SOVEREIGN DEBT AND INCENTIVES TO DEFAULT WITH UNINSURABLE RISKS

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Abstract

We show that sovereign debt is unsustainable if debt contracts are not supported by direct sanctions and default carries only a ban from ever borrowing in financial markets even in the presence of uninsurable risks and time-varying interest rate. This extension of Bulow and Rogoff (1989) requires that the present value of the endowment be finite under the most optimistic valuation. We provide examples where this condition fails and sovereign debt is sustained by the threat of loss of insurance opportunities upon default, even though the most pessimistic valuation of the endowment, the natural debt limit, is finite.

Keywords: Sovereign risk, Ponzi games, Reputational debt, Incomplete markets.

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CONTENTS

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1. INTRODUCTION

The impossibility result of Bulow and Rogoff (1989) asserts that sovereign debt is unsustainable if debt contracts are not supported by direct sanctions and default carries only a ban from ever borrowing in financial markets.[1] The intuition is that, when debt cannot be rolled over, a country can always improve upon contractual arrangements that involve repayments, positive net transfers from the country to foreign investors, by defaulting at a contingency associated with the maximal debt exposure. Under complete markets, debt is unsustainable when it is bounded by a finite present value of the sovereign's future endowment (the natural debt limit). We extend this result to incomplete markets under stronger conditions.

The argument of Bulow and Rogoff (1989) is one of arbitrage: by defaulting, the sovereign can save upon repayments, increase current consumption and replicate the same consumption pattern for the future without borrowing. In the presence of uninsurable risks, however, this arbitrage may not be feasible: even though all securities are available after default, some insurance opportunities may not be replicable without issuing further debt. Hence, the sovereign may prefer to repay the debt in order to maintain access to the opportunities of risk diversification the asset market provides

[1] Bulow and Rogoff (1989) led to a vast literature studying alternative mechanisms enforcing debt repayment in the absence of sanctions. We refer to Aguiar and Amador (2014) and Wright (2011) for a thorough discussion of the literature.

even if incomplete. As illustrated by examples, this implicit threat of insurance loss can sustain sovereign debt, in contrast with the case of complete markets, even when borrowing is bounded by a finite natural debt limit. In order to restore the validity of Bulow and Rogoff (1989), we need to identify conditions for replication after default.

Replication may fail because the cost of providing insurance in an averse contingency may become prohibitively high when debt cannot be issued. This happens when, for instance, assets have to be held for long phases of depreciation before insurance is needed, whereas hedging against an unfavorable state would be feasible when debt is permitted. The cost of providing insurance at future contingencies can be estimated as the largest present value of needed resources, whereas borrowing can only be secured by the smallest present value of future income, the natural debt limit.[2] This cost may be infinite, even when the natural debt limit is finite, thus preventing insurance after default. An analogous situation cannot occur under complete markets because, when the natural debt limit is finite, so is the cost of insurance. Replication under incomplete markets obtains if the most optimistic valuation of future endowment is finite, a property that we refer to as *high implied interest rates* by analogy to Alvarez and Jermann (2000). We show that, under an additional technical assumption, this is sufficient to restore the validity of Bulow and Rogoff (1989).

Our analysis applies to any arbitrary set of securities and, subject to high implied interest rates, to any process of prices. Beyond the generality, the absence of further restrictions on prices allows us to embed our conclusions in a competitive equilibrium framework, where the pricing kernel is in general time-varying, and simple Markov equilibria fail to exist as in Duffie et al. (1994).[3] Allowing for any set of securities clarifies that the structure of the asset market plays no direct role in determining incentives to default. Indeed, for any given asset structure, depending on prices, the condition of high implied interest rates might hold or fail, and so do incentives to default. The extent of insurance is larger the richer the variety of financial

[2]Under incomplete markets, the present value is ambiguous, as asset prices only impose bounds under no arbitrage conditions. We refer to Santos and Woodford (1997), LeRoy and Werner (2001) and our Appendix D for these basic principles.

[3]In a competitive economy with collateral constraints, Gottardi and Kübler (2015) show that Markov equilibria on a finite support do not in general exist with more than two individuals. Most of their analysis also extends to the equilibria with solvency constraints as in Alvarez and Jermann (2000) and Hellwig and Lorenzoni (2009).

assets, but this equally affects the sovereign before and after default. In particular, our approach provides a methodological framework for the analysis of a government trading bonds of several maturities, a case that requires a payoff space induced by several securities; for instance, Aguiar et al. (2016) address the recent debate on sovereign debt maturity.

