# Monetary Equilibria*

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**Abstract.** The introduction of banks that issue money and supply balances and pay out their profits as dividends is the natural modification of the model of general competitive equilibrium that encompasses monetary economies with an operative transactions technology. Monetary policy sets nominal rates of interest and accommodates the demand for balances; alternatively, it sets the supply of balances and rates of interest adjust for money markets to clear. Competitive equilibria exist. Under uncertainty, monetary policy fails to determine the distribution of the rate of inflation or the allocation of resources at equilibrium. If, in addition to rates of interest, monetary policy sets the prices of contingent loans subject to no-arbitrage constraints, or targets the distribution of the terminal level of prices, it lifts the multiplicity.

## 1 Introduction and preview

The almost universal prevalence of monetary exchange and the central place of monetary policy in macroeconomic analysis provide clear incentives to extend the canonical general equilibrium model of Arrow-Debreu-McKenzie (1954) to a monetary economy. The concern dates back to Walras (1902); it was revived by Patinkin (1965) and led to a number of contemporary formulations.

This paper develops a formulation that retains the generality of the canonical model under the simplest possible extension that captures the essentials of monetary exchange.

The formulation proceeds from the following premises:

1. Non-interest-bearing fiat money is dominated by interest-bearing nominal assets as a store of value; the demand for money accordingly results from its role in facilitating transactions: "money buys goods."
2. Fiat money is created at no cost by banks that lend it to individuals and firms at nominal non-negative interest rates.
3. Banks keep balanced accounts, so that outstanding money is the counterpart of assets, claims on individuals and firms; it is "inside money."
4. Banks are owned; profits, equal to interest earned on assets, accrue to shareholders.

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Premise (1) assigns a central role to modeling the transactions technology that underlies the demand for money. Some, notably Ostroy and Starr (1974) or, more recently, Kyotaki and Wright (1989), have looked upstream of market institutions, in order to explain why most transactions take the form of exchanges of goods for money. It is sufficient to take this commonplace observation as a factual starting point. Premise (2) is a simplification that neglects the operating costs of banks; it is a useful first approximation. Taking into account operating costs could lead to models of competitive money supply, not introduced here. Premise (3) rules out "outside money," the focus of much monetary theory; it obviates the artificial conundrum, pointed out by Hahn (1965), that money could not maintain a positive price at equilibrium. It allows for a finite horizon, and it does not invoke overlapping generations, temporary equilibrium, or infinitely lived individuals. Still, the results extend without qualitative modification to infinite horizons, in Bloise, Drèze and Polemarchakis (2000). Importantly, premise (4) preserves Walras' law; failure to adopt that premise has led authors like Dubey and Geanakoplos (1992, 2000) to different conclusions on the multiplicity of equilibria.

The formulation proceeds from the following modeling options:

1. Time is divided into a finite number of "periods," of arbitrary length. Within a period, prices of commodities and rates of interest are constant, but interest accounting on monetary transactions is continuous, with no compounding within the period; for accounting purposes, values are expressed as of the beginning of the period. Analytically, periods are "dates" or points in time.
2. Uncertainty is described, following the canonical model and Debreu (1960), by an event-tree.
   In an abstract economy, an analytically convenient if artificial construct, the event-tree is only implicit.
3. The government is not a separate agent. Thus, a central bank is also owned by individuals.
4. There are no initial positions in monetary claims or assets; individuals have no initial holdings of balances or nominal claims.
5. In line with commodities, money at distinct date events defines distinct moneys, yielding distinct transaction services. There can be multiple currency areas, though the interpretation is not spelled out formally.

To anticipate, the following properties hold at equilibrium:

1. In a single-currency-area economy that extends over $t = 1, \dots, T$ periods of time under certainty, once nominal rates of interest are set, there remains 1 degree of nominal multiplicity, corresponding to the overall or, alternatively, the initial or terminal level of prices; nominal interest rates have real effects.
2. In a single-currency-area economy that extends over $T$ periods of time under uncertainty, with an event-tree that consists of $N$ nodes $s_t$, of

which $S$ are terminal: $N - T + 1 \geq S \geq 1$, if the $N$ rates of interest, $r_{s_t}$, are set, there remain an additional $S$ degrees of nominal multiplicity; one simple statement is that the $S$ terminal price levels could be selected arbitrarily, without affecting the allocation of resources at equilibrium; an alternative statement is that monetary policy can control expected inflation, but not variability of inflation.

3. In an economy, not formally studied here, with $H$ currencies, that extends over an event-tree with $N$ nodes, of which $S$ are terminal, if $HN$ rates of interest, $r_{h,s_t}$, are set, there remain $HS$ degrees of multiplicity, $H$ — fold the number that characterizes a single currency area; exchange rates could be selected arbitrarily, without affecting the allocation of resources at equilibrium; yet, uncovered interest parity and purchasing power parity for commodities that are perfect substitutes in consumption or production hold throughout.

The paper is organised as follows. Section 2 describes a monetary economy, and defines an associated abstract economy. Equilibria in the abstract economy are the subject of section 3, which is self-contained. Section 4 addresses multiplicity of equilibria in the monetary economy and gives an example. Section 5 concludes.

## 2 A monetary economy

### 2.1 The transactions demand for money

The transactions demand for money is usually derived either from constraints on admissible transactions, or from optimizing behaviour in the face of transactions costs. The first approach is exemplified by the "cash-in-advance constraint" introduced by Clower (1967), which specifies that purchases cannot exceed cash balances held at the beginning of the period. If $p$ are prices, $z$ net trades, $z_+$ net purchases, $z_-$ net sales,[1] $\hat{m}$ are initial money balances and $m$ are terminal money balances, the cash-in-advance story unfolds as follows.

An individual acquires cash balances $\hat{m}^i \geq 0$ by borrowing initially from the bank in exchange for bonds at the rate of interest $r$, according to the constraint

$$b^i + \hat{m}^i = 0.$$

Subsequently he purchases commodities according to the constraint

$$pz^i_+ - \hat{m}^i \leq 0.$$

He accumulates end-of-period balances through receipts from the sale of commodities according to the constraint

$$-pz^i_- + m^i = 0.$$

[1] For a scalar, $z_+ = \max\{z, 0\}$ and $z_- = -\max\{-z, 0\}$; for a vector, $z_+ = (\ldots, z_{k+} \ldots)$ and $z_- = (\ldots, z_{k-} \ldots)$.

At the end of the period, or at the beginning of a subsequent, fictitious period that serves for accounting purposes, the individual settles his debt according to the constraint

$$-(1+r)b^i - m^i \leq v^i,$$

where $v^i$ is the dividend income of the individual. The budget constraints reduce to the single, over-all budget constraint in beginning-of-period values

$$pz^i + \hat{r}m^i \leq \tilde{v}^i,$$

where

$$\hat{r} = \frac{r}{1+r}, \quad \tilde{v}^i = \frac{v^i}{1+r},$$

while the cash-in-advance constraint takes the alternative, equivalent form

$$pz_- \leq m.$$

In order to consider the cash-in-advance constraint, one introduces the exchange set correspondence, here defined by

$$\Phi(p) = \mathcal{Z} \cap \{(z,m) : pz_- \leq m\},$$

where $\mathcal{Z}$ is the net trade set, the translation of the consumption set by the endowment of an individual.

Similar reasoning leads to the production set correspondence defined, under a simple cash-in-advance constraint, by

$$\Psi(p) = \mathcal{Y} \cap \{(y,n) : py_+ \leq -n\},$$

where $y$ is the production plan, $y_+$ are net outputs, $-n \geq 0$ is money balances (an input) and $\mathcal{Y}$ the production set of a firm.

The obvious limitation of the cash-in-advance technology is the arbitrary period for which purchases require initial balances.

The second approach avoids this limitation, by allowing multiple cash withdrawals or transactions within a period, or cash holdings carried across periods. It is exemplified by the inventory model of Baumol (1952) and Tobin (1956), where money demand results from an optimal tradeoff between the opportunity (interest) cost of holding cash balances and the cost of multiple transactions.

For illustration purposes, expenditures flow at a constant rate within a single period, and receipts are concentrated at its end. Optimal cash withdrawals are then of equal size and equally spaced. Each withdrawal is the counterpart of a loan from the bank, to be reimbursed with interest at the end of the period. A simple calculation shows that, with $\tau$ withdrawals, the interest charge is $r\overline{m} = rpz_+((1+\tau)/2\tau)$, with $\overline{m}$ the average amount owed to the bank over the period. Still for illustration, the cost of each transaction is one unit of leisure time, and $u(z,t)$ represents the preferences of the individual, with $t = \bar{t} - \tau$ leisure time net of transactions. The exchange set correspondence is

$$\Phi(p) = \mathcal{Z} \cap \left\{(z, m) : 2\overline{m}(\bar{t} - t) = pz_{+}(1 + (\bar{t} - t))\right\},$$

and the budget constraint is

$$pz + r\overline{m} \leq v,$$

where $v \geq 0$ is dividend income.

The first order condition for optimization, ignoring the integer constraint on $\tau$, is

$$\tau = (\frac{\lambda rpz_{+}}{2\frac{\partial u}{\partial t}})^{\frac{1}{2}},$$

where $\lambda$ is the Lagrange multiplier of the budget constraint. The cost of a transaction in terms of income, the marginal rate of substitution between leisure and income, is $(1/\lambda)(\partial u/\partial t)$, whereas $pz_{+}$ measures total spending; this is precisely the Baumol-Tobin formula.

