=1}^{N} Q_{m,n} : \ m = 1, \dots, M\}$, while information sets of individual 2 are columns of the array of information cells, $C = \{C_n = \cup_{m=1}^{M} Q_{m,n} : \ n = 1, \dots, N\}$.

At a state of the world, ω, if $Q(\omega) = Q_{m,n}$[2], individual 1 is informed of $R(\omega) = R_m$, while individual 2 is informed of $C(\omega) = C_n$. It follows that individual 1 *knows* an event, E if $R(\omega) \subset E$, while individual 2 *knows* E if $C(\omega) \subset E$. An event is *common knowledge* if it contains the element of the *meet* of the individual partitions: $(R \wedge C)(\omega) \subset E$. Here, the meet, the finest common coarsening, of the information partitions of individuals is $\{\Omega\}$: the only event that is *common knowledge,* at any state of the world, is Ω [3].

The *join* of the information partitions of individuals 1 and 2 is their coarsest common refinement. Here, $(R \vee C) = (Q_{m.n})_{n=1,\dots N}^{m=1,\dots M}$. Information finer than what is measurable with respect to the join of the individual partitions is not accessible to the individuals.

According to prior beliefs common to individuals, information cells occur with positive probability, $pr(Q_{m,n}) = p_{m,n} > 0$. The prior conditional probability that an event A occurs is $pr(A|Q_{,m,n}), = pr(A \cap Q_{m,n})/pr(Q_{m,n}) = a_{m,n}$. The rectangular array $D = (a_{m.n}, p_{m,n})_{n=1,\dots N}^{m=1,\dots M}$ is a sufficient description of fundamentals [4].

The argument in Aumann (1976) that posteriors that are common knowledge coincide, is, here, evident. Since the only event that is common knowledge is Ω, posterior beliefs that A has occurred are common knowledge at ω only if $pr(A|R(\omega)) = pr(A|R(\omega')), \omega' \in \Omega$, and, as a consequence, $pr((A|R(\omega)) = pr(A|\cup_{\omega'} R(\omega')) = pr(A)$. By a similar argument, $pr(A|C(\omega))$

[2] $Q(\omega)$ is the element of the partition Q that contains ω, and similarly for any partition or state.

[3] For any states of the world, ω and ω', $R(\omega) \cap C(\omega') \neq \emptyset$.

[4] Alternatively, a pair of arrays, $(a_{m.n})_{n=1,\dots N}^{m=1,\dots M}$ and $(p_{m,n})_{n=1,\dots N}^{m=1,\dots M}$.

$= pr(A|C(\omega')), \omega' \in \Omega$, and, as a consequence, $pr((A|C(\omega)) = pr(A| \cup_{\omega'} C(\omega')) = pr(A)$ or $pr(A|R(\omega)) = pr(A|C(\omega))$.

Importantly, for an arbitrary event, A, posteriors need not be common knowledge at ω [5], and a dialog can commence between individual 1 and individual 2 concerning the probability that event A has occurred: $\omega \in A$.

A *dialog* is a sequence of utterances,

$$(q_1, q_2, \ldots, q_t, \ldots, q_T), \quad 0 \leq q_t \leq 1,$$

at t odd by individual 1, and at t even by individual 2. It is finite, $T < \infty$ or infinite, $T = \infty$.

A *bayesian dialog* is defined inductively.

The individual who speaks at t bases his beliefs on information available to him at the end of the preceding round of communication, and q_t is the probability that he attaches to the occurrence of the event A.

At $t = 1$, individual 1 announces

$$q_1 = pr(A|R(\omega)),$$

which reveals that

$$\omega \in \Omega_1 = \{\omega' \in \Omega : pr(A|R(\omega')) = q_1\}.$$

At $t = 2$, individual 1 announces

$$q_2 = pr(A|(C(\omega) \cap \Omega_1)),$$

which reveals that

$$\omega \in \Omega_2 = \{\omega' \in \Omega_1 : pr(A|(C(\omega') \cap \Omega_1)) = q_2\}.$$

At t, if t is odd, individual 1 announces

$$q_t = pr(A|(R(\omega) \cap \Omega_{t-1})),$$

which reveals that

$$\omega \in \Omega_t = \{\omega \in \Omega_{t-1} : pr(A|(R(\omega) \cap \Omega_{t-1})) = q_t\},$$

[5] Posteriors, $pr(A|R(\omega)) = pr(A|C(\omega))$, may well coincide but not be common knowledge.

and a similar argument applies if t is even and it is individual 2 who announces his beliefs.