The restrictions on prices needed to prove the existence of incentives to default obtain at a competitive equilibrium under primitive conditions on fundamentals. In particular, as proved by Santos and Woodford (1997), when the dividend accruing to the market portfolio is at least a constant fraction of the overall endowment, the most optimistic valuation of the aggregate endowment is necessarily finite. This case is of considerable interest on empirical grounds, because of the work of Abel et al. (1989) on dynamic efficiency. Furthermore, when individuals' preferences satisfy a uniform form of impatience, the value of the market portfolio cannot grow unboundedly relative to the aggregate endowment. This guarantees that the most optimistic present value of claims in the remote future vanishes, which is the additional technical property needed to establish Bulow and Rogoff (1989).[4]

Under Markov pricing, the conditions for the validity of Bulow and Rogoff (1989) can be verified by means of a dominant root Perron-Frobenius approach. This requires an extension to incomplete markets of the method for the characterization of dynamic efficiency, most prominently, Aiyagari and Peled (1991). The dominant root estimates the long-term interest rate, and the conditions for Bulow and Rogoff (1989) are satisfied when the long-term interest rate is positive net of growth. Incidentally, this approach is suitable to accommodate time-varying interest rate in traditional frameworks for sovereign debt analysis, for example, Eaton and Gersovitz (1981).

Hellwig and Lorenzoni (2009) argues that sovereign debt is sustainable when the interest rate is sufficiently low so as to provide repayment incentives. At a competitive equilibrium, individuals are able to exactly refinance outstanding obligations by issuing new claims. Hence, as debt can be rolled over, the sovereign has no incentive to default. In contrast, in our examples, debt cannot be rolled over, as it is bounded by a finite natural debt limit, and the sovereign prefers to repay the debt because default would be too costly in terms of implied loss of insurance.

[4] It is worth noticing that these restrictions on prices are exactly those ruling out speculative bubbles on securities in positive net supply for any valuation of their fundamental value consistent with no arbitrage opportunities (Santos and Woodford (1997)).

Pesendorfer (1992) studies repayment incentives of small open economies trading with competitive risk-neutral foreign investors while having access to a limited set of financial assets. His analysis differs from ours in a crucial aspect: the punishment in Pesendorfer (1992) is that defaulters may not hold a negative position in any of the available assets; we instead do allow defaulters to sell assets short, but only insofar as their portfolio does not involve negative payoffs (future net obligations or net liabilities). Our formulation is more in the spirit of Bulow and Rogoff (1989)'s *cash-in-advance* contract: an up-front payment in exchange of positive contingent deliveries in the future. Importantly, under Pesendorfer (1992)'s more severe punishment, sovereign debt could be sustainable even when markets are complete.

Our paper is also related to the recent work of Auclert and Rognlie (2016), which also extends Bulow and Rogoff (1989) to incomplete markets. They only consider a risk-free bond and a constant, strictly positive interest rate, which permits a constructive proof of a replication policy upon default. Our analysis is more general, as we allow for time-varying interest rate and any asset structure. However, due to our minimal assumptions, the replication strategy only obtains abstractly by means of a duality argument. To this extent, the papers are complementary.

The paper is organized as follows. In section 2, we present the intuition for our analysis. In section 3, we lay out the fundamentals of the economy. In section 4, we discuss the restrictions on the asset pricing kernel. In section 5, we restore the validity of Bulow and Rogoff (1989) under restrictions on prices. In section 6, we present examples on the failure of Bulow and Rogoff (1989) in incomplete markets. For completeness, we gather some technical properties of incomplete-markets pricing in Appendices A and D. Furthermore, we present some relevant implications of competitive equilibrium in Appendix B. Finally, we develop the analysis under Markov pricing in Appendix C.

2. ILLUSTRATIVE CASE

We develop the intuition for unsustainable sovereign debt in a simple deterministic economy. Our approach is different from the original argument in Bulow and Rogoff (1989). This alternative method brings out the logic that underlies the incentives to default in a way that can then be immediately extended to uncertainty even under incomplete markets.

Time, t, is discrete, and the initial date is $t = 0$. The sovereign is entitled to an endowment $e = (\ldots, e_t, \ldots) \geq 0$, and consumes $c = (\ldots, c_t, \ldots) \geq 0$. At any period t, the sovereign's preferences are given by an increasing (recursive) utility function. Following Bulow and Rogoff (1989), monotonicity is the only restriction on preferences.

The sovereign has access to international capital markets, where, at any time t, it can trade a one-period discount bond at price $q_t > 0$. The flow budget constraint requires that

$$p_{t+1} v_{t+1} + p_t (c_t - e_t) \leq p_t v_t,$$

where $p = (\ldots, p_t, \ldots) \gg 0$ is a sequence for present-value prices and $v = (\ldots, v_t, \ldots)$ is the evolution of sovereign wealth. For notational convenience, we state the budget constraint in terms of present-value prices, that is, by compounding interest rates over time,

$$q_t = \frac{p_{t+1}}{p_t}.$$

The sovereign holds v_{t+1} units of the one-period bond in period t, each delivering one unit of consumption in the following period. This quantity is a claim if positive (because the sovereign purchases the bond) and a liability if negative (because the sovereign sells short the bond). We assume that $v_t < 0$ in some period t, for otherwise no default incentive would arise.