Many model optimization under transaction costs by introducing "money in the utility function." — typically in the form of real balances. In the illustration of the Baumol -Tobin formula, one could substitute the transactions constraint in the utility function and write

$$u(z, \overline{m}, p) = u(z, \bar{t} - \frac{pz_{+}}{2\overline{m} - pz_{+}});$$

nevertheless, $p$ and $\overline{m}$ would not enter utility in the form of real balances. More importantly, the resulting formulation would be in the nature of a reduced form. Such a formulation is not in the spirit of the canonical model, which distinguishes clearly subjective preferences from objective opportunities.

It could be argued that "liquidity," as measured by real balances, is desirable in itself, as provision for unforeseen outlays. This is the precautionary demand for money. In the canonical model, uncertainty is modeled through alternative states of the environment. Opportunities for outlays contingent on future states are naturally captured by the exchange set correspondence. Liquidity refers to access to a broader set of consumption plans, which is beneficial utility-wise. It would be improper to count these benefits twice.

On these grounds, here, balances, prices and rates of interest do not enter as arguments of preferences , though they did in earlier presentations — Drèze and Polemarchakis (1999, 2000). As asserted there, all results remain valid under that alternative formulation, which remains consistent when it corresponds to a properly defined reduced form.

In order to accomodate a variety of specifications for the transactions technology, we rely on the abstract correspondences $\Psi$ (firms) and $\Phi$ (individuals). And we prove existence of equilibria for an abstract economy, under general assumptions in the tradition of the canonical model.

It is in the nature of the transactions technology that the exchange set correspondence defines liquidity requirements in terms of spot prices, separately for each money, $k$; for example, money at a particular date-event, $s_t = k$, or money in a currency area, $k$. The relevant constraints are of the form $(z_k, m_k) \in \Phi_k(p_k)$, where the vectors $z_k$ and $p_k$ are restrictions of $z$ and $p$ to the commodities that money $k$ buys, distinct from the commodities associated with $k' \neq k$. One then writes $\Phi(p) = \times_k \Phi_k(p_k)$ and states that the economy has a product structure. If, in addition, the exchange sets $\Phi_k$ are homogeneous of degree 0 in $(p_k, m_k)$, one states that the economy is (0-)homogeneous. Homogeneity means that the transactions technology is scale-invariant (free of "money illusion"). Similar definitions apply to the production-set correspondences of firms. Homogeneity is relevant to the transition from the abstract economy to specific applications.

## 2.2 Exchange sets and budget constraints

To define a monetary economy extending over time under uncertainty, the following notation and definitions are introduced.

Dates are $t \in \mathcal{T} = \{1, \ldots, T\}$. States of the world are $s \in \mathcal{S} = \{1, \ldots, S\}$. The resolution of uncertainty is described by $(\mathcal{S}_t : t \in \mathcal{T})$, an increasing sequence of partitions of the set of states of the world. Events at a date are $s_t \in \mathcal{S}_t = \{1, \ldots, S_t\}$. For a date-event, $s_t$, and a date $\hat{t} < t$, the predecessor of $s_t$ at $\hat{t}$ is $s_{\hat{t}}(s_t)$, the date-event $s_{\hat{t}} \in \mathcal{S}_{\hat{t}}$, such that $s_t \subset s_{\hat{t}}$; for $\hat{t} > t$, the set of successors of $s_t$ at $\hat{t}$ is the set $\mathcal{S}_{\hat{t}}(s_t)$, of date-events $s_{\hat{t}} \in \mathcal{S}_{\hat{t}}$, such that $s_{\hat{t}} \subset s_t$. No uncertainty is resolved prior to date 1: $\mathcal{S}_1 = \{\mathcal{S}\}$ and $s_1 = 1$, while all uncertainty is resolved by date $T : \mathcal{S}_T = \{\{s\} : s \in \mathcal{S}\}$. The set of date-events is $\mathcal{N} = \cup_{t \in \mathcal{T}} \mathcal{S}_t$, with cardinality $N$. It is sometimes convenient to augment the set of dates to $\mathcal{T}^* = \{1, \ldots, T, \ T+1\}$. Events at date $T+1$ coincide with events at date $T : \mathcal{S}_{T+1} = \mathcal{S}_T$. The augmented set of date-events is $\mathcal{N}^* = \cup_{t \in \mathcal{T}^*} \mathcal{S}_t$, with cardinality $N + S$.

Commodities are $l \in \mathcal{L} = \{1, \ldots, L\}$. A bundle of commodities at a date-event, $s_t$, is [2] $x_{s_t} = (x_{l,s_t} : l \in \mathcal{L})'$; across date-events, a bundle of commodities is $x = (x_t : t \in \mathcal{T})$. Prices of commodities at a date-event, $s_t$, are $p_{s_t} = (p_{l,s_t} : l \in \mathcal{L})$; across date-events, prices of commodities are $p = (p_t : t \in \mathcal{T})$.

Elementary securities and bonds of one date maturity effect transfers of revenue. The elementary security with payoff one unit of revenue at a date-event, $s_{t+1}$, is traded at the date-event $s_t(s_{t+1})$. At a date-event, $s_t$, elementary securities are [3] $s_{t+1} \in \mathcal{S}_{t+1}(s_t)$, and a portfolio of elementary

[2] For a finite, non-empty set, $\mathcal{K}$, a variable, $x$, which takes different values, $x_k$, for $k \in \mathcal{K}$, and for $\mathcal{K}' \subset \mathcal{K}$, a non-empty subset, $x_{\mathcal{K}'} = \{x_k : k \in \mathcal{K}'\}$; when there is no ambiguity, $x = x_{\mathcal{K}}$. Also, "$'$" denotes the transpose.

[3] It is pedantic to use an index for elementary securities different from the date-event at which they pay off.

securities is $(\eta_{s_{t+1}} : s_{t+1} \in \mathcal{S}_{t+1}(s_t))'$. Prices of assets at a date-event, $s_t$, are $(q_{s_{t+1}} : s_{t+1} \in \mathcal{S}_{t+1}(s_t))$. The bond traded at a date-event, $s_t$, yields interest $r_{s_t}$ at every date event $s_{t+1} \in \mathcal{S}_{t+1}(s_t)$; and $\hat{r}_{s_t} = r_{s_t}(1 + r_{s_t})^{-1}$. At a date-event, $s_t$, holdings of bonds are $b_{s_t}$.

A firm is described by the production set correspondence that assigns to prices of commodities, prices of elementary securities and rates of interest, $(p, q, r)$, production plans, bundles of commodities and balances, $(y, n)$, across date-events.

An individual is described by the exchange set correspondence that assigns to prices of commodities, prices of elementary securities and rates of interest, $(p, q, r)$, net trades of commodities and balances, $(z, m)$, across date-events; the utility function defined over net trades of commodities, $z$, across date-events; and shares in firms and banks.

A monetary economy is defined here for a simple cash-in-advance technology; the extension to other transactions technologies is the subject of remark 2 below. The simple formulation embodies the following conventions.

1. Firms are allowed to distribute negative dividends to raise equity capital; accordingly, they do not trade securities; a simple dividend policy, spelled out below, calls for distributing at the beginning of each period the economic profit of the period.
2. Individuals trade elementary securities against cash at the beginning of each period, before trading in commodities; they trade with the bank, they borrow or lend, for the net amount required to meet the cash-in-advance constraint.
3. Interest accounting does not distinguish the end of a period from the beginning of the next one; the distinction remains essential, however, for information about states and for prices or rates of interest.

At the beginning of period $t$, at node $s_t$, an individual collects dividends $v^i_{s_t}$ (detailed below) and the proceeds of earlier investment $\eta^i_{s_t}$; he inherits terminal balances $m^i_{s_{t-1}(s_t)}$ and a debt to the bank $b^i_{s_{t-1}(s_t)}(1+r_{s_{t-1}(s_t)})$; and he acquires or sells elementary securities at a net cost $\sum_{s_{t+1}\in\mathcal{S}_{t+1}(s_t)} q_{s_{t+1}}\eta^i_{s_{t+1}}$. In order to settle these transactions and to have cash for commodity purchases, he exchanges with the bank bonds $b^i_{s_t}$ for initial balances $\hat{m}^i_{s_t}$, according to the constraint

$$-b^i_{s_t} = \hat{m}^i_{s_t} \geq p_{s_t} z^i_{s_t+} + \sum\nolimits_{s_{t+1}\in\mathcal{S}_{t+1}(s_t)} q_{s_{t+1}}\eta^i_{s_{t+1}} - \eta^i_{s_t} - v^i_{s_t}$$
$$-b^i_{s_{t-1}(s_t)}(1 + r_{s_{t-1}(s_t)}) - m^i_{s_{t-1}}.$$

He then accumulates cash balances through receipts for sales of commodities according to the constraint

$$p_{s_t} z^i_{s_t-} \leq m^i_{s_t}.$$

He enters period $t+1$ with holdings of elementary securities $\eta_{s_{t+1}}$, cash balances $m^i_{s_t}$ and debt to the bank $b^i_{s_t}(1+r_{s_t})$.

