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Economic Theory

Theory and practice of monetary policy

Guest Editors' introduction

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1 Introduction

Money is a store of value that serves as a unit of account and medium of exchange. Monetary policy is the conduct of the monetary authority in the issuance and supply of money balances. The papers in this volume cosider the effects of monetary policy on the allocation of resources and the welfare of individuals at equilibrium.

The functions of money and the modifications of the general equilibrium model necessary to accommodate monetary economies point the way to alternative conduits from monetary policy to resource allocation and welfare.

(1) According to a conundrum, [16], money could not maintain a positive price at general equilibrium. A solution to the conundrum is by now well understood, and alternative formulations, [10, 11], attain this goal, albeit with differing conclusions for the multiplicity or determinacy of equilibria and the role of monetary policy. In particular, they make clear that it is not possible to consider monetary distinct from fiscal policy, and this because of the accounting consistency required by general equilibrium: alternative specifications of the distribution of seignorage have different implications for the determinacy of equilibria, akin to the distinction, [29], between Ricardian and non-Ricardian policies.
(2) Money, as a medium of exchange, entails transaction costs of acquiring and holding balances that insert a wedge between Pareto optimal and competitive allocations; that is, unless transaction costs vanish, [13], which is the case if money, as an asset, earns the no-arbitrage rate of return.
(3) Money as a unit of account assigns a role for monetary policy in the determination of the price level, [24], and, more importantly, the path properties of the price level and other nominal variables.

(a) When individuals have initial claims to output denominated in nominal terms, monetary policy effects redistributions of revenue.
(b) When information is asymmetric, monetary policy can affect the allocation of resources through the information revealed by prices, [26], the rationality of expectations, [18], notwithstanding.
(c) When the asset market is incomplete, [2,7,14] monetary policy can affect the allocation of resources through the menu of available contracts for the transfer of purchasing power across dates and realizations of uncertainty.

(4) Money as a store of value allows monetary policy to alter the allocation of resources at equilibrium, [9,22] even in the absence of other services that money renders, such as liquidity, even if the latter, [4], modify the results.

Simplifying assumptions facilitate the argument; in particular, it is often appropriate (1) to take for granted, and do not attempt to explain, the prevalence of monetary transactions, of exchanges of goods for money, and model the liquidity services of money balances as cash-in-advance constraints; (2) to restrict attention to fiat money, a zero-coupon bond of infinite maturity, whose quantity and value are not linked to any other commodity or asset, but for equilibrium relations; (3) to postulate, in a first instance, a complete asset market, so as to focus on money as a medium of exchange, independently of its role for purposes of intertemporal transfers of revenues and, possibly, insurance.

The papers in this volume address issues that arise concerning the theory and practice of monetary policy.

(1) Lucas Papademos, in [20], "Policy-making in EMU: strategies, rules and discretion," deals with the strategies employed by monetary and economic authorities in the European Monetary Union to maintain price stability and foster higher sustainable growth. With reference to monetary and fiscal policy as well as structural policies, he discusses the role and effectiveness of rules versus discretion in policy implementation; and then, specifically in the context of the euro area economy and European Union's institutional setting, it assesses the formulation and implementation of policies.
(2) Manuel Santos, in [23], "The value of money in a dynamic equilibrium mo- del," considers an economy of multiple and infinitely lived consumer-investors, with uncertainty, an incomplete financial markets and and borrowing constraints; money provides liquidity services captured by cash-in-advance constraints. He demonstrates that, if aggregate wealth is finitely valued, the cash-in-advance constraints bind all agents over infinitely many date events. The result has implications both for the construction of reduced form models as well as for the conduct of monetary and fiscal policy.
(3) Gaetano Bloise, in [5], "A note on the existence of monetary equilibrium over an infinite horizon," considers an economy of multiple, infinitely lived consumer-investors under certainty; money provides liquidity services captured by cash-in-advance constraints. He demonstrates the existence of competitive equilibria when monetary policy sets the supply of balances and rates of interest adjust for markets to clear. The price level is indeterminate as

long as the monetary authority balances its budget by redistributing its profit (seignorage) as dividend payments to individuals.

(4) Pradeep Dubey and John Geanakoplos, in [12], "Determinacy with nominal assets and outside money," demonstrate that monetary equilibria are determinate as long as, along with inside money, there is outside money in circulation. This result, which echoes the distinction between Ricardian and non-Ricardian policies introduced in the fiscal theory of price determination. More importantly, determinacy obtains even in economies with an incomplete asset market in which effects are typically real. In addition, they distinguish between monetary policy that sets rates of interest and accommodate the demand for balances and policy that targets the money supply while the nominal rate of interest adjust for markets to clear; for the latter, the conditions for determinacy are more demanding.

(5) Charles Goodhart, Pojanart Sunirand and Demetres Tsomokos, in [15],"A model to analyze financial fragility," develop a general equilibrium model of a monetary economy with a banking sector of multiple, diverse banks; default is endogenous and equilibria may display financial fragility.The model, which is analytically and even computationally tractable, permits the analysis of alternative monetary and regulatory policies.

(6) Jean-Pascal Bénassy, in [3], "Interest rate rules, inflation and the Taylor Principal: an analytical exploration," considers an economy of overlapping generations, with an operative cash-in-advance constraint, subject to productivity shocks. He demonstrates that the optimal response of the nominal rate of interest to inflation depends, among others, on the extent of price stickiness and the autocorrelation of productivity shocks.

(7) Tomo Nakajima, in [19], "Monetary policy with sticky prices and segmented markets," considers an economy of infinitely lived households, with sticky prices and segmented markets or restricted participation in asset markets subject to shocks in the velocity of circulation of money balances. He demonstrates that market segmentation alters the response of output and inflation to monetary policy. In particular, the local determinacy of equilibrium requires passive (less than proportional) response of the nominal rate of interest to deviations of the inflation rate from the target level.