Bulow and Rogoff (1989) assume that sovereign debt, $-v_t$, never exceeds the market value of a claim on the country's future income stream, that is,

$$-v_t \leq g_t = \frac{1}{p_t} \sum_{r \geq 0} p_{t+r} e_{t+r}.$$

Clearly, this is restrictive only if the claim has finite value or, according to the terminology of Alvarez and Jermann (2000), only if the hypothesis of high implied interest rates is satisfied. It is interesting to notice that, when this limit is violated (that is, $v_t + g_t < 0$ in some period t), the country is necessarily running a Ponzi game. Indeed, the present value of the future income stream evolves according to

$$p_{t+1} g_{t+1} + p_t e_t = p_t g_t.$$

Thus, consolidating with the budget constraint,

$$p_{t+1} (v_{t+1} + g_{t+1}) \leq p_t (v_t + g_t).$$

This reveals that

$$\liminf_t p_t v_t = \lim_t p_t (v_t + g_t) = \inf_t p_t (v_t + g_t),$$

where we use the fact that $\lim_t p_t g_t = 0$ due to the hypothesis of high implied interest rates. Hence,

$$\liminf_t p_t v_t \geq 0 \text{ if and only if } \inf_t (v_t + g_t) \geq 0.$$

Bulow and Rogoff (1989) argue that, when the plan involves liabilities over time (that is, $v_t < 0$ in some period t), the country will have an incentive to default and to revert to *cash-in-advance* contracts. These are budget-balanced plans involving no debt over time, that is, in this simple deterministic economy, budget-balanced plans fulfilling an additional no borrowing constraint. In this sense, reputational debt is unsustainable.

In general, a default incentive for the sovereign balances a benefit (no repayments according to the previous debt obligations) with a cost (reduced trade opportunities, due to the subsequent no borrowing constraint). The intuition of Bulow and Rogoff (1989)'s paradox is instead grounded on a pure arbitrage principle: the sovereign is able to replicate the previous consumption plan without borrowing and, hence, incurs no substantial cost upon default. The crucial step is the construction of this replication policy.

Define $b = (\ldots, b_t, \ldots)$ as the sequence satisfying, at every t,

$$b_t = \sup_{r \geq 0} \frac{1}{p_t} p_{t+r} (-v_{t+r}).$$

In economic terms, b_t is the minimum amount of resources that enables the sovereign to pay back the debt in any arbitrary date beginning from t (investing this amount in period t delivers $(p_t/p_{t+r})\, b_t \geq -v_{t+r}$ after r periods). This value is finite since debt is bounded by the present value of the country's income stream. In particular, observing that $-v_{t+r} \leq g_{t+r}$ at any future period,

$$-v_t \leq b_t \leq \sup_{r \geq 0} \frac{1}{p_t} p_{t+r} g_{t+r} \leq g_t.$$

Furthermore, as debt in present value might be lower at future dates,

$$\text{(B)} \qquad p_t b_t = \max \{-p_t v_t, p_{t+1} b_{t+1}\}.$$

Finally, notice that $b_0 > 0$, as we assume that the country holds a debt in some period t. The construction of this sort of envelope is illustrated by

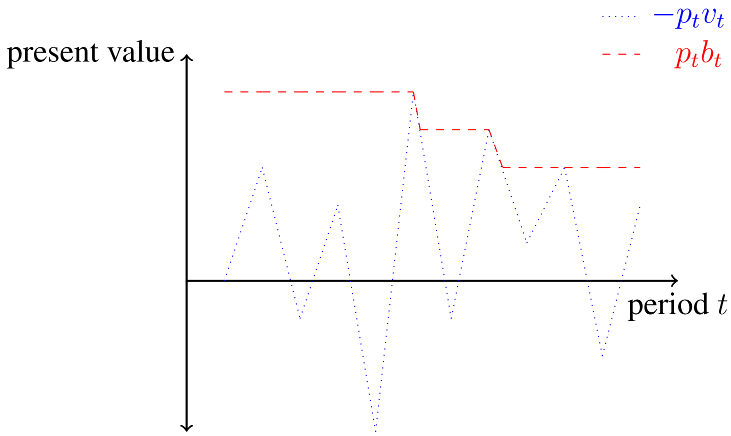

FIGURE 1. Envelope

Figure 1: the drops in the dashed line ($p_t b_t$) reflect contractions in the future peaks of the dotted line ($-p_t v_t$)

Condition (B) reveals that replication is feasible. Indeed, the alternative financial plan $w = v + b \geq 0$ satisfies the budget constraint with no liabilities, that is,

$$p_{t+1}(v_{t+1} + b_{t+1}) + p_t(c_t - e_t) \leq p_t(v_t + b_t).$$

In other terms, at every point in time, b_t is the minimum amount of resources that would allow the sovereign to dispense with liabilities at no cost in terms of future trade opportunities. Furthermore, by condition (B), if the inequality is slack, *i.e.*,

$$p_{t+1}b_{t+1} < p_t b_t,$$

then $w_t = v_t + b_t = 0$. This uncovers a strict benefit from defaulting, and restarting with $w_t = 0$, because current consumption can be increased by the amount $p_t b_t - p_{t+1}b_{t+1} > 0$. In the spirit of Bulow and Rogoff (1989), in this situation, sovereign debt has reached its maximum expansion and, thus, the country begins a repayment policy. Defaulting allows the country to save on these repayments and enjoy higher consumption.