The individual maximizes his utility function subject to the budget constraints

$$p_1 z_1 + \sum_{s_2 \in S_2} q_{s_2}\eta_{s_2} + m_1 + b_1 = v^i_1,$$

$$p_{s_t} z_{s_t} + \sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}}\eta_{s_{t+1}} + m_{s_t} + b_{s_t} = \eta_{s_t} + m_{s_{t-1}(s_t)} + (1 + r_{s_{t-1}(s_t)})b_{s_{t-1}(s_t)} + v^i_{s_t},$$

$$p_{s_T} z_{s_T} + m_{s_T} + b_{s_T} = \eta_{s_T} + m_{s_{T-1}(s_T)} + (1 + r_{s_{T-1}(s_T)})b_{s_{T-1}(s_T)} + v^i_{s_T},$$

$$0 = m_{s_T} + (1 + r_{s_T})b_{s_T}.$$

No-arbitrage in the market for bonds and elementary securities requires that

$$\frac{1}{1+r_{s_t}} = \sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}}.$$

Under this condition, uniform holdings of elementary securities are perfect substitutes for bank loans, and individual demands are indeterminate. A simple way to lift the indeterminacy is the impose, at each $s_t$,

$$m^i_{s_t} + b^i_{s_t}(1 + r_{s_t}) = 0.$$

This implements transfers across periods through holdings of elementary securities, while bank loans provide within-period liquidity. Under this harmless convention, the budget constraints simplify to the "semi-reduced" form

$$p_1 z_1 + \sum_{s_2 \in S_2} q_{s_2}\eta_{s_2} + \hat{r}_1 m_1 = v^i_1,$$

$$p_{s_t} z_{s_t} + \sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}}\eta_{s_{t+1}} + \hat{r}_{s_t} m_{s_t} = \eta_{s_t} + v^i_{s_t},$$

$$p_{s_T} z_{s_T} + \hat{r}_{s_T} m_{s_T} = \eta_{s_T} + v^i_{s_T}.$$

A firm, $j$, chooses a production plan of commodities and balances $(y^j, n^j)$. At the beginning of period $t$, node $s_t$, it exchanges with the bank bonds $b^j_{s_t}$ for initial balances $\hat{n}^j_{s_t} \leq 0$ in the amount needed to pay for inputs $y_{j-}$ and to issue dividends $v^j_{s_t}$, namely

$$b^j_{s_t} = -\hat{n}^j_{s_t} = p_{s_t} y^j_{s_t -} + v^j_{s_t}.$$

It then accumulates cash balances through receipts for sales of outputs,

$$-n^j_{s_t} = p_{s_t} y^j_{s_t +}$$

and uses these terminal balances to repay its debt to the bank:

$$b^j_{s_t}(1 + r_{s_t}) + n^j_{s_t} \;=\; 0.$$

This defines implicitly the dividends (the dividend policy) as

$$v^j_{s_t} \;=\; p_{s_t} y^j_{s_t} + \hat{r}_{s_t} n^j_{s_t}.$$

The bank initially issues balances $\hat{M}_{s_t}$, as demanded by individuals and firms in exchange for bonds. It earns interest $r_{s_t}\hat{M}_{s_t}$ which it pays out as dividends – thereby bringing the total quantity of money to $(1 + r_{s_t})\hat{M}_{s_t} = M_{s_t}$. The dividends may indifferently be expressed as $r_{s_t}\hat{M}_{s_t}$ or $\hat{r}_{s_t} M_{s_t}$.

The dividend income of individual $i$ is

$$v^i_{s_t} \;=\; \sum_{j\in J} \theta^{ij} v^j_{s_t} + \theta^{i,J+1} v^{J+1}_{s_t} \;=\; \sum_{j\in J} \theta^{ij}(p_{s_t} y^j_{s_t} + \hat{r}_{s_t} n^j_{s_t}) + \theta^{i,J+1}\hat{r}_{s_t} M_{s_t}.$$

It is homogeneous of degree 1 in $(p_{s_t}, \cdots, n^j_{s_t}, \cdots, M_{s_t})$.

## 2.3 Consolidation and abstraction

The profit maximisation problem of a firm, $j$, calls for maximising the present value of profits. Prices of elementary securities define cumulative prices of elementary securities

$$\tilde{q}_{s_t} \;=\; 1 \times q_{s_2(s_t)} \times \cdots \times q_{s_{t-1}(s_t)} \times q_{s_t}.$$

The present value of firm $j$'s profits (dividends) is

$$\tilde{v}^j \;=\; \sum_{s_t\in\mathcal{N}} \tilde{q}_{s_t}(p_{s_t} y^j_{s_t} + \hat{r}_{s_t} n^j_{s_t}) \;=\; \sum_{s_t\in\mathcal{N}} \tilde{q}_{s_t} v^j_{s_t}$$

which it maximises subject to the cash-in-advance constraints

$$-n^j_{s_t} \;=\; p_{s_t} y^j_{s_t+}.$$

The problem can also be stated in terms of present-value prices and balances

$$\tilde{p}_{s_t} \;=\; \tilde{q}_{s_t} p_{s_t}, \quad \tilde{n}^j_{s_t} \;=\; \tilde{q}_{s_t} n^j_{s_t};$$

the firm maximises

$$\tilde{p} y^j + \hat{r}\tilde{n}_j,$$

subject to

$$-\tilde{n}^j_{s_t} = \tilde{p}_{s_t} y^j_{s_t+}, \; s_t \in \mathcal{N}.$$

The equivalence of the two formulations follows from the homogeneity of the cash-in-advance constraints in $(p, n)$.

Similarly, the budget constraints of an individual, $i$, can be consolidated into the single present-value constraint

$$\sum_{s_t \in \mathcal{N}} \tilde{q}_{s_t}\{p_{s_t} z_{s_t} + \sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}} \eta_{s_{t+1}} + \hat{r}_{s_t} m_{s_t}\} = \sum_{s_t \in \mathcal{N}} \tilde{q}_{s_t}(\eta_{s_t} + v^i_{s_t})$$

or equivalently (after cancelling the $\eta$-terms)

$$\sum_{s_t \in \mathcal{N}} \{\tilde{p}_{s_t} z_{s_t} + \hat{r}_{s_t} \tilde{m}_{s_t}\} = \sum_{j \in J} \theta^{ij} \tilde{v}^j + \theta^{i,J+1} \tilde{v}^{J+1} := \tilde{v}^i,$$

where $\tilde{m}_{s_t} := \tilde{q}_{s_t} m_{s_t}$ and $\tilde{v}^{J+1} = \sum_{s_t \in \mathcal{N}} \tilde{q}_{s_t} \hat{r}_{s_t} M_{s_t}$. In vector notation,

$$\tilde{p} z + \hat{r} \tilde{m} = \tilde{v}^i.$$

Individual $i$ maximises utility $u^i(z)$ subject to this consolidated budget constraint and to the cash-in-advance constraints

$$\tilde{m}_{s_t} = \tilde{p}_{s_t} z_{s_t -}, s_t \in \mathcal{N},$$

the homogeneity of which (in $p, m$) establishes the equivalence of the original and consolidated formulations.

The consolidated formulation eliminates explicit reference to the financial assets $\eta$ and their prices $q$, to the bonds $b$, and to the nominal spot prices $p$. In particular, the price information guiding individual choices is reduced from the vector $(p, q, r)$ of dimension $NL + (S-1) + N$ (taking no-arbitrage conditions into account), to the vector $(\tilde{p}, r)$ of dimension $NL + N$. This reduction is at the root of the multiplicity property stated in section 4, where we spell out in more detail the links between the two formulations. In general (beyond the simple cash-in-advance specification adopted here), their equivalence rests on the homogeneity property defined in section 2.1.

The full price information $(p, q, r)$ is required to guide the portfolio choices of individuals. Also, in a monetary economy, spot price levels (inflation rates) matter. Yet, it is analytically convenient to study existence and other characteristics of equilibria for an abstract model, suceptible of interpretations encompassing a variety of applications – as defined for instance by alternative transactions technologies. The consolidated economy defined in terms of present-value prices and balances serves that purpose. It is studied in the next section.

Although artificial as a representation of an economy extending over time under uncertainty, the abstract economy describes directly an economy extending over a single period under certainty (spot prices are present-value prices, a.s.o.). In an economy extending over $T$ periods of time under certainty, with present-value prices $\tilde{p}_t = \tilde{q}_t p_t$, the no-arbitrage conditions imply

$q_t = (1 + r_{t-1})^{-1}$, all $t$. Once the nominal rates of interest are given, the cumulative prices $\tilde{q}_t$ are also given, and the translation of spot prices into present-value prices is immediate. The original and abstract economies are identical, under homogeneity.

**Remark 1.** The cash-in-advance economy described above specifies payment of the dividends of the firms and of the bank, for each node $s_t$, *before* individuals acquire the cash balances needed there for commodity purchases. As a consequence, terminal balances of individuals are defined by the simple expression $m^i_{s_t} = p_{s_t} z^i_{s_t -}$, and this expression summarises neatly the implications of the cash-in-advance technology. In particular, homogeneity in $(p, m^i)$ of the exchange set correspondence is immediately verified: $(z_k, m_k) \in \Phi^i_k(p_k)$ iff $(z_k, \lambda m_k) \in \Phi^i_k(\lambda p_k)$. When paid after the acquisition of cash balances for the purchases of commodities, the dividends appear under terminal balances, now defined by

$$m^i_{s_t} = p_{s_t} z_{s_t -} + v^i_{s_t} = p_{s_t} z_{s_t -} + \sum_{j \in J} \theta^{ij} (p_{s_t} y^j_{s_t} + \hat{r}_{s_t} n^j_{s_t}) + \theta^{i,J+1} \hat{r}_{s_t} M_{s_t}.$$

This expression is homogeneous in $(p_{s_t}, m^i_{s_t}, (n^j_{s_t})_{j \in J}, M_{s_t})$. But the exchange set correspondence is now be defined by

$$\Phi^i(p, (y^j, n^j)_{j \in J}, M) = \times_{s_t \in \mathcal{N}} \Phi^i_{s_t}(p_{s_t}, (y^j_{s_t}, n^j_{s_t})_{j \in J}, M_{s_t}).$$

That extension is not introduced explicitly in the definition of the abstract economy, to avoid ancillary complication. See however remark 3 below.