(8) Céline Rochon and Herakles Polemarchakis, in [21], "Debt, liquidity and dynamics," consider economies of overlapping generations in which money, distinct from debt, provides liquidity services. They demonstrate that the dynamic properties of equilibrium paths vary non-trivially with the nominal rate of interest, a policy parameter. In particular, stability of the steady state may require nominal rates of interest above a minimum: equivalently, an increase in the nominal rate of interest may be associated with indeterminacy, while a decrease may be associated with explosive behavior or convergence to an endogenous cycles.

(9) Jeffrey Amato and Hyun Shin, in [1], "Imperfect common knowledge and the information value of prices," consider an economy with differentially informed, monopolistically competitive firms, in which prices serve poorly as aggregators of private information. More importantly, they show that public

information may diminish the ability of prices to convey information; and they draw conclusions concerning the conduct of monetary policy.

(10) Thomas Cooley and Vincenzo Quadrini, in [8], "Monetary policy and the financial decisions of firms," develop a general equilibrium model with heterogeneous, long-lived firms. Importantly, the heterogeneity of firms, described by their size or the amount of equity capital they have, derives from financial decisions in response to idiosyncratic shocks in an imperfect capital market. They demonstrate that the size of firms affects their financial decisions and their response to monetary shocks.

(11) Larry Weiss, in [27], "Inflation indexed bonds and monetary theory," an empirical piece, argues that "tips" — treasury inflation protected securities, provide important new data for analyzing the state of the economy and for assessing the validity and significance of macroeconomic theories. Tip yields contain information for predicting real variables. The inclusion of tip yields supersedes the role of nominal variables - both the ten year nominal bond and federal funds rate, for incrementally predicting (Granger causing) real variables. More importantly, the data support the notion of block exogeneity — the lack of feedback from nominal to real variables. Which would appear to be inconsistent with the idea that monetary policy, as implemented through changes in the federal funds rate, has had measurable real effects.

We collect here basic results on monetary general equilibria by describing a monetary economy with uncertainty (and a complete asset market) that extends over an infinite horizon. A minimal modification of the canonical model of general competitive equilibrium, along the lines of [28], permits consideration of the fundamental properties of monetary economies and of monetary policy. Details are discussed, among others, in [6], though the framework is an elaboration over the classical cash-in-advance economy with a representative individual in [29].

2 A basic economy with money

2.1 Time and uncertainty

Time and uncertainty are described by an event tree. A date-event, σ is an element of a countable set, $\mathcal{S}$, with date t_σ. The set of date-events $\mathcal{S}$ is weakly ordered by $\succeq$; if $\tau \succeq \sigma$, the date-event τ follows the date-event σ, and, in particular, $t_\tau \geq t_\sigma$. The unique initial date-event is ϕ, with date $t_\phi = 0$. For a date event σ, the set of immediate successors is $\sigma_+ = \{\tau \succeq \sigma : t_\tau = t_\sigma + 1\}$, a finite nonempty set; which implies that time horizon is infinite.

2.2 Markets

A finite set $\mathcal{N}$ of physical commodities are traded at every date-event. Thus, the commodity space coincides with the vector space of all real maps on $\mathcal{S} \times \mathcal{N}$. A consumption plan, $x = (\ldots, x_\sigma, \ldots)$, is simply a positive real map on $\mathcal{S} \times \mathcal{N}$, with each x_σ an element of $\mathbb{R}^{\mathcal{N}}$.[1]

A price system consists of prices of commodities, state prices and nominal rates of interest. Prices of commodities, $p = (\ldots, p_\sigma, \ldots)$, are a positive real map on $\mathcal{S} \times \mathcal{N}$, specifying a market value for each commodity at each date-event. State prices, $a = (\ldots, a_\sigma, \ldots)$, are a strictly positive real map on $\mathcal{S}$ – we do not assume any normalization such as $a_\phi = 1$. Nominal rates of interest, $r = (\ldots, r_\sigma, \ldots)$, are a positive real map on $\mathcal{S}$.

To avoid unnecessary notation, we interpret p as present value or Arrow-Debreu prices of commodities. Such prices are expressed in terms of a given unit in account, $p \cdot u = 1$, where u is a given non-zero positive real map on $\mathcal{S} \times \mathcal{N}$ vanishing almost everywhere. Given state prices, a, spot or 'current' prices of commodities are

$$\left(\frac{1}{a_\phi} p_\phi, \ldots, \frac{1}{a_\sigma} p_\sigma, \ldots \right),$$

also in terms of balances at the associated date-event.

As far as the asset market is concerned, it simplifies to assume that portfolios at every date-event consist of elementary one-period Arrow securities and one-period risk-less nominal bonds only. State prices, a, have the usual interpretation: at a date-event, σ, the market value of a portfolio with deliveries only at the immediately following date-events, $(v_\tau : \tau \in \sigma_+)$, is

$$\frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau v_\tau.$$

The absence of arbitrage opportunities requires that, at every date-event,

$$\left(\frac{1}{1 + r_\sigma} \right) a_\sigma = \sum_{\tau \in \sigma_+} a_\tau.$$

This no-arbitrage relation between nominal rates of interest, r, and state prices, a, we assume always holds.

2.3 Fundamentals

There is a finite set, $\mathcal{I}$, of individuals. Each individual, i, is characterized by a preference relation, $\succeq^i$, on consumption plans; an initial endowment of commodities, e^i, that is itself a consumption plan; and a positive share, ν^i, into transfers, so that, across individuals, $\sum_i \nu^i = 1$. Fundamentals are, thus, $(\ldots, (\succeq^i, e^i, \nu^i), \ldots)$.

An allocation, $x = (\ldots, x^i, \ldots)$, is a collection of consumption plans, a positive real map on $\mathcal{S} \times \mathcal{N} \times \mathcal{I}$. It is balanced if, at every date-event,

$$\sum_i x^i_\sigma = \sum_i e^i_\sigma.$$

[1] Throughout, for a real(-valued) map $f : X \to \mathbb{R}$, where X is any arbitrary set, positive (strictly positive) means $f(x) \geq 0$ ($f(x) > 0$) for all x in X. In addition, f^+ and f^- are, respectively, the positive and the negative part of f, so that $f = f^+ - f^-$.

We omit technical details. We should, however, mention fairly general restrictions on fundamentals important for the existence of equilibria. Preferences are continuous (in the relative product topology), convex and strictly monotone; the endowment of each individual is bounded and uniformly strictly positive.