For default not to be profitable, the sequence b must be rolling-over exactly, that is,

$$p_{t+1}b_{t+1} = p_t b_t.$$

This implies that sovereign debt is recurrently at a peak over time (or the peak is never reached) and, therefore, a real repayment policy never begins. No default incentive emerges, but this contradicts the assumption that debt is bounded by the present value of the future income stream. More precisely, in this case,

$$0 < p_0 b_0 = \lim_t p_t b_t \leq \lim_t p_t g_t = 0,$$

where the fact that $\lim_t p_t g_t = 0$ is an implication of the hypothesis of high implied interest rates.

We develop this line of reasoning in order to provide a proof under incomplete markets. In particular, we construct an analogous envelope of future liabilities. Under incomplete markets, however, valuation is ambiguous for streams that are not tradable in the market and so is the construction of the envelope. We show that the process corresponding to the most pessimistic (largest) valuation of future liabilities, when finite, is suitable for the replication policy. To ensure that this is the case, we need the largest present value of the country's future income to be finite. This rules out any incentive to default, but the sovereign might still be borrowing recurrently, as in the deterministic economy discussed above. To eliminate this residual case, we need to slightly strengthen the hypothesis on the pricing kernel: the optimistic valuation of future claims should vanish in the remote future.

3. FUNDAMENTALS

3.1. **Uncertainty.** Trading occurs at each date-event in the set $\mathcal{S}$ along an infinite horizon. Time is indexed by t in $\mathbb{T} = \{0, 1, 2, \ldots\}$. We use the common notation s^t to denote one of the date-events in $\mathcal{S}$ that may be reached in period t in $\mathbb{T}$. Date-events in $\mathcal{S}$ are endowed with a partial ordering $\succeq$, that is, whenever date-event s^{t+r} (s^{t-r}) in $\mathcal{S}$ succeeds (precedes) date-event s^t in $\mathcal{S}$, we write $s^{t+r} \succeq s^t$ ($s^t \succeq s^{t-r}$). Thus, $\{s^{t+1} \in \mathcal{S} : s^{t+1} \succ s^t\}$ is the *finite* set of immediate successors of date-event s^t in $\mathcal{S}$, whereas $\{s^t \in \mathcal{S} : s^{t+1} \succ s^t\}$ is the *unique* predecessor of date-event s^{t+1} in $\mathcal{S}$. There is a unique initial date-event s^0 in $\mathcal{S}$. The set $\mathcal{S}$, endowed with the partial order $\succeq$, is the event-tree.

The *continuation tree* at date-event s^t in $\mathcal{S}$ is $\mathcal{S}(s^t) = \{s^{t+r} \in \mathcal{S} : s^{t+r} \succeq s^t\}$. A *finite contingent truncation* $\mathcal{F}(s^t)$ of the continuation tree $\mathcal{S}(s^t)$ is a finite set of $\succeq$-unordered elements in $\mathcal{S}(s^t)$ such that any date-event in $\mathcal{S}(s^t)$ admits either a (weak) successor or a (weak) predecessor in $\mathcal{F}(s^t)$. Figure 2 presents an example of a contingent truncation in a simple binomial tree: the continuation tree is initiated at the date-event corresponding to the first solid circle, while a contingent truncation is identified by the thick circles.

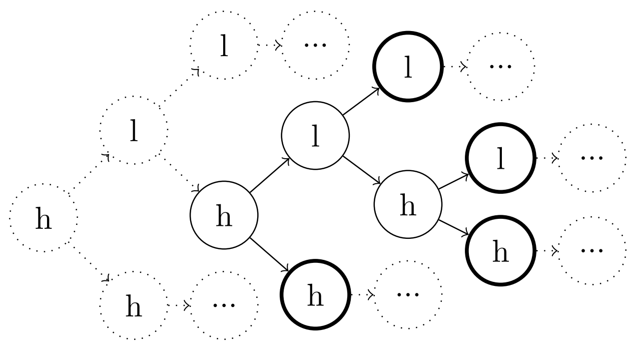

FIGURE 2. Finite contingent truncation

3.2. **Basic notation.** Denote by L the linear space of all maps $x : \mathcal{S} \to \mathbb{R}$. A map x in L is positive, $x \geq 0$, when $x\left(s^t\right) \geq 0$ for every date-event s^t in $\mathcal{S}$; non-trivially positive, $x > 0$, when positive and $x\left(s^t\right) > 0$ for some date-event s^t in $\mathcal{S}$; strictly positive, $x \gg 0$, when $x\left(s^t\right) > 0$ for every date-event s^t in $\mathcal{S}$. The positive cone is $L^+ = \{x \in L : x \geq 0\}$. For an element x on L, $\left(x\left(s^t\right)\right)_{s^t \in \mathcal{D}}$ is also regarded as an element of the linear space L for any subset $\mathcal{D}$ of $\mathcal{S}$, that is, as an element of L such that $x\left(s^t\right) = 0$ at every date-event s^t not in the subset $\mathcal{D}$ of $\mathcal{S}$.