**Remark 2.** For simplicity, the description above of a monetary economy has been confined to the cash-in-advance transactions technology. The extension to more general technologies, including in particular inventory models à la Baumol-Tobin, is easy to outline. At the beginning of period $t$, at node $s_t$, an individual, $i$, still collects dividends $v^i_{s_t}$ and proceeds of earlier investment $\eta^i_{s_t}$; he inherits terminal balances $m^i_{s_{t-1}(s_t)}$ and a debt to the bank, provisionnally denoted $d^i_{s_{t-1}(s_t)}$; and he acquires elementary securities at a net cost $\sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}} \eta_{s_{t+1}}$. He then exchanges with the bank bonds $b^{i0}_{s_t}$ for initial balances $\hat{m}^i_{s_t}$ according to the constraint

$$-b^{i0}_{s_t} = \hat{m}^i_{s_t} \geq \max\{0, \sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}} \eta^i_{s_{t+1}} - \eta^i_{s_t} - v^i_{s_t} - m^i_{s_{t-1}(s_t)} - d^i_{s_{t-1}(s_t)}\}.$$

Next, net trades $z^i_{s_t}$ and new financial transactions, say $b^{i\nu}_{s_t}, \nu = 1, 2, \cdots$, occur. The timing of net trades, the timing, nature and number of new financial transactions result from individual optimisation constrained by the exchange correspondence. The amount of cash on hand rises (falls) additively with negative (positive) components of $p_{s_t} z^i_{s_t}$ and $b^{i\nu}_{s_t}$ – without ever becoming negative. Terminal balances are

$$m^i_{s_t} = -p_{s_t} z^i_{s_t} - \sum_{\nu \geq 0} b^{i\nu}_{s_t} := -p_{s_t} z^i_{s_t} - b^i_{s_t},$$

where $b^i_{s_t}$ is the terminal level of the principal due ($b^i_{s_t} < 0$) or owned ($b^i_{s_t} > 0$) as a consequence of all transactions with the bank within period $t$. The interest associated with $b^i_{s_t}$ is $r_{s_t} \bar{b}^i_{s_t}$, where $\bar{b}^i_{s_t}$ is the average debt to the bank over the period, as implicitly defined by the sequence of transactions. Thus, the terminal debt to the bank is $d^i_{s_t} = b^i_{s_t} + r_{s_t} \bar{b}^i_{s_t}$. It replaces the term $b^i_{s_t}(1 + r_{s_t})$ appearing in the previous subsection. In the illustration of section 2.1, $b^i_{s_t} = p_{s_t} z_{s_{t+}}$ and $\bar{b}_{s_t} = p_{s_t} z_{s_{t+}}((1+\tau)/2\tau)$. This simple dissociation of the principal of the debt from the interest liability should permit reconciliation of our abstract formulation with a variety of specifications of the transactions technology.

# 3 An abstract economy

## 3.1 Notation and definitions

Commodities are $l \in \mathcal{L} = \{1, \ldots, L\}$, a finite, non-empty set. A bundle of commodities is $z = (z_l : l \in \mathcal{L})'$. Prices of commodities are [4] $p = (p_l : l \in \mathcal{L}) > 0$.

Media of exchange, moneys, are $k \in \mathcal{K} = \{1, \ldots, K\}$, a finite, non-empty set. Balances are $m = (m_k : k \in \mathcal{K})'$. Rates of interest are $r = (r_k : k \in \mathcal{K}) \geq 0$.

At prices of commodities and rates of interest $(p, r)$, the value of a bundle of commodities and balances $(z, m)$ is

$$(p, r)(z, m) = pz + rm.$$

Agents in the economy are monetary authorities, banks, $k \in \mathcal{K} = \{1, \ldots, K\}$, firms, $j \in \mathcal{J} = \{1, \ldots, J\}$, and individuals, $i \in \mathcal{I} = \{1, \ldots, I\}$, finite, non-empty sets.

Associated with the monetary authority $k$, there is the medium of exchange, $k$, whose supply, $M^k \geq 0$, is under its control. Across monetary authorities, the supply of balances is $M = (M^k : k \in \mathcal{K})'$.

A currency, $k$, serves as a medium of exchange for commodities $\mathcal{L}_k \subset \mathcal{L}$. For $k \neq \tilde{k}$, distinct currencies, $\mathcal{L}_k \cap \mathcal{L}_{\tilde{k}} = \emptyset$ : commodities for which different currencies serve as media of exchange are distinct; for a bundle of commodities, $z$, and for a currency, $k$, the projection to components $k \in \mathcal{L}_k$, is $z_k$; similarly, for prices, $p_k$ is the projection to components $k \in \mathcal{L}_k$.

[4] "$\gg$," "$>$," "$\geq$"and "$\ll$," "$<$," "$\leq$"are vector inequalities.

A firm is described by the production set correspondence, $\Psi^j$, that assigns to prices of commodities a subset, $\Psi^j(p)$, of bundles of commodities and non-positive balances: for $(y, n) \in \Psi^j(p)$, the production plan of commodities is $y$, while balances, an input, are $n \leq 0$.

An individual is described by the triple, $(\Phi^i, u^i, \theta^i)$, of an exchange set correspondence, an ordinal utility function and shares in banks and firms. The exchange set correspondence, $\Phi^i$, assigns to prices of commodities a subset, $\Phi^i(p)$, of bundles of commodities and non-negative balances: for $(z, m) \in \Phi^j(p)$, the net trade or exchange of commodities is $z$ while balances are $m \geq 0$. The utility function has domain the net-trade set, which includes $\Phi^i(p)$ for each $p$. Shares in firms and banks are $\theta^i = (\theta^{i,\mathcal{J}}, \theta^{i,\mathcal{K}}) = (\theta^{i,j} : j \in \mathcal{J}, \theta^{i,J+k} : k \in \mathcal{K})$.

Across firms, the production set correspondence is $\Psi^{\mathcal{J}} = \times_{j \in \mathcal{J}} \Psi^j$; across individuals, the exchange set correspondence is $\Phi^{\mathcal{I}} = \times_{i \in \mathcal{I}} \Phi^i$.

The allocation correspondence, $A$, assigns to prices of commodities and rates of interest a subset $A(p, r) \subset \Psi^{\mathcal{J}}(p, r) \times \Phi^{\mathcal{I}}(p, r)$ of allocations, $a = ((y, n)^j : j \in \mathcal{J}, (z, m)^i : i \in \mathcal{I})$, production plans and balances for firms and net trades and balances for individuals. At an allocation, the aggregate excess demand for commodities is $z^a = \sum_{i \in \mathcal{I}} z^i - \sum_{j \in \mathcal{J}} y^j$; the allocation is *feasible* if $z^a = 0$. If the supply of balances is $M$, the aggregate excess demand for balances is $m^a = \sum_{i \in \mathcal{I}} m^i - \sum_{j \in \mathcal{J}} n^j - M$; the allocation is feasible for supply of balances $M$ if it is feasible and, in addition, $m^a = 0$.

The autarkic allocation is $a^0(p, r)$, with $(y, n)^j = 0$, for every firm and $(z, m)^j = 0$, for every individual.

An allocation, $a \in A(p, r)$, is weakly Pareto superior to an allocation, $\hat{a} \in A(\hat{p}, \hat{r})$, if $u^i(z^i) \geq u^i(\hat{z}^i)$ for every individual; it is Pareto superior if the preference is strict: $u^i(z^i) > u^i(\hat{z}^i)$, for some individual.

A feasible allocation, $a \in \mathbb{A}(p, r)$, is *Pareto optimal* if there do not exist prices of commodities and rates of interest, $(\hat{p}, \hat{r})$ and a feasible allocation, $\hat{a} \in \mathbb{A}(\hat{p}, \hat{r})$, that is Pareto superior; it is weakly Pareto optimal if no feasible allocation is strictly Pareto superior. The allocation is Pareto optimal at prices of commodities and rates of interest $(p, r)$ if there do not exist a feasible allocation, $\hat{a} \in \mathbb{A}(p, r)$, that is Pareto superior. Similarly, for supply of balances $M$.

## 3.2 Assumptions

*Standard assumptions* extend to an economy in which money serves as a medium of exchange.

**Assumption 1** *The production set correspondence is closed and convex valued, and it allows for no-production:* $(0, 0) \in \Psi^j(p)$.

**Assumption 2** *The production set correspondence is continuous.*

**Assumption 3** *If* $-\hat{n}_{\hat{k}} \geq p_{\hat{k}} y_{\hat{k}+}$, *and* $\hat{n} = (\ldots, \hat{n}_k, \ldots)$, *with* $\hat{n}_k = n_k$, *for* $k \neq \hat{k}$, *then* $(y, n) \in \Psi^j(p) \Rightarrow (y, \hat{n}) \in \Psi^j(p)$.

The firm attains satiation in balances in a medium of exchange if they exceed the revenue from sales.

The aggregate production set correspondence is $\Psi^a = \sum_{j \in \mathcal{J}} \Psi^j$.