2.4 Sequential trade

An individual enters a date-event with nominal wealth, w^i_σ, inherited from previous financial investments, and receives a transfer, h^i_σ. He trades in assets, $(v^i_\tau : \tau \in \sigma_+)$, and balances, m^i_σ, subject to a budget constraint

$$m^i_\sigma + \frac{1}{a_\sigma}\sum_{\tau\in\sigma_+} a_\tau v^i_\tau \leq w^i_\sigma + h^i_\sigma.$$

He then trades in commodities against balances, subject to a cash-in-advance constraint,

$$\frac{1}{a_\sigma} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq m^i_\sigma.$$

At each successor date-event, $\tau \in \sigma_+$, his nominal wealth is given by the deliveries of the portfolio plus balances received from transactions in commodities,

$$w^i_\tau = v^i_\tau + m^i_\sigma - \frac{1}{a_\sigma} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right).$$

The effect of the cash-in-advance constraint is that proceedings from sales of commodities do not earn an interest, as they must be carried over in the form of balances. A traditional wealth constraint,

$$-\frac{1}{a_\tau}\sum_{\nu\succeq\tau} a_\nu h^i_\nu - \frac{1}{a_\tau}\sum_{\nu\succeq\tau}\left(\frac{1}{1+r_\nu}\right) p_\nu \cdot e^i_\nu \leq w^i_\tau,$$

allows the individual to contract any amount of nominal debt that can be honored in finite time. Finally, the initial nominal wealth, w^i_ϕ, is predetermined.

Under well-know conditions, this sequence of constraints reduces to a single intertemporal budget constraint of the form

$$\sum_{\sigma\succeq\phi} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq \delta^i + \sum_{\sigma\succeq\phi}\left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^-,$$

where

$$\delta^i = a_\phi w^i_\phi + \sum_{\sigma\succeq\phi} a_\sigma h^i_\sigma.$$

The associated demand for balances, m^i, is given by any positive real map on $\mathcal{S}$ that satisfies, at every date-event,

$$p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq a_\sigma m^i_\sigma,$$
$$\left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ = \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma.$$

Indeed, when nominal interest vanishes, balances and one-period safe bonds become perfect substitutes for intertemporal transfers of wealth.

As the asset market is sequentially complete, the intertemporal budget constraint depends only on present value prices of commodities, p, nominal rates of interest, r, and outside nominal claims, δ^i, consisting of an initial predetermined nominal wealth, w^i_ϕ, and the present value of nominal transfers, h^i. State prices affect trade opportunities directly only through outside nominal claims, δ^i.

2.5 Equilibrium

An *equilibrium* consists of a balanced allocation, x, prices, p, state prices, a, a supply of balances, m, a positive real map on $\mathcal{S}$, public transfers, h, a real map on $\mathcal{S}$, public liabilities, w, a real map on $\mathcal{S}$, and public bonds, b, a real map on $\mathcal{S}$, such that the following conditions are satisfied:

(a) The consumption plan of every individual is optimal subject to the single intertemporal budget constraint, so that

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^-,$$

while

$$z^i \succ^i x^i \quad \Rightarrow$$

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^+ > \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^-,$$

where

$$\delta^i = \nu^i \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^-;$$

(b) money markets clear, so thet, at every date-event,

$$p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- \leq a_\sigma m_\sigma,$$
$$\left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma;$$

(c) asset markets clear, so that, at every date-event, [2]

$$\sum_{\tau \succeq \sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau m_\tau = a_\sigma w_\sigma + \sum_{\tau \succeq \sigma} a_\tau h_\tau;$$

(d) public liabilities consist of balances and bonds only, so that, at every date-event,

$$w_\tau = m_\sigma + b_\sigma, \quad \tau \in \sigma_+.$$

To avoid the distributional effects of the overall price level, for every individual, $\delta^i = \nu^i \delta$, where $\delta = \sum_i \delta^i$. This reflects a primitive assumption of proportionality between initial nominal wealth, $w^i_\phi = \nu^i w_\phi$, and transfers, $h^i = \nu^i h$. In a more general formulation, the distribution of outside claims across individuals, $(\ldots, \delta^i, \ldots)$, would vary with the overall price level.

An equilibrium is said to be *fundamental* if, at every date-event,

$$a_\sigma p_\sigma \cdot \sum_i \left(x^i - e^i \right)^- = a_\sigma m_\sigma;$$

at a fundamental equilibrium, balances are held for transaction purposes only. An equilibrium is said to be *with trade* if, at every date-event,

$$p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma \right)^- > 0.$$

Condition (c), in the definition of equilibrium, implies that,at every date-event,

$$\left(\frac{r_\sigma}{1 + r_\sigma} \right) m_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau w_\tau = w_\sigma + h_\sigma,$$

which can be interpreted as the sequential public budget constraint. From condition (d), it follows that

$$w_\tau = w_\sigma + h_\sigma + r_\sigma \left(w_\sigma + h_\sigma - m_\sigma \right), \quad \tau \in \sigma_+$$

or, alternatively,

$$\left(\frac{1}{1 + r_\sigma} \right) b_\sigma = w_\sigma + h_\sigma - m_\sigma, \quad \tau \in \sigma_+.$$

At a date-event, a public authority supplies balances, m_σ, issues transfers, h_σ, and supplies bonds, b_σ, so as to cover inherited public liabilities, w_σ, consisting of matured bonds and pre-existing balances.

[2] To simplify presentation, the fact that $\sum_{\tau \succeq \sigma} a_\tau h_\tau$ is well-defined (it converges) is part of the definition of an equilibrium.

3 Alternative monetary policies

Different hypotheses on the conduct of monetary policy and on the nature of public liabilities correspond to different sections of the equilibrium set, E, subject, possibly, to additional restrictions. The classification that follows, even if neither exhaustive nor the most general, captures relevant aspects of several monetary doctrines.

