The Economic Implications of an Incomplete Asset Market

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When the asset market is incomplete, the role of prices extends beyond conveying the aggregate scarcity of commodities. In conjunction with the asset structure, they determine the attainable reallocations of revenue. This affects nontrivially the existence, optimality, and determinacy of competitive equilibrium allocations, as well as the revelation of information by prices. Further, it accounts for diverse phenomena, among them the preservation of memory in macroeconomics aggregates.

A simple exchange economy extends over two periods. Uncertainty, indexed by finitely many states of nature, $s=1,\dots,S$, is resolved in the second period. Commodities, $l=1,\dots,L$, are traded in spot markets in the second period after the uncertainty has been resolved and assets have paid off. A commodity bundle is $x=(\dots,x(s),\dots)=(\dots,x_l(s),\dots)$. Commodity prices are $p=(\dots,p(s),\dots)=(\dots,p_l(s),\dots)$.

Assets, $a=1,\dots,A$, are traded in the first period and pay off in the second. A portfolio is $y=(\dots,y_a,\dots)$. Assets are real: the payoff of an asset is a commodity bundle $r_a=(\dots,r_a(s),\dots)$. At commodity prices p, the payoff of an asset, more precisely the payoff of an asset in terms of revenue, is

$$r_a(p)=p'\Box r_a=(\dots,p'(s)r_a(s),\dots).$$

For example, the payoff of a futures contract in commodity l is $r_a(p)=(\dots,p_l(s),\dots)$.

The asset structure is the matrix $R=(\dots,r_a,\dots)$. Asset payoffs are $R(p)=(\dots,r_a(p),\dots)$.

A reallocation of revenue is $\tau=(\dots,\tau(s),\dots)$. The attainable reallocations of revenue coincide with the column span of the matrix of asset payoffs, $[R(p)]=\{\tau:\tau=R(p)y$, for some portfolio, $y\}$.

An individual expresses excess demand for commodities, $z=(\dots,z(s),\dots)$, so as to maximize utility. The reallocation of revenue necessary to finance the excess demand is $p'\Box z=(\dots,p'(s)z(s),\dots)$. An individual faces the attainability constraint $p'\Box z\in[R(p)]$, in addition to the budget constraint $p'z=0$.

The attainable reallocations of revenue depend nontrivially on commodity prices. For example, if the asset structure consists of forward contracts in two commodities, and if the relative prices of these two commodities coincide in two states of nature:

$$p_l(s)/p_{l'}(s)=p_l(s)/p_{l'}(s'),$$

independent reallocations of revenue into these two states are not attainable.

The asset market is complete if and only if all reallocations of revenue are attainable; otherwise it is incomplete. Evidently, for the asset market to be complete, it is necessary but not sufficient that $A=S$. At competitive equilibrium prices, the commodity markets clear; the asset market clears as a residual. Associated with competitive equilibrium prices there is an allocation of commodities; also an allocation of assets.

Competitive equilibrium allocations typically fail to be constrained Pareto optimal. Since individuals optimize under the attainability constraint, it is not surprising that competitive equilibrium allocations typically fail to be optimal. What is surprising is that competitive allocations typically fail to satisfy an appropriate criterion of constrained optimality.

At a given allocation of assets, the attainability constraint takes the form $p'\Box z=R(p)\bar{y}$, where $\bar{y}$ is the portfolio assigned to

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the individual. At competitive equilibrium prices for a given allocation of assets, commodity markets clear. Associated with competitive equilibrium prices for a given allocation of assets, there is an allocation of commodities. A competitive equilibrium allocation of commodities is constrained pareto suboptimal, if and only if there exists a competitive equilibrium allocation for a given allocation of assets that Pareto dominates it.

When the asset market is incomplete, competitive equilibrium allocations of commodities are typically constrained suboptimal. The reason is as follows. A reallocation of assets and the resulting reallocation of revenue across individuals and across states of nature have two effects on the welfare of an individual: a direct, revenue effect; and an indirect relative prices effect due to the change in relative prices required to attain market clearing following the reallocation of revenue at each state of nature. Since at a competitive equilibrium individuals optimize in the asset market, the revenue effect following a reallocation of assets cannot be Pareto improving. Nevertheless, when the asset market is incomplete, the marginal rates of substitution of revenue across states of nature for different individuals typically do not coincide. The relative price effect may then yield a Pareto improvement. (See John Geanakoplos and myself, 1986.)

Constrained suboptimality imposes on a policymaker who intervenes in the asset market the constraint to employ only the instruments available to the market, the existing assets. The claim of constrained suboptimality is strengthened by the result that the observable behavior of individuals (their demand functions in the asset market and the spot commodity markets), suffices in order to recover the utility functions of individuals even when the asset market is incomplete and hence to determine improving interventions. (See Geanakoplos and myself, 1990.)

Competitive equilibria may not exist. This is due to the discontinuous dependence of the attainable reallocations of revenue on commodity prices. For example, if the asset structure consists of forward contracts in two commodities, discontinuities occur at commodity prices at which the relative price of these two commodities coincide at all states of nature:

$$p_l(s)/p_{l'}(s) = p_l(s')/p_{l'}(s'), \text{ for all } s, s'.$$

At such prices, the two-dimensional set of attainable reallocations of revenue changes discontinuously to a one-dimensional set. The resulting discontinuity in demand may cause competitive equilibria not to exist. Nevertheless, for a family of economies indexed by the asset structure and the endowments of individuals, competitive equilibria typically do exist. (See Darrell Duffie and Wayne Shafer, 1985.)