**Assumption 4** *The aggregate production set correspondence does not allow for free lunches in the production of commodities:* $(y, n) \in \Psi^a(p) \Rightarrow y \not> 0$, *neither for reversibility in the production of commodities:* $(y, n) \in \Psi^a(p)$ *and* $(y, \hat{n}) \in -\Psi^a(p) \Rightarrow y = 0$.

**Assumption 5** *The aggregate production set correspondence allows for the free disposal of commodities:* $(y, n) \in \Psi^a(p)$, *and* $\hat{y} \leq y$, *while* $p_l = 0$ *for* $\hat{y}_l < y_l$, *implies* $(\hat{y}, n) \in \Psi^a(p)$.

**Assumption 6** *The exchange set correspondence is closed and convex valued, it is bounded from below: there exists* $\underline{\underline{z}}^i$, *such that* $(z, m) \in \Phi^i(p) \Rightarrow (z, m) \geq (\underline{\underline{z}}^i, 0)$, *and it allows for no - exchange:* $(0, 0) \in \Phi^i(p)$.

**Assumption 7** *The exchange set correspondence is continuous.*

**Assumption 8** *There exists* $\underline{z}^i \ll 0$, *such that* $(\underline{z}^i, \underline{m}) \in \Phi^i(p)$, *where* $\underline{m}_k = p_k \underline{z}_{k-}$. *The shares in banks and firms are non-negative:* $\theta^i = (\theta^{i,\mathcal{K}}, \theta^{i,\mathcal{J}}) \geq 0$.

For rates of interest $\mathbf{1}^K \geq r \geq 0$, no-exchange is not a minimum wealth point: $p\underline{z} + r\underline{m} = \sum_{k \in \mathcal{K}} (1 - r_k) p_k \underline{z}_k < 0$.

**Assumption 9** *The utility function is quasi-concave: for every net trade, the set* $U^i(z) = \{\hat{z} : u^i(\hat{z}) \geq u^i(z)\}$ *is convex.*

**Assumption 10** *The utility function is continuous.*

**Assumption 11** *There is no local satiation in the net trades of commodities and balances: for every net trade and for every* [5] $\varepsilon > 0, U^i_\varepsilon(z) = \{\hat{z} : u^i(\hat{z}) > u^i(z),$ *and* $\|(\hat{z} - z)\| < \varepsilon\} \neq \emptyset$.

**Assumption 12** *If* $\hat{m}_{\hat{k}} \geq p_{\hat{k}} z_{\hat{k}-}$, *and* $\hat{m} = (\ldots, \hat{m}_k, \ldots)$, *with* $\hat{m}_k = m_k$, *for* $k \neq \hat{k}$, *then* $(z, m) \in \Phi^i(p) \Rightarrow (z, \hat{m}) \in \Phi^i(p)$.

The individual attains satiation in balances in a medium of exchange if they exceed the revenue from sales.

**Assumption 13** *Every firm and every bank is owned* [6] $: \sum_{i \in \mathcal{I}} \theta^i = \mathbf{1}^{J+K}$.

An economy that satisfies standard assumptions can, but need not be homogeneous in the prices of commodities and balances.

[5] "$\| \ \|$"denotes the Euclidean distance.

[6] $\mathbf{1}^N$ is the vector of 1's of dimension $N$; $\mathbf{1}^N_n$ is the $n$-th unit vector of dimension $N$.

### 3.3 Equilibria

At rates of interest $r$, the profit of a bank if it issues balances $M^k$ is

$$v^k(r^k, M^k) = r^k M^k.$$

At prices of commodities and rates of interest $(p, r)$, the profit of a firm if it chooses production plan of commodities and balances $(y, n)$ is

$$v^j(y, n, p, r) = (p, r)(y, n).$$

The profit maximization problem of the firm is

$$\max v^j, \text{ s.t. } (y, n) \in \Psi^j(p).$$

The set of solutions to the profit maximization problem is $(y^j, n^j)(p, r)$, the production correspondence is $(y^j, n^j)$, and the maximal profit of the firm is $\hat{v}^j(p, r) = (p, r)(y^j, n^j)(p, r)$; whenever $(y^j, n^j)(p, r) = \emptyset$, $v^j(p, r) = -\infty$.

At prices of commodities, rates of interest and supply of balances $(p, r, M)$, the profit income of an individual is

$$\hat{v}^i(p, r, M) = \sum_{k \in \mathcal{K}} \theta^{i,k} v^k(r^k, M^k) + \sum_{j \in \mathcal{J}} \theta^{i,j} \hat{v}^j(p, r).$$

The utility maximization problem of the individual is

$$\max u^i, \text{ s.t. } (z, m) \in \Phi^i(p), \ (p, r)(z, m) \leq \hat{v}^i(p, r, M).$$

The set of solutions to the optimization problem of the individual is $(z^i, m^i)(p, r, M)$, and $(z^i, m^i)$ is his exchange correspondence.

The aggregate excess demand correspondence is $(z^a, m^a) = \sum_{i \in \mathcal{I}}(z^i, m^i) - \sum_{j \in \mathcal{J}}(y^j, n^j) - (0, M)$.

*Competitive equilibrium* prices of commodities, rates of interest and supply of balances, $(p^*, r^*, M^*)$, are such that $0 \in (z^a, m^a)(p^*, r^*, M^*)$. Competitive equilibrium prices of commodities and rates of interest, $(p^*, r^*)$, are such that $(p^*, r^*, M)$ are competitive equilibrium prices of commodities, rates of interest and supply of balances, for some supply of balances, $M$. For rates of interest $r$, competitive equilibrium prices of commodities, $p^*$, are such that $(p^*, r)$ are competitive equilibrium prices of commodities and rates of interest. For supply of balances $M$, competitive equilibrium prices of commodities and rates of interest, $(p^*, r^*)$ are such that $(p^*, r^*, M)$ are competitive equilibrium prices of commodities, rates of interest, and supply of balances.

Associated with competitive equilibrium prices of commodities and rates of interest, $(p^*, r^*)$, there is a feasible allocation $a(p^*, r^*)$.

Monetary authorities set rates of interest and accommodate the demand for balances; alternatively, monetary authorities set the supply of balances and rates of interest adjust to attain equilibrium.

**Proposition 1.** *For rates of interest $1^K \geq r \geq 0$ and for any scalar $c > 0$, there exist competitive equilibrium prices of commodities, $p^*$, with $\sum_{l \in \mathcal{L}} p_l^* = c$.*

**Proof** Rates of interest are $r = (\ldots, r_k, \ldots) \geq 0$.

The set of prices of commodities is $\mathcal{P}_c = \{p > 0 : \sum_{l \in \mathcal{L}} p_l = c\}$.

There exists $b > 0$, such that, for prices of commodities and rates of interest $(p, r) \in \mathcal{P} \times \{r\}$, if $z^a \leq 0$, then, for every firm, $\|y^j\| < b$, and, for every individual, $\|z^i\| < b$.

The truncated optimization problem of a firm is the profit maximization problem subject to the constraints

$$\|y\| \leq b, \quad -n \leq 1^K cb.$$

The set of solutions is $(y_b^j, n_b^j)(p, r)$, the production correspondence is $(y_b^j, n_b^j)$, the maximal profit is $\hat{v}_b^j(p, r)$, and the profit function is $\hat{v}_b^j$.

The profit income of an individual is

$$v_b^i(p, r, M) = \sum_{k \in \mathcal{K}} \theta^{i,k} v^k(r^k, M^k) + \sum_{j \in \mathcal{J}} \theta^{i,j} \hat{v}_b^j(p, r).$$

The truncated optimization problem of the individual is the utility maximization problem with profit income $v_b^i(p, r, M)$ and subject to the constraints

$$\|z\| \leq b, \quad m_k \leq 1^K cb.$$

The set of solutions is $(z_b^i, m_b^i)(p, r, M)$, and the modified exchange correspondence is $(z_b^i, m_b^i)$.

If, at a solution to the truncated optimization problems of firms and individuals, for every firm $\|y^j\| < b$, and, for every individual, $\|z^i\| < b$, then, since the economy is convex, the truncation is not a binding constraint.

The aggregate, truncated excess demand correspondence is

$$(z_b^a, m_b^a) = \sum_{i \in \mathcal{I}} (z_b^i, m_b^i) - \sum_{j \in \mathcal{J}} (y_b^j, n_b^j) - (0, M).$$

The truncated set of balances is $\mathcal{M}_{c,b} = \{M \geq 0 : M \leq 1^K(cb(I + J))\}$.

The set of aggregate, truncated excess demand for commodities is $\mathcal{Z}_b^a = \{z : \|z\| \leq b(I + J)\}$.

For every firm, the truncated production correspondence is non - empty, compact, convex valued and upper hemi-continuous; the profit function is non-negative. For every individual, the truncated exchange correspondence is non-empty, compact, convex valued and upper hemi-continuous. The aggregate, truncated excess demand correspondence is non-empty, compact, convex valued and upper hemi-continuous; it satisfies $(p, r)(z_b^a, m_b^a) \leq 0$.

The correspondence $f = (f_1, f_2, f_3)$, with domain $\mathcal{P}_c \times \mathcal{Z}_b^a \times \mathcal{M}_{c,b}$ is defined by $f_1 = \arg\max \{p z_b^a : p \in \mathcal{P}_c\}$, $f_2 = z_b^a(p, r, M)$, and $f_3 = m_b^a(p, r, M)$.

There exists $(p^*, z^{a*}, m^{a*})$, a fixed point, and an associated allocation, $a(p^*, r) = ((y^{j*}, n^{j*}) : j \in \mathcal{J}, (z^{i*}, m^{i*}) : i \in \mathcal{I})$.