Equilibria *with balanced transfers*, E_w, satisfy the additional restriction $w = 0$. This implies that, at every date-event,

$$h_\sigma = \left(\frac{r_\sigma}{1 + r_\sigma}\right) m_\sigma.$$

Such equilibria are those studied by Drèze and Polemarchakis [10]. A central bank runs a balanced budget by redistributing its profit to individuals according to given shares. There are no public liabilities, as balances and public bonds are balanced, $m+b = 0$. At an equilibrium with balanced transfers, conditions (c)-(d) are trivially satisfied, and they do not pose any restriction beyond conditions (a)-(b).

Equilibria *with canonical transfers*, E_b, satisfy the additional restriction $b = 0$. This implies that $m = w + h$ and, thus, at every date-event,

$$m_\tau = m_\sigma + h_\tau, \quad \tau \in \sigma_+.$$

The money supply varies only by direct transfers to individuals. There are no public bonds, $b = 0$, though public liabilities correspond to the outstanding stock pre-exiting balances at all date-events, $w = m - h$.

Equilibria *without transfers*, E_h, are those that satisfy the additional restrictions $h = 0$. Under this assumption, public liabilities evolve so as to satisfy, at every date-event,

$$w_\tau = w_\sigma - r_\sigma\left(m_\sigma - w_\sigma\right), \quad \tau \in \sigma_+$$

or, alternatively,

$$\left(\frac{1}{1 + r_\sigma}\right) b_\sigma = w_\sigma - m_\sigma.$$

This is the framework privileged by the Fiscal Theory of the Price Level, which interprets w_ϕ as an initial predetermined public liability. At equilibrium, a given initial public liability, w_ϕ, must be covered by intertemporal seignorage,

$$\sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma}\right) a_\sigma m_\sigma = a_\phi w_\phi.$$

Interestingly, equilibria without transfers also coincide with (our reconstruction of) those of Dubey and Geanakoplos [11] extended to an infinite horizon. The initial public liability, w_ϕ, is there interpreted as a pre-existing stock of outside money; over time, w is outside money, whereas $n = m - w$ is inside money; inside money is backed by bonds issued by the central bank.

One may consider $E(r)$ or $Et(m)$, the set of equilibria with a given path of nominal rates of interest, r, or of the money supplies, m, that corresponds to the monetary policy that pegs the interest rate or the money supply. This can be coupled with hypotheses on the conduct of monetary policy; for instance, $E_h(r, w_\phi)$ is the set of equilibria without transfers, a given initial public liability, w_ϕ, and given path of nominal rates of interest, r, as prescribed by the Fiscal Theory of Price Determination.

4 Issues of existence

Classical issues of existence of a monetary equilibrium can be understood through (the intertemporal analogue of) Walras' Law,

$$\sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) p_\sigma \cdot \sum_i \left(x_\sigma^i - e_\sigma^i \right)^- = \delta.$$

This condition establishes that intertemporal transaction costs exhaust aggregate outside claims, δ, independently of how such claims are attributed to individuals (initial wealths and/or transfers). The validity of Walras' Law follows from the assumption of a finite set of impatient individuals, so that it would fail in overlapping generations economies. Be it an identity or an equilibrium restriction, as currently disputed, it helps to organize one's thoughts about the failure of existence of an equilibrium.

The essence of the problem originally posed by Hahn [16], in this perspective, reduces to the requirement that, at equilibrium, there be trade at some date-event or, equivalently, that $\delta > 0$.[3] This would not happen if the initial allocation is Pareto efficient. Even if trade allows for a welfare improvement, individuals might not be able to benefit from trade, as the latter is subject to costs associated with nominal rates of interest. As initially well understood by Dubey and Geanakoplos [11], $\delta > 0$ requires that gains to trade dominate transaction costs.

Proposition 1 (Gains to trade) *An equilibrium involves trade at some date-event, $\delta > 0$, whenever the initial allocation, $(\ldots, e^i, \ldots)$, is weakly Pareto dominated by an allocation, $(\ldots, z^i, \ldots)$, that satisfies, at every date-event,*

$$\sum_i \left(z_\sigma^i - e_\sigma^i \right)^+ \leq \left(\frac{1}{1 + r_\sigma} \right) \sum_i \left(z_\sigma^i - e_\sigma^i \right)^- .$$

The condition in Proposition 1 involves an evaluation of gains to trade at given nominal rates of interest, and, as a consequence, it is immediately applicable under an interest rate peg. For a money supply policy, it requires some elaboration. Needless to notice, in monetary economies with a representative individual, such a condition is implied by assumptions on marginal utilities at the autarchy.

[3] A further elaboration of the basic argument presented here accounts for trade occurring at all date-events.

The need for gains to trade for the existence of a monetary equilibrium is not related to the length of the time horizon, finite or infinite. A finite horizon, however, reveals an additional inconsistency, which has been erroneously identified with the Hahn problem. Indeed, suppose that the supply of balances is constant, $m = (\dots, \mu, \dots)$, and there is no intervention by the public authority, so that $w = (\dots, \mu, \dots)$ and $h = 0$. This corresponds to the most classical case studied in monetary economics. Walras' Law requires that

$$\sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \mu = a_\phi \mu.$$

Truncating the economy at some date t (and using date $t + 1$ for clearing debts across individuals only), simple accounting yields

$$\sum_{\sigma \succeq^t \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \mu < a_\phi \mu = \sum_{\sigma \succeq^t \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \mu + \sum_{\sigma \succeq_{t+1} \phi} a_\sigma \mu,$$

where $\sigma \succeq^t \phi$ ($\sigma \succeq_t \phi$) means $\sigma \succeq \phi$ and $t_\sigma \leq t$ ($t_\sigma = t$). It is, thus, evident that Walras' Law cannot be satisfied over a finite horizon, independently of gains to trade, unless negative transfers (taxes) are introduced (Lerner [17]). This requires

$$\sum_{\sigma \succeq^t \phi} a_\sigma h_\sigma = - \sum_{\sigma \succeq_{t+1} \phi} a_\sigma \mu = - \left(a_\phi - \sum_{\sigma \succeq^t \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \right) \mu.$$

Interestingly, with non-vanishing nominal rates of interest, it is unnecessary to tax away the entire amount of balances at the initial date, as such a value is progressively eroded by transaction costs. With vanishing nominal rates of interest, taxing away the entire amount of balances at some date would be necessary also on an infinite horizon, an obvious fact that has been obscured by the focus of analysis on overlapping generations economies.