3.3. **Consumption and preferences.** We let e be the element of L^+ representing the sovereign's endowment process of the single commodity, where $e\left(s^t\right)$ in $\mathbb{R}_+$ is the available value at date-event s^t in $\mathcal{S}$. The sovereign's preferences on consumption plans c in L^+ are defined by a *contingent utility function* $U : L^+ \to L$, where $U(c)\left(s^t\right)$ is the utility value beginning from date-event s^t in $\mathcal{S}$. It is assumed that $U(\hat{c})\left(s^t\right) > U(\tilde{c})\left(s^t\right)$ whenever $\left(\hat{c}\left(s^{t+r}\right)\right)_{s^{t+r} \in \mathcal{S}(s^t)} > \left(\tilde{c}\left(s^{t+r}\right)\right)_{s^{t+r} \in \mathcal{S}(s^t)}$. Strict monotonicity is the only restriction on preferences.

3.4. **Markets.** To simplify notation, at no loss of generality, incomplete markets are represented by a linear subspace V of L such that v is in V if and only if $\left(v\left(s^{t+1}\right)\right)_{s^{t+1} \succ s^t}$ is also in V for every s^t in $\mathcal{S}$. In other terms, the space of tradable claims decomposes sequentially in a collection of components for every date-event, each with deliveries only at subsequent date-events. For a tradable claim v in V, we use the canonical decomposition $v = v^+ - v^-$, separating *claims* v^+ in L^+ from *liabilities* v^- in L^+. These are interpreted as *net* positions, since the portfolio composition is not explicit. We maintain the common assumption that some strictly positive element u on L is also in V, that is, available financial instruments allow for a (possibly risky) strictly positive transfer. The presence of a risk-free bond would be sufficient to ensure this, though it is more demanding than necessary.

For a better understanding, as an example, consider the case where a finite set J of securities is traded at every date-event s^t in $\mathcal{S}$.[5] Each security j in J is represented as a payoff process R_j in L, interpreted as promises to deliver at successor nodes. A portfolio z in L^J specifies holdings of available securities at all contingencies, with $z_j\left(s^t\right)$ in $\mathbb{R}$ being the holding of security j in J at date-event s^t in $\mathcal{S}$. In this case, the space of tradable contingent claims V consists of all v in L such that, for some portfolio process z in L^J, at every date-event s^t in $\mathcal{S}$,

$$\left(v\left(s^{t+1}\right)\right)_{s^{t+1}\succ s^t}=\left(\sum_{j\in J}R_j\left(s^{t+1}\right)z_j\left(s^t\right)\right)_{s^{t+1}\succ s^t}.$$

In other terms, each process v in V corresponds to the payoff of some trading plan z in L^J. Notice that, in this notation, an elementary Arrow security at date-event s^t in $\mathcal{S}$ is a security j in J with payoff $R_j\left(\hat{s}^{t+1}\right)=1$ at a specific successor $\hat{s}^{t+1}\succ s^t$ and payoff $R_j\left(s^{t+1}\right)=0$ at any other successor $s^{t+1}\succ s^t$. A risk-free bond, instead, is a security j in J with constant payoff $R_j\left(s^{t+1}\right)=1$ at every successor $s^{t+1}\succ s^t$.

To maintain the analogy with complete markets, the market pricing of securities is represented as an element φ on V.[6] Thus, at every date-event s^t in $\mathcal{S}$, the market value of any portfolio with deliveries $\left(v\left(s^{t+1}\right)\right)_{s^{t+1}\succ s^t}$ in V is given by

$$\sum_{s^{t+1}\succ s^t}\varphi\left(s^{t+1}\right)v\left(s^{t+1}\right).$$

No arbitrage implies that, whenever $\left(v\left(s^{t+1}\right)\right)_{s^{t+1}\succ s^t}$ in V is a (non-trivial) positive claim, then its market value must be strictly positive. In other terms, any (non-trivially) positive claim is costly on the market.

[5] More generally, the set of traded securities might be varying over time, as for instance in Magill and Quinzii (1994), Hernández and Santos (1996), Levine and Zame (1996) and Santos and Woodford (1997). Securities might be of any maturity and in positive (or even negative) net supply. All these features are irrelevant for the budget restrictions, as assets are priced under no arbitrage conditions.

[6] This is at no loss of generality, as we assume that the Law of One Price is satisfied (see Ch. 2 and Ch. 17 in LeRoy and Werner (2001)).