At the fixed point, $m^{a*} = 0$ and $p^* z^{a*} \leq 0$.

It follows that $z^{a*} \leq 0$, $p^* z^{a*} = 0$, and $m^{a*} = 0$.

For every firm, $(y^{j*}, n^{j*}) \in (y^j, n^j)(p^*, r)$, while $(z^{i*}, m^{j*}) \in (z^i, m^i)(p^*, r, M^*)$, for every individual.

In order to show that $p^*$ are competitive equilibrium prices of commodities at rates of interest $r$, it remains to show that $z^{a*} = 0$; but, since $z_l^{a*} < 0 \Rightarrow p_l^* = 0$, the free disposal of commodities accommodates this. □

The restriction to rates of interest $0 \leq r_k \leq 1$, is natural: rates of interest in the abstract economy typically correspond to rates of interest $\hat{r}_k = r_k/(1 + r_k) \leq 1$ in the underlying economy.

**Corollary 1.** *For supply of balances* $\overline{M} \geq 0$, *and for* $\overline{c} > 0$, *there exist competitive equilibrium prices of commodities and rates of interest,* $(p^*, r^*)$, *with* $\sum_{l \in \mathcal{L}} p_l^* = \overline{c}$.

**Remark 3.** In a more general specification, the exchange set of an individual varies with rates of interest, the supply of balances by banks, and the production plans and holdings of balances by firms: one writes $\Phi^i(p, r, \ldots M^k, \ldots, y^j, n^j, \ldots)$. This allows for financial policies of firms and banks that modify the transactions constraints faced by individuals. Readers may verify that the proof as written covers the more general specification, with the single caution that the last statement in assumption 5 should be modified to read: "and it allows for no-exchange, when the profits of firms are non-negative:

$$(p, r)(y^j, n^j) \geq 0 \text{ for all } j \in J \text{ implies } (0, 0) \in \Phi^i(p, (y^j, n^j)_{j \in J}, M)."$$

The reasoning is similar to that applicable in the canonical model, to the effect that the budget correspondence is non-empty valued.

**Remark 4.** The abstract economy is a general model covering applications to various transactions technologies. In applications, the prices entering the definitions of the production and exchange set correspondences are typically spot nominal prices, distinct from the present-value prices entering the definitions of profits and of consolidated budget constraints – a distinction not present in the abstract economy. The transition from the abstract economy to economies with a specific transactions technology is well-defined whenever that technology has a homogeneous product structure, implying that the production and exchange set correspondences can be written indifferently in terms of spot, or in terms of present-value prices and balances.

The economy has a *product structure* if:

the production set correspondence of a firm is $\Psi^j = \mathcal{Y}^j \cap (\times_{k \in \mathcal{K}} \Psi^j_k)$, where $\mathcal{Y}^j$ is the production set, independent of the prices of commodities, while $\Psi^j_k$ describes transactions constraints in commodities served by currency $k$;
the exchange set correspondence of an individual is $\Phi^i = \mathcal{Z}^i \cap (\times_{k \in \mathcal{K}} \Phi^i_k)$, where $\mathcal{Z}^i$ is the net trade set, independent of the prices of commodities, while $\Phi^i_k$ describes transactions constraints in commodities served by currency $k$.

The economy is (0-)*homogeneous* in prices of commodities and balances if it has a product structure and, for any scalar, $c > 0$, and any currency, $k$,

$$(y_k, n_k) \in \Psi^j_k(p_k) \Rightarrow (y_k, cn_k) \in \Psi^j_k(cp_k);$$
$$(z_k, m_k) \in \Phi^i_k(p_k) \Rightarrow (z_k, cm_k) \in \Phi^i(cp_k).$$

### 3.4 Characteristics of equilibria

**Multiplicity** The economy displays *multiplicity of degree I* if there exists an open set of competitive equilibrium prices of commodities, rates of interest and supplies of balances of dimension $I$.

Competitive equilibrium prices of commodities, rates of interest and supplies of balances $(p^*, r^*, M^*)$ and $(p^{**}, r^{**}, M^{**})$ are *essentially distinct* if, for an allocation, $a(p^*, r^*)$, associated with $(p^*, r^*, M^*)$, for any allocation, $a(p^{**}, r^{**})$, associated with $(p^{**}, r^{**}, M^{**})$, for some individual, $u^i(z^{i**}) \neq u^i(z^{i*})$.

The economy displays *real multiplicity of degree R* if there exists an open set of essentially distinct competitive equilibrium prices of commodities, rates of interest and supplies of balances of dimension $R$.

Competitive equilibrium prices of commodities, rates of interest and supplies of balances $(p^*, r^*, M^*)$ and $(p^{**}, r^{**}, M^{**})$ are *equivalent* if, for any allocation, $a(p^*, r^*)$, associated with $(p^*, r^*, M^*)$, there exists an allocation, $a(p^{**}, r^{**})$, associated with $(p^{**}, r^{**}, M^{**})$, such that, for every individual, $u^i(z^{i**}) = u^i(z^{i*})$.

The economy displays *nominal multiplicity of degree N* if there exists an open set of equivalent competitive equilibrium prices of commodities, rates of interest and supplies of balances of dimension $N$.

**Corollary 2.** *The economy displays $K+1$ degrees of multiplicity; if the economy is homogeneous, 1 degree of multiplicity is nominal.*

Further specification of the role of balances in the production correspondences of firms and and the exchange correspondences of individuals is required to identify degrees of real multiplicity. If individuals hold initial nominal positions, all $K + 1$ degrees of multiplicity may be real.

**Monetary equilibria** The exchange set correspondence captures, among others, the role of balances in the exchange of commodities for an individual; the production set correspondence captures, among others, the role of balances in the exchange of commodities for a firm.

Competitive equilibrium prices of commodities, rates of interest and supply of balances, $(p^*, r^*, M^*)$, are *monetary* if

$$r_k^* M_k^* \neq 0, \quad \text{for some } k \in \mathcal{K}.$$

*Money is essential* if, for some individual, if $(z, m) \in \Phi^i(p, r)$, then $m = 0 \Rightarrow pz_- = 0$; or, for some firm, if $(y, n) \in \Psi^j(p, r)$, then $n = 0 \Rightarrow py_+ = 0$.

This is the case when cash-in-advance constraints are operative.

Under assumptions implying that no production-no trade cannot be an equilibrium, if money is essential, every equilibrium is monetary.

**Optimality** An allocation associated with competitive equilibrium prices of commodities, rates of interest and supplies of balances $(p^*, r^*, M^*)$ is optimal at prices of commodities and rates of interest $(p^*, r^*)$, for supplies of balances $M^*$. This conditional optimality result is a straightforward extension of the classical welfare theorem for economies with preferences and technologies invariant with respect to prices or rates of interest. In light of the control of the supplies of balances or of the rates of interest by monetary authorities, this conditional notion of optimality is not appropriate.

A stronger characterization of optimality is not possible, as long as the dependence of transactions on prices and rates of interest remains unrestricted.

The production set correspondence is independent of the prices of commodities and rates of interest as long as the firm attains satiation in balances if $(y, \ldots, -p_k y_{k+}, \ldots) \in \Psi^j(p) \Rightarrow (y, \ldots, -\hat{p}_k y_{k+}, \ldots) \in \Psi^j(\hat{p})$.

The production set $\Psi^j$, is defined by $y \in \Psi^j \Leftrightarrow (y, \ldots, -p_k y_{k+}, \ldots) \in \Psi^j(p)$.

The exchange set correspondence is independent of the prices of commodities and rates of interest as long as the individual attains satiation in balances if $(z, \ldots, p_k z_{k-}, \ldots) \in \Phi^i(p) \Rightarrow (z, \ldots, \hat{p}_k z_{k-}, \cdots) \in \Phi^i(\hat{p})$.

The exchange set $\Phi^i$, is defined by $z \in \Phi^i \Leftrightarrow (z, \cdots \hat{p}_k z_{k-}, \ldots) \in \Phi^i(p)$.

The economy is *separable* between the consumption or production of commodities and the exchange of commodities if the production set correspondences of firms and the exchange set correspondences of individuals are independent of the prices of commodities, as long as satiation in balances is attained.

A distribution of revenue is $d = (\tau^i : i \in \mathcal{I})$, such that $\sum_{i \in \mathcal{I}} \tau^i = 0$.

With a distribution of revenue, the budget constraint of an individual is

$$pz + rm = \hat{v}^i(p, r, M) + \tau^i.$$

**Corollary 3.** *If the economy is separable, and $a(p^*, r^*)$ is a competitive equilibrium allocation, there exists a distribution of revenue, $d^*$, and an associated competitive equilibrium allocation $a(p^{**}, r^{**})$, with rates of interest $r^{**} = 0$, that is weakly Pareto superior. A competitive equilibrium allocation, $a(p^{**}, r^{**})$, associated with rates of interest $r^{**} = 0$, is Pareto optimal.*

**Proof** It suffices to show that there exists $d^*$ such that, for each $i, p^{**}z^{i*} \leq p^{**}z^{i**} = v^{i**} + \tau^{i*}$. If $\tau^{i*} = p^{**}z^{i*} - v^{i**}$, then

$$\sum_{i \in I} \tau^{i*} = p^{**} \sum_{i \in I} z^{i*} - p^{**} \sum_{j \in J} y^{j**} \leq$$

$$p^{**} \sum_{i \in I} z^{i*} - p^{**} \sum_{j \in J} y^{j*} = 0. \qquad \square$$

This result establishes existence of a suitable distribution of revenue, but is of little help to a compute it, since it depends upon prices $p^{**}$ that clear markets given the distribution. A weaker, but more operational result, is that, *starting from a realised* competitive equilibrium allocation $a(p^*, r^*)$, one can define *explicitly* a weakly Pareto-improving distribution $d^*$, namely $\tau^{i*} = \theta^{i,J+1} v^{J+1*} - rm^{i*} + \sum_{j \in J} \theta^{ij} rn^{j*}$.