The argument in Geanakoplos and Polemarchakis (1982), that posteriors converge to values that are common knowledge coincide is, evident. If $\Omega_{t-1} = \Omega_t = \bar{\Omega}$, then $\{R(\omega') \cap \bar{\Omega} : \omega' \in \bar{\Omega}\}$ is a partition of $\bar{\Omega}$ and the posterior beliefs of individual 1 with this information partition are constant over $\bar{\Omega}$ and equal to q_{t-1}, while, by a similar argument, $\{R(\omega') \cap \bar{\Omega} : \omega' \in \bar{\Omega}.\}\}$ is a partition of $\bar{\Omega}$ and the posterior beliefs of individual 1 with this information partition are constant over $\bar{\Omega}$ and equal to q_{t-1}. It is then Aumann (1976) that beliefs are common knowledge and coincide: $q_{t-1} = q_t = \bar{q}$. Since partitions are finite, convergence must occur in finitely many steps, with the number of iterations bounded by the minimum of the cardinality of the two partitions.

Proposition. *Any finite dialog,* $(q_1, q_2, \ldots, \bar{q}_t, \ldots, \bar{q}, \bar{q})$ *that terminates in consensus is a bayesian dialog.*

Proof. ———-

□

States of the world, ω, are drawn from a measurable space. An individual has prior beliefs, μ, and receives information sequentially: there is a sequence of measurable sets, $Q_1, \ldots, Q_n, \ldots, Q_N$, such that, the state that realizes is $\omega^* \in Q_N$, and, at stage t, the individual is informed of

$$Q^t = \cup_{n=t}^N Q_n, \quad t = 1, \ldots, t, \ldots, N.$$

If A is a measurable set, and if

$$q_n = \mu(Q_n), \quad a_n = \mu(A \cap Q_n) \quad \text{and} \quad p_n = \mu(A|Qn) = \frac{\mu(A \cap Q_n)}{\mu(Q_n)},$$

then,

$$a_n = p_n q_n;$$

it follows that

$$p^t = \mu(A|Q^t) = \frac{\mu(A \cap Q^t)}{\mu(Q^t)} = \frac{\sum_{n=t}^N \mu(A \cap Q_n)}{\sum_{n=t}^N \mu(Q_n)} = \frac{\sum_{n=t}^N p_n q_n}{\sum_{n=t}^N q_n}.$$

A monologue at state of the world, ω^*, concerns the probability that a set, A, has occurred; it is the sequence of posterior beliefs,

$$p^1, \ldots, p^t, \ldots, p^N.$$

Evidently, monologue is fully determined by the sequence of pairs of probabilities , $(a_1, q_1), \ldots, (a_n, q_n), \ldots, (a_N, q_N)$, with $a_1 \leq q_1, \ldots, a_n \leq q_n$, $\ldots, a_N \leq q_N$.

We restrict attention to monologues at states of the world in terminal information sets that are not null: $q_N = \mu(Q_N) > 0$, and, as a consequence,

$$q^t = \mu(Q^t) > 0, \quad t = 1, \ldots, t, \ldots, N.$$

Evidently, if the sequence

$$(a_1, q_1), \ldots, (a_n, q_n), \ldots, (a_N, q_N),$$

generates the monologue

$$p^1, \ldots, p^t, \ldots, p^N,$$

then, the sequence

$$(a_1, q_1), \dots, (a_n, q_n), \dots, (a'_N, q'_N), (a'_{N+1}, q'_{N+1})$$

generates the monologue

$$p^1, \dots, p^t, \dots, p^N, p^{N+1}$$

if

$$\begin{aligned} &a'_N + a'_{N+1} = a_N, \\ &q'_N + q'_{N+1} = q_N, \quad \text{and} \\ &\tfrac{a'_{N+1}}{q'_{N+1}} = p_{N+1}. \end{aligned} \tag{1}$$

Claim 1. *For any* p^{N+1}*, any monologue,*

$$p^1, \dots, p^t, \dots, p^N,$$

can be extended to a monologue

$$p^1, \dots, p^t, \dots, p^N, p^{N+1}.$$

Proof. The system of equations (1) is a system of 3 equations in 4 unknowns: $a'_N, a'_{N+1}, q'_N, q'_{N+1}$ with domain restrictions. If

$$q'_N = (1-x)q_N, \quad q'_{N+1} = (1-x)q_N \quad , 0 < x < 1,$$

then, it suffices to solve the equation

$$(1-x)\frac{a'_N}{q'N} + xp^{N+1} = p^N, \quad 0 < x < 1 \quad 0 < \frac{a'_N}{q'_N} < 1.$$

There is a continuum of solutions; the solution that maximizes q_{N+1} is the solution with maximal

$$x = \max\{\frac{p^N}{p^{N+1}}, \frac{1-p^N}{1-p^{N+1}}\}.$$

□

It follows that

Any finite monologue is a bayesian monologue.

Any monologue is a bayesian monologue.

The substantive question that remains for an infinite monologue is to guarantee that

$$q_\infty = \lim_{n\to\infty} q_n > 0.$$

In a note (private communication) Samet, Hoffman, and Di Tillio (2016) gave a parametric example of a finite dialog that diverges for arbitrarily many rounds, before agreement at a terminal stage; this is another in the line of the examples above. They also give a parametric example of a dialog that diverges for arbitrarily many rounds, along the lines of the examples above. They also raised the question whether a dialog that continues forever (for countable infinitely many rounds) can involve beliefs that diverge for ever; and they prove it cannot, as follows from the convergence result in Nielsen (1984).

