Incomplete markets

H. M. Polemarchakis * Sergio Turner †

S. Durlauf and L. Blume (eds.), *The New Palgrave: A Dictionary of Economic Theory and Doctrine, Second Edition* MacMillan Press, forthcoming

When thew asset market is incomplete, prices do not only convey the aggregate scarcity of commodities. In conjunction with the asset structure, they determine the attainable reallocations of revenue. This affects the existence, optimality and determinacy of competitive equilibrium allocations.

More importantly, it causes competitive allocations to fail the criterion of constrained optimality which allows for the incompleteness of the asset markets. The constrained suboptimality of competitive equilibria in economies with an incomplete asset market makes an important methodological point: intervention is often said to be unnecessary, because competitive equilibrium is Pareto efficient; since incompleteness is ubiquitous, such a view is untenable; intervention may be counterproductive because the fiscal authority does not know enough, about tastes and technology, to set the right taxes and subsidies; not because there are no beneficial taxes and subsidies. The argument shows that such taxes exist; it does not indicate how to compute them.

An elementary exchange economy extends over two periods. Uncertainty, described by states of nature $s = 1, \ldots, S$ is resolved in the second period. Commodities $l = 1, \ldots, L$ are traded in spot markets in the second period after the uncertainty has been resolved and assets have paid off. A commodity bundle is $x = (\ldots, x(s), \ldots) = (\ldots, x_l(s), \ldots)$; prices of commodities are $p = (\ldots, p(s), \ldots) = (\ldots, p_l(s), \ldots)$.

Assets $a = 1, \ldots, A$ are traded in the first period an d pay off in the second. A portfolio of assets is $y = (ldots, y_a, \ldots)$. Assets are real: the payoff of an asset is a commodity bundle $r_a = (\ldots, r_a(s), \ldots) = (\ldots, r_{a,l}(s), \ldots)$. At commodity prices p, the payoff of an asset in terms of revenue is

$$r_a(p) = p' \Box r_a = (\ldots, p'(s) r_a(s), \ldots).$$

For example, the payoff of a futures contract in commodity l is $r_a(p) = (\ldots, p_l(s), \ldots)$. The asset structure is described by the matrix of payoffs of assets $R = (\ldots, r_a, \ldots)$. Asset payoffs in terms of revenue are $R(p) = (\ldots, r_a(p), \ldots)$.

*Department of Economics, University of Warwick; `h.polemarchakis @warwick.ac.uk`
†Department of Economics, Brown University; `sergio_turner@brown.edu`

A reallocation of revenue is $\tau = (\ldots, \tau(s), \ldots)$. The attainable reallocations of revenue coincide with the column span of the matrix of payoffs of assets in terms of revenue, $[R(p)] = \{\tau : \tau = R(p)y, \text{for some portfolio of assets,} y\}$.

The reallocation of revenue that finances an excess demand for commodities, $z = (\ldots, z(s), \ldots)$, is $p' \Box z = (\ldots, p'(s)z(s), \ldots)$. An individual expresses excess demand so as to maximize utility subject to the budget constraint, $p'z = 0$ and the attainability constraint $p \Box z \in [R(p)]$.

The attainable reallocations of revenue may depend non-trivially on relative commodity pricers. For example, if there are two assets, forward contacts in commodities l and l^prime, and if the relative price of these commodities in states of nature s and s' coincide, $p_l(s)/p_{l'}(s) = p_l(s')/p_{l'}(s')$, then, independent transfers of revenue in these states are not attainable. Attainable reallocations of revenue are independent of relative commodity prices if asset payoffs are denominated in the same numéraire commodity: $r_{a,l}(s) = 0$, for $a = 1, \ldots, A$ and $l = 1, \ldots, l^* - 1, l^* + 1, \ldots, L$; the numéraire commodity may vary across states of nature, but not across assets in a given state; or, it may be a bundle of commodities.

The asset market is complete if all reallocations of revenue are attainable, independently of spot commodity prices. For the asset market to be complete it is necessary but not sufficient that $A = S$.

At a competitive equilibrium prices, commodity markets clear; asset markets clear as a residual. Associated with competitive equilibrium prices there is an allocation of commodities; also an allocation of portfolios of assets.