It is optimal for individuals and firms to attain satiation in balances.

# 4 Time, uncertainty and multiplicity

## 4.1 Multiplicity

A simple cash-in-advance economy was defined in section 2 by:

(i) individuals, $i$, solving

$$\max_{z,m} u^i(z),$$

subject to

$$m_{s_t} = p_{s_t} z_{s_{t-1}}, \quad s_t \in \mathcal{N},$$

and the semi-reduced form constraints

$$p_1 z_1 + \sum_{s_2 \in S_2} q_{s_2} \eta_{s_2} + \hat{r}_1 m_1 = v_1^i,$$

$$p_{s_t} z_{s_t} + \sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}} \eta_{s_{t+1}} + \hat{r}_{s_t} m_{s_t} = \eta_{s_t} + v_{s_t}^i,$$

$$p_{s_T} z_{s_T} + \hat{r}_{s_T} m_{s_T} = \eta_{s_T} + v_{s_T}^i;$$

(ii) firms, $j$, solving

$$\max_{y,n} \sum_{s_t \in \mathcal{N}} \tilde{q}_{s_t} (p_{s_t} y_{s_t} + \hat{r}_{s_t} n_{s_t}) = \sum_{s_t \in \mathcal{N}} \tilde{q}_{s_t} v_{s_t}^j$$

subject to

$$-n_{s_t} = p_{s_t} y_{s_{t+}}, s_t \in \mathcal{N};$$

(iii) the bank, announcing the interest rates $r$ and distributing profits

$$v_{s_t}^{J+1} = \hat{r}_{s_t} M_{s_t}, s_t \in N.$$

Using the notation $\tilde{p}_{s_t} = \tilde{q}_{s_t} p_{s_t}, \tilde{m}_{s_t}^i = \tilde{q}_{s_t} m_{s_t}^i, \tilde{n}_{s_t}^j = \tilde{q}_{s_t} n_{s_t}^j, \tilde{M}_{s_t} = \tilde{q}_{s_t} M_{s_t}, \tilde{v}_{s_t}^i = \sum_{s_t \in N} \tilde{q}_{s_t} v_{s_t}^i, \tilde{v}_{s_t}^j = \sum_{s_t \in N} \tilde{q}_{s_t} v_{s_t}^j$ and $\tilde{v}^{J+1} = \sum_{s_t \in N} \tilde{q}_{s_t} v_{s_t}^{J+1}$, reducing the individual budget constraints to their present value consolidation yields the alternative formulation

(i') $\max_{z,\tilde{m}}\ u^i(z)$ subject to $\tilde{m}_{s_t} = \tilde{p}_{s_t} z_{s_{t-}}$ and $\tilde{p}z + \hat{r}\tilde{m} = \tilde{v}^i$;

(ii') $\max_{y,\tilde{n}}\ \tilde{p}y + \hat{r}\tilde{n} = \tilde{v}^j$ subject to $-\tilde{n}_{s_t} = \tilde{p}_{s_t} y_{s_{t+}}$;

(iii') $\tilde{v}^{J+1} = \hat{r}\tilde{M}$.

This alternative formulation fits the definition of an abstract economy. Hence, by proposition 1, there exist, for arbitrary $\hat{r} \geq 0$ and $c > 0$, competitive equilibrium prices $\tilde{p}^*$ (with $\tilde{p}^* 1^{NL} = c$) and an associated allocation $a(\tilde{p}^*, \hat{r}) = ((z^{i*}, \tilde{m}^{i*})_{i\in I}, (y^{j*}, \tilde{n}^{j*})_{j \in J}, \tilde{M}^*)$.

Actually, there exists a multiplicity of degree $S-1$ of competitive equlibria with $(p^*, \tilde{q}^*, r)$ such that $\tilde{q}_{s_t}^* p_{s_t}^* = \tilde{p}_{s_t}^*, s_t \in N$, and $(r/(1+r)) = \hat{r}$. The construction of the set of these equilibria, starting from some $(\tilde{p}^*, a(\tilde{p}^*, \hat{r}))$, is straight forward. It is convenient and more transparent to exhibit at once the multiplicity of degree $S$ of equilibria obtained by varying as well the constant $c$ in proposition 1. The construction then proceeds as follows.

1. Select arbitrarily positive constants $c_{s_T} > 0, s_T \in S_T$, and define the nominal spot price vector $p_{s_T}^* > 0$ by $p_{s_T}^* \tilde{q}_{s_T}^* = \tilde{p}_{s_T}^*$, with $\tilde{q}_{s_T}^* > 0$, such that $\sum_{l \in \mathcal{L}} p_{\ell s_T}^* = c_{s_T}$.
2. For each $s_{T-1}$, for each $s_T \in S_T(s_{T-1})$, define $q_{s_T}^*$ (the nominal spot price at $s_{T-1}$ of elementary security $s_T$) by $q_{s_T}^* = \alpha_{s_T} \tilde{q}_{s_T}^*$, with $\alpha_{s_T} > 0$, such that $\sum_{s_T \in S_T(s_{T-1})} q_{s_T}^* = (1 + r_{s_{T-1}})^{-1}$. Define $\tilde{q}_{s_{T-1}}^* = \tilde{q}_{s_T}^* / q_{s_T}^*$ (any $s_T \in S_T(s_{T-1})$, the ratio is unique) and $p_{s_{T-1}}^* = \tilde{p}_{s_{T-1}}^* / \tilde{q}_{s_{T-1}}^*$.
   This general step may be repeated backwards for each $s_{T-2}, s_{T-3}, \cdots$ down to $s_1$. This yields vectors $p^*, \tilde{q}^*, q^*$ such that, for each $s_t$, $\tilde{p}_{s_t}^* = \tilde{q}_{s_t}^* p_{s_t}^*$ and $\sum_{s_{t+1} \in S_{t+1}(s_t)} q_{s_{t+1}}^* = (1 + r_{s_t})^{-1}$, where $\tilde{p}_{s_t}^*$ is a competitive equilibrium price vector of the abstract economy, for some normalising constant $c$ (implicitly determined by the arbitrary constants $c_{s_T}$). From now on, the variables $(\tilde{m}^{i*}, \tilde{n}^{j*}, M^*)$ are taken congruent with that normalising constant.
3. For each $s_t \in N$, for each $i$, or $j$, define $m_{s_t}^{i*} = \tilde{m}_{s_t}^{i*} / \tilde{q}_{s_t}^*, n_{s_t}^{j*} = \tilde{n}_{s_t}^{j*} / \tilde{q}_{s_t}^*$; define $M_{s_t}^* = \tilde{M}_{s_t}^* / \tilde{q}_{s_t}^*$. By homogeneity (by inspection, in this

simple case), for each $i$, resp. $j$, $(z^{i*}, m^{i*}) \in \Phi^i(p^*)$ and $(y^{j*}, n^{j*}) \in \Psi^j(p^*)$. Also, define $v^{j*}_{s_t} = p^*_{s_t} y^{j*}_{s_t} + \hat{r}_{s_t} n^{j*}_{s_t}, v^{J+1*}_{s_t} = \hat{r}_{s_t} M_{s_t}$.

4. For each $s_T \in S_T$, for each $i$, define $\eta^{i*}_{s_T} - p^*_{s_T} z^{i*}_{s_T} + = \hat{r}_{s_T} m^*_{s_T} - v^{i*}_{s_T} = \hat{r}_{s_T} m^*_{s_T} - \sum_{j \in J} \theta^{ij}(p^*_{s_T} y^{j*}_{s_T} + \hat{r}_{s_T} n^{j*}_{s_T}) - \theta^{iJ+1} \hat{r}_{s_T} M^*_{s_T}$. Equilibrium on the commodity markets and the definition of $M^* = \sum_{i \in I} m^{i*} - \sum_{j \in J} n^{j*}$ imply $\sum_{i \in I} \eta^{i*}_{s_T} = 0$ (equilibrium on the market for elementary security $s_T$, as per Walras law).
5. For each $s_{T-1} \in S_{T-1}$, for each $i$, define $\eta^{i*}_{s_{T-1}} = p^*_{s_{T-1}} z^{i*}_{s_{T-1}} + \sum_{s_T \in S_T(s_{T-1})} q^*_{s_T} \eta^{i*}_{s_T} + (r_{s_T}/(1 + r_{s_T})) m^{i*}_{s_{T-1}} - v^{i*}_{s_{T-1}}$. Again, $\sum_{i \in I} \eta^{i*}_{s_{T-1}} = 0$. This general step may be repeated backwards for each $s_{T-2}, s_{T-3}, \cdots$, down to $s_1$.

In short: since prices of elementary securities, $q_{s_t}$, can be set arbitrarily, subject to the arbitrage conditions, given non - negative rates of interest $r_{s_t}$, there exists a competitive equilibrium allocation with arbitrary values for $q_{s_T}$, the prices of elementary securities associated with terminal nodes or, equivalently, the level of prices, $\sum_{l \in \mathcal{L}} p_{l, s_T}$, at the terminal nodes.