Alternatively, assets are nominal: their payoffs are denominated in abstract units of account, inside fiat money in zero net supply.

The prices of units of account are $\delta = (\ldots, \delta(s), \ldots)$; equivalently, this is the purchasing power of units of account. The payoff of an asset in units of accounts is $r_a = (\ldots, r_a(s), \ldots)$. The payoff of an asset is $r_a(\delta) = \delta' \square r_a$. The matrix of asset payoffs is $R(\delta) = (\ldots, r_a(\delta), \ldots)$. The attainable reallocations of revenue are $[R(\delta)] = \{\tau: \tau = R(\delta)y, \text{ for some portfolio, } y\}$.

The attainability constraint takes the form $p' \square z \in [R(\delta)]$. At competitive equilibrium prices for commodities and units of account, the markets for commodities and units of account clear; the asset market clears as a residual. Associated with competitive equilibrium prices, there is an allocation of commodities; also of units of account and of assets.

In order to eliminate nominal differences across equilibria, let some commodity be numeraire, $p_{l*}(s) = 1$, for all s. Since units of account are in zero net supply and provide no liquidity or other services, the market for units of account clears at all prices. Evidently, the price of units of account or their purchasing power can be normalized to equal one at each state of nature; then, the budget constraint at each state of nature is not homogeneous.

Indeterminacy occurs, and neutrality (the dichotomy between real and nominal variables) fails when competitive equilibrium commodity allocations associated with dis-

tinct prices for units of account are distinct. When the asset market is complete, indeterminacy does not occur. This is evident since all reallocations of revenue are attainable and the attainability constraint vanishes.

Suppose, for simplicity, that asset $a=1$ pays off one unit of account in state $s=1$ and zero in every other state, while assets $a=2,\ldots,A$ pay off zero in state $s=1$; equivalently, consumption in state $s=1$ can be interpreted as consumption during the first, contracting period.

When the asset market is incomplete, variations in the prices of units of account in states $s=2,\ldots,S$ that are not colinear typically alter the attainable reallocations of revenue across these states. Since the individual's holdings of assets $a=2,\ldots,A$ are not constrained by the budget constraint, the associated competitive equilibrium allocations of commodities are distinct as well; this is the case provided individuals are sufficiently numerous and diverse. (See David Cass, 1985.) The degree of indeterminacy, the dimension of a set of distinct competitive equilibrium allocations of commodities in $S-2$. (See Geanakoplos and Andrew Mas-Colell, 1989.)

Indeterminacy occurs even when the unit of account is outside money in positive net supply that provides liquidity services. It suffices that individual precautionary demand for money balances be determined before the resolution of uncertainty in the second period. The indeterminacy of competitive equilibrium commodity allocations translates into the indeterminacy of asset prices in the first period. When individuals are differentially informed, rational expectations equilibria may be completely noninformative. (See my 1990 paper with Paolo Siconolfi.)

Consider an exchange economy of overlapping generations. A single perishable consumption good is available each period. An aggregate individual is born each period and lives for three periods. The economy is stationary: individuals are identical other than for the calendar time of their birth. The endowment of the individuals in the three periods of his life is $(\theta,1,0)$, and his intertemporal, von Neumann-Morgenstern utility function over consumption in the three periods of his life, (x_1,x_2,x_3) is $\ln x_2+\delta\ln x_3$, $1>\delta>0$.

The endowment of the aggregate individual in the first period of his life is a positive random variable that is i.i.d. in time according to the frequency function $\pi(\theta)$. It follows from the time invariance and intertemporal separability of individual preferences and of the aggregate shock that an optimal and time-invariant allocation is i.i.d. in time; aggregate output is

$$y_t^* = y^*(\theta_t) = \theta_t + 1 - x_2(\theta_t)$$
$$= (\delta/(1+\delta))(1+\theta_t).$$

This is a competitive equilibrium allocation if the asset market is complete: before economic activity commences, elementary securities are traded for the transfer of revenue across time periods and realizations of uncertainty; all individuals, irrespective of the calendar time and realization of uncertainty of their birth participate in this initial exchange.

Alternatively, with outside fiat money in fixed supply, M, as the store of value, at a time-invariant allocation aggregate output is

$$\hat{y}_t = \hat{y}(\theta_t, \hat{m}_t)$$
$$= \frac{(\theta_t+\delta+\theta_t\delta)(M-\hat{m}_t)}{M+\delta(M-\hat{m}_t)},$$

where $\hat{m}_t$ is the money held by individuals in the second period of their lives and $M-\hat{m}_t$ by those in the third period. Evidently, since $\hat{m}_t = \hat{p}_{t-1}\theta_{t-1}$, aggregate output is serially correlated; so is the price level,

$$\hat{p}_t = \hat{p}(\theta_t, \hat{m}_t)$$
$$= \frac{M+\delta(M-\hat{m}_t)}{\theta_t+\delta+\delta\theta_t}.$$

Also, a Philips trade off exists between the deviation of output from its optimal level and the rate of inflation. (See Jayasri Dutta and myself, 1990.)

Here, the incompleteness of the asset market is primitive. It is an open question to derive the incompleteness of the asset market from primitive differences of information among individuals.

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