Dubey and Geanakoplos [11] avoid the inconsistency over a finite horizon by distinguishing between outside money, $\nu > 0$, and inside money, $\mu - \nu > 0$. Outside money is held by individual at the initial date and there are no other transfers. No inconsistency arises as Walras' Law now only requires

$$\sum_{\sigma \succeq^t \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \mu = a_\phi \nu > \sum_{\sigma \succeq^t \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \nu.$$

Drèze and Polemarchakis [10] avoid the inconsistency by interpreting ν as the intertemporal seignorage that is transferred to individuals, so that Walras' Law,

$$\sum_{\sigma \succeq^t \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \mu = a_\phi \nu,$$

becomes an identity. In either institutional framework, no inconsistency arises over a finite horizon, though consequences for the determinacy of equilibria do differ.

5 Non-neutrality of money

Interest in monetary equilibria derives, mostly, from the possible effectiveness of monetary policy. In our framework, monetary policy, broadly interpreted, can be of allocative relevance only insofar as it affects nominal rates of interest, as different nominal rates, in general, support different equilibrium allocations. This is a consequence of our assumption of a complete asset market, along with the simplification that the overall price level entails no distributive effects. Both assumptions only serve to simplify the exposition and focus on issues proper to monetary economies.

Proposition 2 (Effectiveness of monetary policy) *At a fundamental equilibrium with trade, at every date-event,*

$$\left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \sum_{\tau \in \sigma_+} p_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- \frac{m_\sigma}{m_\tau}.$$

Variations in the rates of growth of the supply of balances, up to their ‘expected’ value, are associated with changes in variables of allocative relevance, such as relative prices, nominal interest and net trades. The condition in Proposition 2 is the equivalent of the characterization of equilibrium in terms of first-order conditions in a cash-in-advance economy with a representative individual.

6 Degrees of multiplicity

We want to characterize the degree of multiplicity of equilibria. To this purpose, we uniquely decompose state prices into nominal rates of interest and a measure on the underlying state space. We then introduce a notion of abstract equilibrium. This reformulation of equilibrium conditions suggests the existence of a multiplicity of abstract equilibria indexed by the measures on the state space. Abstract equilibria are then mapped into equilibria with balanced transfers and without transfers, respectively, with obvious consequences for their multiplicity.

An equivalent measure (with some abuse of terminology) is any strictly positive real map, π, on $\mathcal{S}$ such that, at every date-event,

$$\pi_\sigma = \sum_{\tau \in \sigma_+} \pi_\tau.$$

As usual, π can be interpreted as a measure on states of nature, by letting π_σ be the measure of the (measurable) set $\{\tau \in \mathcal{S} : \tau \succeq \sigma\}$. Here, ‘equivalent’ is only to suggest that such a measure assigns a strictly positive weight to every date-event.

Given nominal rates of interest, r, a correspondence between arbitrage-free state prices and equivalent measures is given, at every date-event, by

$$a_\sigma = \psi_\sigma(r)\, \pi_\sigma,$$

where $\psi_\phi(r) = 1$ and, at every date-event,

$$\psi_\tau(r) = \left(\frac{1}{1+r_\sigma}\right) \psi_\sigma(r), \quad \tau \in \sigma_+.$$

In the decomposition, $\psi(r)$ represents intertemporal changes in nominal values, whereas π captures their dispersion across date-events.

Given nominal rates of interest, r, and prices, p, variations in equivalent measures translate into variations in levels of current prices. Rates of inflation, i, measured in the aggregate endowment, e, are given, at every date-event, by

$$1 + i_\tau = \frac{a_\sigma}{a_\tau} \frac{p_\tau \cdot e_\tau}{p_\sigma \cdot e_\sigma} = (1 + r_\sigma) \frac{\pi_\sigma}{\pi_\tau} \frac{p_\tau \cdot e_\tau}{p_\sigma \cdot e_\sigma}, \quad \tau \in \sigma_+.$$

Hence, π captures the variability of rates of inflation across date-events, whereas expected inflation is determined by the no-arbitrage condition, consistently with Fisher Equation,

$$\sum_{\tau \in \sigma_+} (1 + i_\tau) \frac{\pi_\tau}{\pi_\sigma} = (1 + r_\sigma) \sum_{\tau \in \sigma_+} \frac{p_\tau \cdot e_\tau}{p_\sigma \cdot e_\sigma} = \frac{1 + r_\sigma}{1 + \rho_\sigma},$$

where ρ represents real rates of interest, measured at the aggregate endowment.

An *abstract equilibrium* consists of a balanced allocation, x, prices, p, nominal rates of interest, r, a supply of balances, m, and an equivalent measure, π, such that the following conditions are satisfied:

(α) The consumption plan of every individual is optimal subject to the single intertemporal budget constraint, so that

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1 + r_\sigma}\right) p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^-,$$

while

$$z^i \succ^i x^i \quad \Rightarrow$$

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^+ > \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1 + r_\sigma}\right) p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^-,$$

where

$$\delta^i = \nu^i \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^-;$$

(β) money markets clear, so that, at every date-event,

$$p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- \leq \psi_\sigma(r) \pi_\sigma m_\sigma,$$

$$\left(\frac{r_\sigma}{1 + r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \left(\frac{r_\sigma}{1 + r_\sigma}\right) \psi_\sigma(r) \pi_\sigma m_\sigma.$$

It is clear that restrictions (α)-(β) correspond to restrictions (a)-(b), with the only difference that state prices are decomposed into nominal rates of interest and an equivalent measure. They serve to show that equivalent measures account for all degrees of indeterminacy.

Proposition 3 (Fundamental indeterminacy) *The set of equilibria with balanced transfers coincide (up to identifications of no allocative relevance) with the set of abstract equilibria.*

A direct consequences of Proposition 3 is that, under a policy that sets interest rates, at an equilibrium with balanced transfers, state prices are completely indeterminate, up to no arbitrage. These degrees of multiplicity are fully captured by equivalent measures. A policy that sets the supply of balances partly obscures an analogous conclusion. There are technical issues of existence of an equilibrium that we do not enter into. However, when an equivalent measure π is given, there is no over-determination in conditions (α)-(β).