**Proposition 2.** *In the economy that extends over finite time under uncertainty, if the economy is 0-homogeneous, then, for nominal rates of interest $r \geq 0$, there exist, for any scalars $c_{S_T} = (\ldots, c_{s_T}, \ldots) \gg 0$, competitive equilibrium contingent prices of commodities, $\tilde{p}^*$, and associated spot prices, $p^*$, with $\sum_{l \in \mathcal{L}} p^*_{l, s_T} = c_{s_T}$, for every state of the world $s_T$. The allocations of commodities associated with alternative scalars, $c_{S_T}$ are equivalent.*

It thus takes more than standard monetary policy to control the variability of price levels across future events.

### 4.2 Example

A parametric economy illustrates the results.

Dates are $t = 1, 2$.

States of the world $s \in \mathcal{S}$, a finite set, realize at date 2; date-events are 1 and $s_2 \in \mathcal{S}_2 = \mathcal{S}$.

There is one commodity; a quantity of the commodity at a date event is $z_1$ or $z_{s_2}$. The prices of the commodities at a date-event are $p_1$ or $p_{s_2}$.

Elementary securities and bonds of one date maturity effect transfers of revenue between date 1 and the events at date 2. Holdings of the elementary security with payoff one unit of revenue at the date-event $s_2$, are $\eta_{s_2}$, and its price is $q_{s_2}$; rates of interest are $r_1$ and $r_{s_2}; \hat{r} = r(1 + r)^{-1}$.

There is no production. Individuals are $i = 1, 2$.

Money serves as a medium of exchange at each date-event. Balances at a date-event are $m_1$ and $m_{s_2}$. The supply of balances is $M_1$ and $M_{s_2}$.

The preferences of individual 1 over net trades are described by the intertemporal von Neumann-Morgenstern utility function defined by $u^1(z_1, \cdots,$

$z_{s_2}, \ldots) = \ln(e_1 + z_1) + \delta \mathrm{E} \ln(z_{s_2})$, and of individual 2 by $u^2(z_1, \ldots, z_{s_2}, \ldots) = \ln(z_1) + \delta \mathrm{E} \ln(e_{s_2} + z_{s_2})$, where expectations are with respect to a strictly positive probability measure, $\pi = (\ldots, \pi_{s_2}, \ldots)$ over the set of states of the world or events at date 2. Equivalently, individuals 1 and 2 have logarithmic utility functions over consumption, and endowments $e_1$ and $e_{s_2}$, respectively. The share of individual 1 in the bank at each date-event is $\theta^1 \geq 0$, and that of individual 2 is $\theta^2 = 1 - \theta^1 \geq 0$.

A cash-in-advance constraint is operative at each date - event.

The sequence of budget constraints for an individual is

$$p_1 z_1 + \sum_{s_2} q_{s_2} \eta_{s_2} + \hat{r}_1 m_1 = v_1^i$$

$$p_{s_2} z_{s_2} + \hat{r}_{s_2} m_{s_2} = \eta_{s_2} + v_2^i.$$

Given their specialised endowments, agents trade as follows.

At date 1, individual 1 buys elementary securities in amounts $\eta^1_{s_2} > 0$, and pays for them with cash borrowed from the bank; individual 2 sells elementary securities in amounts $\eta^2_{s_2} < 0$, and uses the proceeds to buy the commodity from individual 1, who in turn uses the proceeds to settle his loan from the bank. The bank lends $\sum_{s_2} q_{s_2} \eta^1_{s_2}$ to individual 1 at a profit $v_1 = \hat{r}_1 \sum_{s_2} q_{s_2} \eta^1_{s_2}$ which it distributes.

At date 2, state $s_2$, individual 1 collects $\eta^1_{s_2}$ and uses the proceeds to buy an amount $z^1_{s_2}$ of the commodity from individual 2. Individual 2 borrows $\eta^2_{s_2}$ from the bank to honour his elementary security committment. He then sells an amount $z^2_{s_2}$ of the commodity and uses the proceeds to settle his loan from the bank. The bank lends $-\eta^2_{s_2}$ to individual 2 at a profit $v_{s_2} = -\hat{r}_{s_2} \eta^2_{s_2}$ which it distributes.

The banks sets the interest rates $r_1$ and $r_{s_2}$. In order to exhibit a readable analytical solution, we impose $r_{s_2} = r_2$ for all $s_2$ in $\mathcal{S}_2$. In line with proposition 2, we take the spot prices of the commodity at date 2, $p_{s_2}$, as given and solve for the prices of elementary securities $q_{s_2}$, for $p_1$, and for the trades in assets and commodities. We invite readers to consider the simpler formulae that obtain when $\hat{r}_1 = \hat{r}_2 = 0$, or when $\mathcal{S} = 1$ (certainty).

After endogenising the profit distribution, the equilibrium net trades are given by

$$-z_1^1 = z_1^2 = \sum_{s_2 \in \mathcal{S}_2} q_{s_2} \eta^1_{s_2} / p_1 (1 + \hat{r}_1 + \hat{r}_1 \theta^1)$$

$$z^1_{s_2} = -z^2_{s_2} = \eta^1_{s_2} (1 + 2\hat{r}_2) / p_{s_2} (1 + \hat{r}_2 + \hat{r}_2 \theta^2).$$

The solution for $q_{s_2}$ is transparent. Define $(ep)^H = (\sum_{s_2 \in \mathcal{S}_2} \pi_{s_2} / e_{s_2} p_{s_2})^{-1}$, the harmonic mean of the (exogenous) value of period 2 endowments. Then:

$$q_{s_2} = \frac{\pi_{s_2}}{e_{s_2} p_{s_2}} \frac{(ep)^H}{1 + r_1}.$$

As expected, $q_{s_2}$ is homogeneous of degree $-1$ in $p_{s_2}$; the product $q_{s_2} p_{s_2}$ is determined by primitives. The solution for $\eta^1_{s_2}$ is unfortunately less transparent:

$$\eta^1_{s_2} = e_{s_2} p_{s_2} (1 + \hat{r}_1)(1 + \hat{r}_2 + \hat{r}_2 \theta^2) / \left\{ \delta(1 + 2\hat{r}_1) + (1 + \hat{r}_2)(1 + \hat{r}_1 + \hat{r}_1 \theta^1) \right\}.$$

The distribution of bank profits is the source of the complication. Note however that $q_{s_2} \eta^1_{s_2}$ is constant across the states $s_2$, under our shortcut of setting $r_{s_2} = r_2$ across states. The solution for $p_1$ is disheartening, though it simplifies to $p_1 = (ep)^H / e_1$, when $r_1 = r_2 = 0$. In general

$$p_1 = \frac{(ep)^H}{e_1(1 + 2r_1)} \frac{(1 + 2\hat{r}_2)(1 + \hat{r}_1 + \hat{r}_1 \theta^1) + \delta(1 + \hat{r}_1)(1 + \hat{r}_2 + \hat{r}_2 \theta^2)}{(1 + \hat{r}_2)(1 + \hat{r}_1 + \hat{r}_1 \theta^1) + \delta(1 + 2\hat{r}_1)(1 + \hat{r}_2 + \hat{r}_2 \theta^2)}.$$

It is of interest to exhibit the ratio, across individuals, of the marginal rates of substitution (MRS's) between consumption at different date-events. For two states $s_2, s'_2$, the MRS's of the two individuals are the same, and equal to $\frac{\eta^1_{s_2} p_{s'_2}}{\eta^1_{s'_2} p_{s_2}}$. On the other hand, the MRS's between consumption at $s_1$ and $s_2$ differ by a factor $(1 + \hat{r}_1)$, reflecting the misallocation induced by the cash-in-advance constraint.

## 5 Conclusion

In economies with an operative transactions technology, and money that provides liquidity services, competitive equilibria exist; which assures the consistency of the specification.

Nevertheless, competitive equilibrium allocations are multiple; and they may be Pareto comparable. If monetary policy sets the rates of interest (either directly, or indirectly through the money supply), then under certainty all future rates of inflation are determined, but the overall price level remains indeterminate. Inflation rates are positively correlated with nominal interest rates, pointing to the limitations of equilibrium theory to study out of equilibrium situations, a frequent concern of monetary authorities. Under uncertainty, expected rates of inflation are determined, but the variability of inflation is indeterminate, even though markets for nominal securities are complete. This leaves room for alternative determinants of inflation paths, for instance: (i) a more comprehensive monetary policy, setting not only interest rates but also quantities of money, or exchange rates, or prices of elementary securities; (ii) a process of nominal price adjustments, driven by price-setting agents, nominal contracts a.s.o.; (iii) nominal constraints associated with initial positions (e.g. public debt), policy committments (e.g., pensions) or the like.

The multiplicity of equilibria delimits the scope of monetary policy and the degrees of freedom available to policy makers. The welfare implications of alternative choices of policy parameters demonstrate the real effects of monetary policy.

Wealth effects that reflect imperfect "Ricardian equivalence," and price rigidities or structural constraints on price adjustments also make for effective monetary policy.

Speculatively, nominal rigidities (price-level stickiness) might be the saving grace of monetary economies: they prevent price-level jumps, which otherwise could generate excessive inflation variability, unchecked by monetary policy. Downward rigidities at controlled rates of expected inflation place on upper bound on inflation variances.

The failure of monetary policy to determine the distribution of the rate of inflation parallels the results in Lucas and Stokey (1987), Sargent and Wallace (1975) or Woodford (1994).

An extension encompasses a market for assets which is incomplete.

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