An implicit *price* p in P is a strictly positive element of L satisfying, for every tradable claim v in V, at every date-event s^t in $\mathcal{S}$,

$$\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right) v\left(s^{t+1}\right) = \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) v\left(s^{t+1}\right).$$

By the assumption of no arbitrage, implicit prices exist and form a (non-empty) convex cone P. Observe that only the ratios are relevant for the determination of such implicit prices. This provides an equivalent representation of the asset pricing kernel. Indeed, as prices are invariant on the space of tradable claims V, at every date-event s^t in $\mathcal{S}$, the market value of claims v in V is given by

$$\inf_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) v\left(s^{t+1}\right) = \sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) v\left(s^{t+1}\right).$$

Here we take these basic facts as a primitive framework. They are well-established results in the literature (see, for instance, Magill and Quinzii (1996), Santos and Woodford (1997) and LeRoy and Werner (2001)). To make the paper self-contained, the relevant theorems are collected in Appendix D.

3.5. **Natural debt limit.** To disentangle incentives to default, we impose restrictions on borrowing that rule out Ponzi schemes. In particular, as in Bulow and Rogoff (1989), and in line with a well-established tradition in the literature (see Aguiar and Amador (2014)), we assume that borrowing is bounded by a finite natural debt limit.

The natural debt limit is the maximum amount of debt that can be repaid in (almost) finite time out of the future endowment. As established by Hernández and Santos (1996), Levine and Zame (1996) and Santos and Woodford (1997), when markets are incomplete, this limit is determined as the worst valuation of future endowment. Thus, sovereign debt is restricted, at every date-event s^t in $\mathcal{S}$, by

$$v\left(s^t\right) \geq -\inf_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+r}\in\mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right).$$

This solvency constraint guarantees the existence of a repayment policy in finite time conditional on the set of available securities. Under limited commitment, however, the sovereign might default on debt obligations even when a complete repayment is feasible.

In general, without any further assumptions on prices, the natural debt limit might be infinite, because any debt can be repaid in finite time. In our analysis, we restrict prices so as to rule out this situation, that is, we assume that

$$\text{(E)} \qquad \inf_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}} p\left(s^t\right) e\left(s^t\right) \text{ is finite.}$$

When individuals can commit, debt is only restricted by their repayment capacity and the natural debt limit is necessarily finite, because otherwise unbounded debt would prevent the existence of optimal plans (see (Santos and Woodford 1997, Proposition 2.3)). Under limited commitment, as shown by Hellwig and Lorenzoni (2009), repayment incentives might require low interest rates and, thus, condition (E) might fail at a competitive equilibrium. We shall later on illustrate some primitive assumptions on fundamentals that imply a finite natural debt limit even under limited commitment.

3.6. **Self-enforcing contracts.** A *contract* c in L^+ is sustained by a financial plan v in V if, at every date-event s^t in $\mathcal{S}$,

$$\sum_{s^{t+1} \succ s^t} \varphi\left(s^{t+1}\right) v\left(s^{t+1}\right) + \left(c\left(s^t\right) - e\left(s^t\right)\right) \leq v\left(s^t\right).$$

Obviously, the budget constraint is vacuous when Ponzi games are not ruled out. Thus, we say that a contract c in L^+ is *budget-feasible* if it is sustained by a financial plan v in V subject to the finite natural debt limit.

The sovereign can default at any date-event. Upon default all securities remain available subject to a no liability restriction. Thus, after default, financial plans w are restricted to $V \cap L^+$. A budget-feasible contract c in L^+ is *immune to default* if, at every date-event s^t in $\mathcal{S}$, for every alternative contract $\hat{c}$ in L^+ which is sustained by a restricted financial plan w in $V \cap L^+$ with $w\left(s^t\right) = 0$, $U\left(c\right)\left(s^t\right) \geq U\left(\hat{c}\right)\left(s^t\right)$. In other terms, a contract is immune to default whenever, at every date-event, a country would not benefit from defaulting and trading subject to the no liability restriction thereafter.

For the understanding of default incentives, it is important to identify the role of liabilities in providing insurance opportunities. To this purpose, we say that a contract c in L^+ is *replicable* whenever it is sustained by a financial plan w in $V \cap L^+$. When a contract is not replicable, default involves the implicit cost of restricting insurance opportunities. Notice that default might be profitable even though the contract is not replicable, because the cost of restricted insurance opportunities is overwhelmed by the gain from saved repayments.

4. HIGH INTEREST RATES

4.1. **Conditions on prices.** We here present the condition under which we establish that sovereign debt is unsustainable when markets are incomplete. This condition ensures that the government can replicate the same consumption pattern after default. When it fails, debt might be sustainable as the government might prefer to repay in order to preserve insurance opportunities. We provide examples of this under condition (E), that is, a finite natural debt limit (see section 6).