With numéraire assets, competitive equilibria exist under standard assumptions, and, generically, they are strongly regular: competitive equilibrium allocations are described, locally, by finitely many continuously differentiable functions of the distribution of endowments; in addition, equilibrium allocations in commodity spot markets are described, locally, by finitely many continuously differentiable functions of the allocation of distribution of portfolios of assets.

When the asset market is incomplete, competitive equilibrium allocations generically are not Pareto optimal. More importantly, they fail a less demanding criterion of constrained optimality which allows for the incompleteness of the asset markets: there exist reallocations of portfolios that lead to Pareto improvements in welfare after spot commodity prices adjust and markets clear.

The intuition for constrained optimality is as follows:

A reallocation of assets and, as a consequence, of revenue across across individuals and across states of nature has two effects on the welfare of an individual: a direct, revenue effect; and an indirect relative price affect as spot commodity prices adjust to maintain equilibrium. Since individuals optimize in the asset market, the revenue effect cannot be Pareto improving. But, when the asset market is incomplete, the marginal rates of substitution ——————————
—————————— —————————————————————— ——————————
——————————————

An example

There are two types of individuals, $i = \alpha, \beta$; each type consists of a continuum of individuals of unit mass.

Dates are 0 and 1.

One commodity is exchanged and consumed at date 0, and quantities of the commodity are x, while two commodities, $l = a, b$, are exchanged and consumed at date 1, and quantities of the commodities are x_a and x_b.

The intertemporal utility function of an individual of type β is

$$u^\beta = x + (1-\gamma)\ln x_a + \gamma \ln x_b, \quad 0 < \gamma < 1,$$

and his endowment at date 1 consists of b units of commodity b.

The intertemporal utility function of an individual of type α is

$$u^\alpha = x + \gamma \ln x_a + (1-\gamma)\ln x_b,$$

and his endowment at date 1 consists only of commodity a; but, importantly, it is subject to idiosyncratic shocks: it is $a \pm \varepsilon$, with equal probability.

At date 1, equal proportions of individuals of type α have endowments $a+\varepsilon$, and $a-\varepsilon$, and, as a consequence, there is no aggregate risk.

With quasi-linear preferences, it is not necessary to specify the endowments of individuals at date 0.

At date 0, the consumption good is numéraire, while q is the price of a risk-free bond of that matures at date 1.

At date 1, commodity a is numéraire, while the price of commodity b is p.

With holdings of the bond y for individuals of type α and $-y$ for individuals of type β, the equilibrium price at date 1 is

$$p(y) = \frac{(1-\gamma)a + (1-2\gamma)y}{(1-\gamma)b},$$

which depends non-trivially on asset holdings as long as $\gamma \neq 1/2$.

At date 1, the marginal utility of revenue for individuals of type β is

$$\lambda^\beta = \frac{1}{pb - y},$$

while, for individuals of type α, it varies with the realization of the idiosyncratic shock – the personal state of each individual – and is

$$\lambda^\alpha(\varepsilon) = \frac{1}{a+\varepsilon+y}, \quad \text{or} \quad \lambda^\alpha(-\varepsilon) = \frac{1}{a-\varepsilon+y},$$

with equal probability.

The optimization of individuals of type β at date 0 requires that

$$q = \frac{1}{pb-y} = \frac{(1-\gamma)}{(1-\gamma)a - \gamma y},$$

while optimization of individuals of type α at date 0 requires that

$$q = (\frac{1}{2})\frac{1}{a+\varepsilon+y} + (\frac{1}{2})\frac{1}{a-\varepsilon+y} = \frac{a+y}{(a+y)^2-\varepsilon^2};$$

as a consequence, at equilibrium,

$$y^* = \frac{-a+\sqrt{a^2+4\varepsilon^2(1-\gamma)}}{2}.$$

Policy is a pair (dx, dy) of transfers of revenue and bonds to individuals of type α.