A complementary result occurs without transfers. Indeed, the hypothesis of no transfers poses additional equilibrium restrictions that typically allow for a full determination of equivalent measure.

Proposition 4 (The fiscal theory of the price level) *The set of equilibria without transfers coincides (up to identifications of no allocative relevance) with the set of abstract equilibria that satisfy, for some public liabilities, w, the additional restrictions given, at every date-event, by*

$$\psi_\sigma(r)\,\pi_\sigma w_\sigma = \omega_\sigma$$

and

$$\psi_\tau(r)\,\pi_\sigma w_\tau = \sum_{\tau\in\sigma_+}\omega_\tau, \qquad \tau\in\sigma_+,$$

where, at every date-event,

$$\omega_\sigma = \sum_{\tau\succeq\sigma}\left(\frac{r_\tau}{1+r_\tau}\right)p_\tau\cdot\sum_i\left(x_\tau^i - e_\tau^i\right)^- .$$

A full determination of equivalent measure, π, is obtained when outside claims, ω, are strictly positive. Indeed, one easily verifies that, in this case, the additional restrictions in Proposition 4 exactly peg one equivalent measure, for a given strictly positive initial value, w_ϕ, of public liabilities.

7 Canonical transfers

Different hypotheses on transfers reflect different institutional frameworks. Most of these differences are rhetorical. At a fundamental equilibrium with balanced transfers, at every date-event,

$$\sum_{\tau\succeq\sigma}\left(\frac{r_\tau}{1+r_\tau}\right)p_\tau\cdot\sum_i\left(x_\tau^i - e_\tau^i\right)^- = m_\sigma + \frac{1}{a_\sigma}\sum_{\tau\succeq\sigma}\sum_{\nu\in\tau_+}a_\nu\left(m_\nu - m_\tau\right).$$

This simple observation establishes that the set of fundamental equilibria with balanced transfers and with canonical transfers basically coincide.

Proposition 5 (Canonical transfers) *The set of fundamental equilibria with balanced transfers coincides (up to identifications of no allocative relevance) with the set of fundamental equilibria with canonical transfers (and no initial public liabilities).*

Implications of Proposition 5 are of some interest. As a matter of mere fact, given a money supply, m, a fundamental equilibrium exists with balanced transfers if and only if it exists with canonical transfers. Fundamental equilibria with balanced transfers can then be regarded as auxiliary to the study of equilibria with canonical transfers. In particular, the latter inherit all the degrees of indeterminacy exhibited by the former.

8 Fundamental value of money

We here propose a classical argument for money be priced at its fundamental value at equilibrium (Santos [23]).

Proposition 6 (Fundamental value of money) *If money supply is uniformly strictly positive, at a fundamental equilibrium,*

$$a_\sigma = \sum_{\tau \succeq \sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau$$

at every date-event.

Using the aggregate endowment, e, as numéraire, current values of money, q, and real state prices, α, satisfy, at every date-event, σ, $\alpha_\sigma q_\sigma = a_\sigma$ and $\alpha_\sigma = p_\sigma \cdot e_\sigma$. No arbitrage imposes, at every date-event,

$$\alpha_\sigma q_\sigma = a_\sigma = \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma + \sum_{\tau \in \sigma_+} a_\tau = \left(\frac{r_\sigma}{1 + r_\sigma} \right) \alpha_\sigma q_\sigma + \sum_{\tau \in \sigma_+} \alpha_\tau q_\tau .$$

Thus, once its real dividends are properly identified with nominal rates of interest, money is priced as any other long-term asset. At a fundamental equilibrium,

$$q_\sigma = \frac{1}{\alpha_\sigma} \sum_{\tau \succeq \sigma} \left(\frac{r_\sigma}{1 + r_\sigma} \right) \alpha_\tau q_\tau ,$$

so that money is priced at its fundamental value.

9 Inside and outside money

It is interesting to remark a consequence of the fact that money is priced at its fundamental value.

Proposition 7 (Inside and outside money) *If money supply is uniformly strictly positive, at a fundamental equilibrium without transfers, at every date-event,*

$$\sum_{\tau \succeq \sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau \left(m_\tau - w_\sigma \right) = 0.$$

Implications of Proposition 7 deserve a brief discussion. Suppose that, following Dubey and Geanakoplos [11], one interprets w as outside money and m as the total supply of money across date-events. The supply of money is varied only by issuing bonds, so that there are no direct transfers to individuals. If, say, the supply of balances is constant, $m = (\dots, \mu, \dots)$, and there is initial inside money, $w_\phi = \nu$, at a fundamental equilibrium,

$$\sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma \left(\mu - \nu \right) = 0.$$

It follows that either nominal rates of interest vanish everywhere or $\mu = \nu$. It is unclear how the interplay between inside and outside money should determine the nominal interest at equilibrium, as suggested by the analysis of Dubey and Geanakoplos [11] over a finite horizon.

10 Efficiency

An equilibrium fails Pareto efficiency. Also, in general, a Pareto efficient allocation cannot be supported as an equilibrium for a given supply of balances or for given nominal rates of interest. Thus, neither of the Welfare Theorems holds.

Proposition 8 (Limited efficiency) *An equilibrium allocation,* $(\dots, x^i, \dots)$*, is not Pareto dominated by an alternative allocation,* $(\dots, z^i, \dots)$*, satisfying, at every date-event,*

$$\sum_i z_\sigma^i \leq \left(\frac{1}{1 + r_\sigma} \right) \sum_i e_\sigma^i .$$

Hence, nominal rates of interest capture the displacement from a Pareto optimal allocation of resources. In particular, in the case of purely extrinsic (or sunspot) uncertainty, Proposition 8 provides bounds on the variability of the allocation, when nominal rates of interest are pegged by a monetary authority.

We are not aware of any notion of constrained efficiency suitable for the analysis of monetary economies.