To extend Bulow and Rogoff (1989), we further restrict prices by assuming that

$$\text{(F)} \qquad \sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}} p\left(s^t\right) e\left(s^t\right) \text{ is finite.}$$

That is, the value of the endowment is (uniformly) finite for all prices consistent with the absence of arbitrage opportunities. In analogy with the terminology used in complete markets (see Alvarez and Jermann (2000)), we refer to this property as *high implied interest rates*.

Under complete markets, high implied interest rates deliver the continuity of the pricing kernel in a topology which is coherent with impatience: the value of residual claims in the remote future vanishes. We need a similar property for Bulow and Rogoff (1989) under incomplete markets, namely,

$$\text{(H)} \qquad \lim_{t \to \infty} \sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) = 0,$$

where $\mathcal{S}^t$ contains all date-events in $\mathcal{S}$ at date t in $\mathbb{T}$. When the pricing kernel satisfies condition (H), we say that it exhibits *uniformly high implied interest rates*.

We clarify the relation occurring among restrictions (E), (F) and (H) in Appendix A. These conditions are all equivalent under complete markets, whereas they are progressively more restrictive when markets are incomplete: condition (H) implies condition (F) which implies condition (E). In a non-growing economy, condition (H) is satisfied, for instance, when the (possibly time-varying) interest rate is uniformly positive. Condition (F) fails when the endowment obeys a random-walk process, with sufficiently large variance, even if the interest rate is positive and, hence, the natural debt limit is finite (that is, condition (E) holds true).

The nature of condition (H), as a reinforcement of condition (F), is rather technical: it rules out prices attaching relatively high value to the remote future (see, again, Appendix A). In general, condition (F) does not imply condition (H) when markets are incomplete. Furthermore, we provide a complete characterisation of this failure of coincidence (Proposition A.2), which only happens in rather singular non-stationary situations. In particular, for a Markov pricing kernel, condition (H) is no more restrictive than condition (F) (see Appendix C).

4.2. **Competitive equilibrium.** The hypothesis of uniformly high implied interest rates restricts prices and, hence, endogenous variables. However, it can be derived from assumptions on fundamentals at a competitive equilibrium under incomplete markets.[7] To this purpose, it is relevant to notice that the analysis of Santos and Woodford (1997) applies independently of the nature of borrowing constraints, provided that these constraints do not induce mandatory savings. Thus, in particular, debt can be limited by endogenous borrowing constraints preventing default, as in Zhang (1997), Alvarez and Jermann (2000) and Hellwig and Lorenzoni (2009). Though we do not pursue this line of research, some details of equilibrium implications are presented in Appendix B.

Condition (F) is enforced when infinite-maturity securities in positive net supply (such as Lucas trees) are traded in the market and their overall dividend is at least a fraction of the aggregate endowment. The price of a security is necessarily finite at a competitive equilibrium and, by no arbitrage, dominates its fundamental value, that is, the present value of dividends for any consistent process of state prices (*i.e.*, any p in P). This imposes a uniform bound on the present value of dividends and, so, as dividends are at least a share of the endowment, on the present value of the endowment. Notice that this implication holds true as far as the asset market is arbitrage-free.[8]

[7] Competitive equilibrium under incomplete markets is studied, among others, by Magill and Quinzii (1994), Hernández and Santos (1996), Levine and Zame (1996) and Santos and Woodford (1997).

[8] It is worth remarking that, in an economy *without* outside assets, condition (F) might fail at a competitive equilibrium even when borrowing is restricted by the natural debt limit and, hence, condition (E) holds true.

To ensure that also condition (H) holds true at a competitive equilibrium requires an additional hypothesis on preferences. The purpose of this restriction is to avoid that the market value of the aggregate portfolio grows unboundedly relative to the endowment. At each contingency, impatience imposes a bound on the market value of securities, because individuals would otherwise profit by dismissing a small fraction of their portfolio, increasing their current consumption by a large amount and balancing their budget with a permanent small contraction of future consumption. When impatience is *uniform* across contingencies, this bound cannot grow over time relative to the aggregate endowment, hence enforcing condition (H) in an economy where aggregate dividends are at least a share of the endowment.

4.3. **Verificability.** Verifying for high implied interest rates, even under complete markets, is not straightforward when prices are not recursive. However, when the pricing is Markov (that is, prices evolve according to a simple Markov process), the determination of high implied interest rates simplifies considerably. In fact, we extend the dominant root, or Perron-Frobenius, approach (*e.g.*, Aiyagari and Peled (1991)) to incomplete markets. This analysis is developed in Appendix C.

We show that condition (F) holds true if and only if the dominant root is less than unity. This root is related to the yield to maturity of a long-term discount bond (a sort of a long-term interest rate) and it does not exceed unity exactly when the long-term yield is strictly positive. Furthermore, under Markov pricing, condition (H) is as restrictive as condition (F) and, so, has not autonomous role.