Recently, Di Tillio, Lehrer, and Samet (2020) extended the argument an infinite dialog. For a simultaneous dialog,

———-

Corollary. ———-

Proof. ———-

□

Experts At a state of the world, ω, individual 1 is an expert concerning the event A if

$$Q_{m,n} \subset R(\omega) \Rightarrow pr(A|R(\omega)) = pr(A|Q_{m,n}) :$$

no information in the join would cause him to alter his beliefs.

A *dialog with an expert* is a dialog

$$(\bar{q}, q_2, \ldots, \bar{q}, q_{2t}, \ldots, \bar{q}), \quad 0 \leq \bar{q}, q_T \leq 1,$$

at t odd, $q^{2t+1} = \bar{q}$, by individual 1, and at t even, q^{2t}, by individual 2.

Corollary. *Any finite dialog with an expert,* $(\bar{q}, q^2, \ldots, \bar{q}, q^{2t}, \ldots, \bar{q}, \bar{q})$ *that terminates in consensus is a bayesian dialog with an expert.*

Proof. It suffices to set $p_{i,n} = \bar{q}$ for all n, and apply the construction in the proposition starting with $m \geq 2$.

□

Prior beliefs Restrictions on the prior beliefs of individuals imply restrictions on possible dialogs. Most simply, suppose the space of staters of the world is partitioned into 2 equiprobable information cells: $M = 1$, and $N = 2$. Then, a Bayesian dialog is $\{q_1, \bar{q}\}$, with, importantly,

$$q_1 \in (\frac{1}{2}\bar{q}, \frac{1}{2}\bar{q} + \frac{1}{2}).$$

Order matters Consider the array of conditional probabilities

$$\begin{pmatrix} a & 1 & 0 \\ \\ 0 & 0 & 1 \end{pmatrix}, \quad a \neq 0, 1.$$

with uniform probability of 1/6 across information cells, and suppose that the state of the word that realizes is $\omega \in Q_{3,3}$.

Informed of $R_3 = R(\omega)$, individual 1 announces $q^1 = 1/3$.

With $a \neq 0$, this reveals his information set to individual 2, who now knows that $\omega \in Q_{2,3}$ and announces $q^2 = 1$.

In turn, this prompts individual 1 to revise his posterior to $q^3 = 1$.

Alternatively, individual 2 speaks first. Informed of $C_3 = C(\omega)$, he announces $q^1 = 1/2$.

With $a \neq 1$, this allows individual 1 to deduce that $C(\omega) \in \{C_2, C_3\}$ or $C(\omega) \neq C_1$, and, as a consequence, announce $q^2 = 1/2$.

The announcement of individual 2 reveals no information; the beliefs of individuals are common knowledge, and they coincide.

But, this common posterior belief that assigns probability 1/2 to the event, reflects information less precise than the information the individuals would have access to at the end of a dialog initiated by individual 1.

Public information may obfuscate Consider the arrays of prior and conditional probabilities

$$\begin{pmatrix} 0 & \frac{1}{6} & 0 \\ \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \\ 0 & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ \\ 0 & 1 & 0 \\ \\ 0 & 0 & 1 \end{pmatrix}.$$

and suppose that the state of the word that realizes is $\omega \in Q_{3,3}$.

Informed of $R_3 = R(\omega)$, individual 1 announces $q^1 = 1/2$.

This reveals his information set to individual 2, who now knows that $\omega \in Q_{3,3}$ and announces $q^2 = 1$.

In turn, this prompts individual 1 to revise his posterior to $q^3 = 1$.

The dialog ends in consensus and common knowledge.

Importantly, the common knowledge information available to individuals coincides with the information pooling would have led them to since $R_3 \cap C_3 = Q_{3,3}$.

Suppose now that, following the realization of ω, a public authority, policy maker announces that

$$\omega \notin \{Q_{1,3} \cup Q_{2,3}\}.$$

Note that this is something both individuals know; of course, it is not common knowledge, and, as a consequence, it affects the exchange of information between the individuals.

With the truncated array of conditional probabilities

$$\begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix},$$

the posterior beliefs of the individuals are common knowledge and there is no information to exchange.

The point, as in Dutta and Polemarchakis (2012) is that this common posterior belief that assigns probability 1/2 reflects information that is less precise than the information the individuals would have access to by exchanging beliefs, through a dialog, without the prior public announcement.

Silence Here, a dialog is an alternating sequence of utterances; formally, an interlocutor cannot remain silent when it is his turn to speak. One can interpret silence by an interlocutor at t as the repetition of his utterance at $t-2$: that is, $q^t = q^{t-2}$. It is important, nevertheless, that different rounds of responses be distinct, which may not be the case if both interlocutors remain silent.

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