11 Extensions

We here present notions of abstract equilibrium that are suitable for the analysis of monetary economies with, respectively, incomplete markets and overlapping generations. As this is only meant to be suggestive, we largely omit details.

11.1 Incomplete markets

It simplifies to assume that the asset structure is given by nominal one-period assets and that a risk-less nominal bond can be traded at every date-event. Under this assumption, a dual representation of the asset structure can be given in terms of equivalent measures. Write $\pi \sim \pi'$ whenever $\pi_\phi = \pi'_\phi$ and both equivalent measures give the same market value to all portfolios, that is, at a date-event, σ, for all revenues generated by some tradable portfolio, $(v_\tau : \tau \in \sigma_+)$,

$$\sum_{\tau \in \sigma_+} \frac{\pi_\tau}{\pi_\sigma} v_\tau = \sum_{\tau \in \sigma_+} \frac{\pi'_\tau}{\pi'_\sigma} v_\tau,$$

Markets are incomplete when such an equivalence relation does not coincide with the identity.

An *abstract equilibrium* consists to a balanced allocation, x, prices, p, nominal rates of interest, r, a supply of balances, m, and an equivalent measure, π, such that the following conditions are satisfied:

(α') The consumption plan of every individual is optimal subject to the single intertemporal budget constraint, so that, for every $\pi' \sim \pi$,

$$\sum_{\sigma \succeq \phi} \frac{\pi'_\sigma}{\pi_\sigma} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1+r_\sigma}\right) \frac{\pi'_\sigma}{\pi_\sigma} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^-,$$

and, for some $\pi' \sim \pi$,

$$z^i \succ^i x^i \quad \Rightarrow$$

$$\sum_{\sigma \succeq \phi} \frac{\pi'_\sigma}{\pi_\sigma} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^+ > \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1+r_\sigma}\right) \frac{\pi'_\sigma}{\pi_\sigma} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^-,$$

where[4]

$$\delta^i = \nu^i \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^-;$$

[4] To simplify, we assume that transfers are distributed to individuals so as not to affect their intertemporal trading opportunities; that is so as to lie in the span of the asset structure.

(β') money markets clear, so that, at every date-event,

$$p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- \leq \psi_\sigma(r)\,\pi_\sigma m_\sigma,$$

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \left(\frac{r_\sigma}{1+r_\sigma}\right) \psi_\sigma(r)\,\pi_\sigma m_\sigma.$$

Apparently, incomplete markets do not alter the main conclusions on degrees of multiplicity of equilibria. However, equivalent measures do, now, entail additional effects of real allocative relevance, as evident from condition (α').

11.2 Overlapping generations

The main modification required to encompass overlapping generations economy (that is, a non-necessarily finite set of individuals) lies in a potential failure of intertemporal Walras' Law.

An *abstract equilibrium* consists to a balanced allocation, x, prices, p, nominal rates of interest, r, a supply of balances, m, a distribution of outside claims across individuals, $(\ldots, \delta^i, \ldots)$, and an equivalent measure, π, such that the following conditions are satisfied:

(α'') The consumption plan of every individual is optimal subject to the single intertemporal budget constraint, so that

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ \leq \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^-,$$

and

$$z^i \succ^i x^i \Rightarrow$$

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^+ > \delta^i + \sum_{\sigma \succeq \phi} \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^-;$$

(β'') money markets clear, so that, at every date-event,

$$p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- \leq \psi_\sigma(r)\,\pi_\sigma m_\sigma,$$

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \left(\frac{r_\sigma}{1+r_\sigma}\right) \psi_\sigma(r)\,\pi_\sigma m_\sigma.$$

In general, one cannot draw the conclusion that

$$\sum_{\sigma \succeq \phi} p \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \sum_i \delta^i = \delta.$$

If intertemporal seignorage (the left-hand side above) is finite, equilibria might exhibit positive (negative) debt whenever δ exceeds (is exceeded by) the intertemporal value of seignorage. A clear distinction between money and debt can be made (Rochon and Polemarchakis [21]).

Appendix

Proofs

Proof of Proposition 1. If not, $\delta = 0$, the equilibrium allocation, $(\ldots, x^i, \ldots)$, coincides with the initial allocation, $(\ldots, e^i, \ldots)$. As $z^i \succ^i e^i$ for every individual,

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^+ > \sum_{\sigma \succeq \phi} \left(\frac{1}{1 + r_\sigma}\right) p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^-$$

or

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^+ > \sum_{\sigma \succeq \phi} p_\sigma \cdot \left(\frac{1}{1 + r_\sigma}\right) \left(z^i_\sigma - e^i_\sigma\right)^- .$$

Summing over individuals, one obtains

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \sum_i \left(z^i_\sigma - e^i_\sigma\right)^+ > \sum_{\sigma \succeq \phi} p_\sigma \cdot \left(\frac{1}{1 + r_\sigma}\right) \sum_i \left(z^i_\sigma - e^i_\sigma\right)^- ,$$

which is a contradiction as p is positive. □

Proof of Proposition 2. It follows from the fact that, at every date-event,

$$\left(\frac{1}{1 + r_\sigma}\right) a_\sigma = \sum_{\tau \in \sigma_+} a_\tau$$

and

$$a_\sigma = \frac{1}{m_\sigma} p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- .$$

The second condition is meaningful as we restrict attention to fundamental equilibria with trade. □

Proof of Proposition 3. It follows from the obvious identifications given by

$$a_\sigma = \psi_\sigma (r) \pi_\sigma$$

and

$$h_\sigma = \frac{1}{\psi_\sigma (r) \pi_\sigma} \left(\frac{r_\sigma}{1 + r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- ,$$

at every date-event. □

Proof of Proposition 4. Market clearing for assets requires, at every date-event, σ,

$$\psi_\sigma (r) \pi_\sigma w_\sigma = \omega_\sigma ,$$

where ψ is constructed given nominal rates of interest, r, as explained above. In addition, the fact that public liabilities consist of balances and bonds only implies

$$\psi_\tau(r)\,\pi_\sigma w_\tau = \omega_\sigma - \left(\frac{r_\sigma}{1+r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- = \sum_{\tau\in\sigma_+} \omega_\tau, \text{ for all } \tau \in \sigma_+.$$