The welfare effects of a policy are

$$du^\alpha = dx + qdy - \left((\frac{1}{2})\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + (\frac{1}{2})\lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon) \right) p'dy,$$

and

$$du^\beta = -dx - qdy - \lambda^\beta(x_b^\beta - b)p'dy.$$

Pareto improving interventions exist if the matrix

$$\begin{pmatrix} 1 & q - \left((\frac{1}{2})\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + (\frac{1}{2})\lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon) \right) p' \\ -1 & -q - \lambda^\beta(x_b^\beta - b)p' \end{pmatrix}$$

is nonsingular, which is the case, for $\varepsilon \neq 0$: singularity of the matrix would occur if and only if

$$\frac{1}{2}\lambda^\alpha(\varepsilon)x_b^\alpha(\varepsilon) + \frac{1}{2}\lambda^\alpha(-\varepsilon)x_b^\alpha(-\varepsilon) = -\lambda^\beta(x_b^\beta - b),$$

which is equivalent to

$$\frac{1-\gamma}{p} = -\frac{1}{pb-y}\left(\frac{\gamma\,(pb-y)}{p} - b \right),$$

or $y = 0$, which occurs only in the absence of idiosyncratic shocks, with $\varepsilon = 0$.

For the purposes of illustration, suppose that $\gamma = 1/10$, $a = b = 1$ and $\varepsilon = 1/2$. Let the endowments at date 0 be fixed so that, in the absence of trade in assets, both types of individuals obtain utilities equal to zero. Graph 1 shows ex-ante date-1 utility levels for both types of individuals, after trade in commodities, for different values of asset holdings. As long as $u^\alpha > 0$ and $u^\alpha + u^\beta > 0$, trade in assets constitutes a Pareto improvement: type α individuals can compensate type β individuals at date 0 and still improve their expected utility. Competitive trade in the asset induces one such Pareto improvement: equilibrium value of y is around 0.18, with $u^\alpha + u^\beta$ near 0.04. That trade, however, is not constrained efficient: pushing y up to 0.57 would take $u^\alpha + u^\beta$ to almost 0.07. That is, type α individuals could improve about 0.03 beyond the competitive outcome, still compensating type β individuals. That trade in assets is only constrained efficient, and falls short of the first-best outcome ($u^\alpha + u^\beta$ of 0.14). It shows, nonetheless, that welfare costs associated to constrained inefficiency may be substantial.

The incompleteness of the asset market affects the revelation of information at equilibrium; and it it accounts for diverse phenomena, among them the preservation of memory in macroeconomic aggregates.

References

1. Bloise, G., J. H. Drèze and H. M. Polemarchakis (0000), "Monetary equilibria over an infinite horizon," *Economic Theory,* 00, 000-000.

2. Balasko, Y. and D. Cass (1989), "The structure of financial equilibrium, I: exogenous yields and unrestricted participation," *Econometrica,* 57, 135-162.

3. Cass, D. (1985), " On the 'number'of equilibrium allocations with financial markets," Working Paper No. 85-00, CARESS, University of Pennsylvania.

4. Duffie, D., and W. Shafer (1985; 1986) "Equilibrium in incomplete markets, I: a basic model of generic existence; II: generic existence in stochastic economies," *Journal of Mathematical Economics,* 14, 285-300; 15, 199 -216.

5. Geanakoplos, J. D. and A. Mas-Colell (1989), "Real indeterminacy with financial assets," *Journal of Economic Theory,* 47, 22-38.

6. Geanakoplos, J. D. and H. M. Polemarchakis (1990), "Observability and optimality," *Journal of Mathematical Economics,* 19, 153-165.

7. Geanakoplos, J. D. and H. M. Polemarchakis (1986), "Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete," in D. Starrett, W. P. Heller and R. M. Starr (eds.), *Uncertainty, Information and Communication: Essays in Honour of K. J. Arrow, Volume III,* Cambridge University Press, 65-96.

8. Hart, O. D. (1975), "On the Optimality of equilibrium when the market structure is incomplete," *Journal of Economic Theory,* 11, 418-443.

9. Kübler, F., Chiappori, P.-A., I. Ekeland and H. M. Polemarchakis (2002), "The identification of preferences from the equilibrium prices of commodities and assets," *Journal of Economic Theory,* 102, 000-000.

10. Radner, R. (1972), "Existence of equilibrium of plans, prices and price expectations," *Econometrica,* 40, 289-303.