When outside claims, ω, are strictly positive, such restrictions jointly imply, at every date-event, σ,

$$\pi_\tau = \pi_\sigma \frac{\omega_\tau}{\sum_{\tau\in\sigma_+}\omega_\tau}, \text{ for all } \tau \in \sigma_+.$$

Since $w_\phi > 0$ is given, the above condition pegs π_ϕ and, hence, state prices, a, and public liabilities, w, are uniquely determined. □

Proof of Proposition 5. The crucial observation is that, at equilibrium,

$$\sum_{\sigma\succeq\phi} p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^+ = \sum_{\sigma\succeq\phi} p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^-$$

is finite. This follows from the aggregation of budget constraints across individuals and market clearing for commodities. Hence, at a fundamental equilibrium, using market clearing for balances, one obtains, at every date-event,

$$\sum_{\tau\succeq\sigma} p_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- = \sum_{\tau\succeq\sigma} a_\tau m_\tau.$$

In addition, notice that

$$\begin{aligned}\sum_{\tau\succeq^t\sigma}\left(\frac{r_\tau}{1+r_\tau}\right) a_\tau m_\tau &= \sum_{\tau\succeq^t\sigma}\left(\frac{r_\tau}{1+r_\tau}\right) p_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- \\ &= a_\sigma m_\sigma + \sum_{\tau\succeq^t\sigma}\sum_{\nu\in\tau_+} a_\nu\left(m_\nu - m_\tau\right) - \sum_{\tau\succeq_t\sigma}\sum_{\nu\in\tau_+} a_\nu m_\nu,\end{aligned}$$

where $\tau \succeq^t \sigma$ ($\tau \succeq_t \sigma$) means $\tau \succeq \sigma$ and $t_\tau - t_\sigma \leq t$ ($t_\tau - t_\sigma = t$). As the equilibrium is fundamental,

$$\lim_{t\to\infty}\sum_{\tau\succeq_t\sigma}\sum_{\nu\in\tau_+} a_\nu m_\nu = 0,$$

which proves that, at every date-event,

$$\sum_{\tau\succeq\sigma}\left(\frac{r_\tau}{1+r_\tau}\right) p_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- = m_\sigma + \sum_{\tau\succeq\sigma}\sum_{\nu\in\tau_+} a_\nu\left(m_\nu - m_\tau\right).$$

To prove the claim, simply define $h_\phi = m_\phi$ and, at every date-event,

$$h_\tau = m_\tau - m_\sigma, \text{ for all } \tau \in \sigma_+.$$

Also, let $w = m - h$ and $b = 0$. It is then clear that, at every date-event,

$$\sum_{\tau \succeq \sigma} \left(\frac{r_\sigma}{1 + r_\sigma}\right) a_\tau m_\tau = a_\sigma m_\sigma + \sum_{\tau \succ \sigma} a_\tau h_\tau = a_\sigma w_\sigma + \sum_{\tau \succeq \sigma} a_\sigma h_\sigma.$$

The conclusive argument is then obvious. □

Proof of Proposition 6. Observe that, at every date-event,

$$a_\sigma = \sum_{\tau \succeq^t \sigma} \left(\frac{r_\tau}{1 + r_\tau}\right) a_\tau + \sum_{\tau \succeq_{t+1} \sigma} a_\tau,$$

where $\tau \succeq^t \sigma$ ($\tau \succeq_t \sigma$) means $\tau \succeq \sigma$ and $t_\tau - t_\sigma \leq t$ ($t_\tau - t_\sigma = t$). So, it suffices to show that

$$\lim_{t \to \infty} \sum_{\tau \succeq_{t+1} \sigma} a_\tau = 0.$$

At a fundamental equilibrium with money supply $m \geq \mu > 0$, at every date-event,

$$a_\sigma \mu \leq a_\sigma m_\sigma = p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^-$$

Hence,

$$\lim_{t \to \infty} \sum_{\tau \succeq_{t+1} \sigma} a_\tau \leq \lim_{t \to \infty} \frac{1}{\mu} \sum_{\tau \succeq_{t+1} \sigma} a_\tau m_\tau = \lim_{t \to \infty} \frac{1}{\mu} \sum_{\tau \succeq_{t+1} \sigma} p_\tau \cdot \left(x^i_\tau - e^i_\tau\right)^- = 0,$$

which proves the claim. □

Proof of Proposition 7. Observing that

$$\sum_{\tau \succeq \sigma} \left(\frac{r_\tau}{1 + r_\tau}\right) a_\tau m_\tau - a_\sigma w_\sigma = 0$$

and

$$\sum_{\tau \succeq \sigma} \left(\frac{r_\tau}{1 + r_\tau}\right) a_\tau w_\sigma - a_\sigma w_\sigma = 0,$$

the claim easily proves to be true. □

Proof of Proposition 8. Suppose not. It follows that, for every individual, i,

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot z^i_\sigma + \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma}\right) p_\sigma \cdot e^i_\sigma \geq$$

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot z^i_\sigma + \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma}\right) p_\sigma \cdot \left(z^i_\sigma - e^i_\sigma\right)^- \geq$$

$$\nu^i \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma}\right) p_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma \succeq \phi} p_\sigma \cdot e^i_\sigma \geq \sum_{\sigma \succeq \phi} p_\sigma \cdot e^i_\sigma,$$

with at least one strict inequality. Summing over individuals,

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \sum_i z^i_\sigma + \sum_{\sigma \succeq \phi} p_\sigma \cdot \left(\frac{r_\sigma}{1 + r_\sigma} \right) \sum_i e^i_\sigma =$$

$$\sum_{\sigma \succeq \phi} p_\sigma \cdot \sum_i z^i_\sigma + \sum_{\sigma \succeq \phi} \left(\frac{r_\sigma}{1 + r_\sigma} \right) p_\sigma \cdot \sum_i e^i_\sigma > \sum_{\sigma \succeq \phi} p_\sigma \cdot \sum_i e^i_\sigma,$$

which is a contradiction. □

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