Most of the literature on sovereign debt focuses on time-invariant interest rate with risk-neutral pricing (see Aguiar and Amador (2014) and Wright (2011)). Some recent quantitative work, however, points out that risk premia might be relevant for understanding sovereign bond prices and that this requires to consider non-risk-neutral pricing kernels (see, for instance, Arellano (2008) and Arellano and Ramanarayanan (2012)). Our analysis provides a simple condition to verify whether interest rates are high in such frameworks.

5. UNSUSTAINABLE DEBT

We here show that, under uniformly high implied interest rates, sovereign debt is unsustainable. This extends Bulow and Rogoff (1989)'s result (see also Martins-da-Rocha and Vailakis (2016)) under more restrictive assumptions than those for complete markets. Importantly, we provide an alternative argument which applies independently of the extension to market incompleteness.

We show that, under condition (F), the contract is replicable without debt (although in general the replication policy can only be identified abstractly by means of a duality argument). This implies that the default option is profitable unless the sovereign debt is not contracting over time in present value. This residual circumstance cannot occur when condition (F) holds uniformly (*i.e.*, under condition (H)), as it would otherwise imply a Ponzi scheme. In complete markets, the uniformity follows by assumption since prices have a sequential representation. In incomplete markets, a non-sequential price, as the limit of admissible sequential prices, cannot be ruled out, unless some uniformity of valuation is imposed on the pricing kernel (see Appendix A for a thorough discussion). Importantly, when the pricing kernel is Markovian, the uniformity always obtains under high implied interest rates (see Appendix C)

Proposition 5.1 (Sovereign debt paradox)**.** *Under uniformly high implied interest rates, a budget-feasible contract c in L^+ is immune to default only if it involves no liabilities. That is, any financial plan v in V sustaining this contract subject to the finite natural debt limit must be positive.*

Proof. Let g in L^+ be given, at every date-event s^t in $\mathcal{S}$, by

$$g\left(s^t\right) = \sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+r}\in\mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right).$$

Consider a budget-feasible contract c in L^+ which is sustained by a financial plan v in V subject to the finite natural debt limit. Given a date-event s^t in $\mathcal{S}$, define

$$b\left(s^t\right) = \sup_{p\in P} \sup_{\mathcal{F}(s^t)\subset\mathcal{S}(s^t)} \frac{1}{p\left(s^t\right)} \sum_{s^{t+r}\in\mathcal{F}(s^t)} p\left(s^{t+r}\right)\left(-v\left(s^{t+r}\right)\right) \geq -v\left(s^t\right),$$

where $\mathcal{F}\left(s^t\right)$ is any finite contingent truncation of $\mathcal{S}\left(s^t\right)$ (for the definition, see section 3.1). Notice that, for fixed price p in P, given any contingent truncation $\mathcal{F}\left(s^t\right)$ of $\mathcal{S}\left(s^t\right)$,

$$\frac{1}{p\left(s^t\right)} \sum_{s^{t+r}\in\mathcal{F}(s^t)} p\left(s^{t+r}\right)\left(-v\left(s^{t+r}\right)\right) \leq \frac{1}{p\left(s^t\right)} \sum_{s^{t+r}\in\mathcal{F}(s^t)} p\left(s^{t+r}\right) g\left(s^{t+r}\right) \leq g\left(s^t\right),$$

where we use the fact that the process g in L^+ dominates the natural debt limit and satisfies the weak roll-over condition

$$\sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) g\left(s^{t+1}\right) + p\left(s^t\right) e\left(s^t\right) \leq p\left(s^t\right) g\left(s^t\right).$$

Therefore, the process b in L obeys $-v \leq b \leq g$. Furthermore, the fact that truncations are contingent implies that b in L fulfils the following property:

$$b(s^t) = \max\left\{-v(s^t), \sup_{p\in P} \frac{1}{p(s^t)} \sum_{s^{t+1}\succ s^t} p(s^{t+1}) b(s^{t+1})\right\}.$$

This happens because $\mathcal{F}\left(s^t\right)$ is a non-trivial finite contingent truncation of $\mathcal{S}\left(s^t\right)$ if and only if

$$\mathcal{F}\left(s^t\right) = \bigcup_{s^{t+1}\succ s^t} \mathcal{F}\left(s^{t+1}\right),$$

where $\mathcal{F}\left(s^{t+1}\right)$ is a finite contingent truncation of $\mathcal{S}\left(s^{t+1}\right)$ for every $s^{t+1} \succ s^t$ in $\mathcal{S}$.

By the Theorem of Duality (see Appendix D), there exists a tradable claim b^* in V such that, at every date-event s^t in $\mathcal{S}$,

$$b\left(s^t\right) \leq b^*\left(s^t\right)$$

and

$$\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right) b^*\left(s^{t+1}\right) = \sup_{p\in P} \frac{1}{p(s^t)} \sum_{s^{t+1}\succ s^t} p(s^{t+1}) b(s^{t+1}).$$

Therefore, at every date-event s^t in $\mathcal{S}$,

$$\begin{aligned}
\sum_{s^{t+1}\succ s^t} \varphi\left(s^{t+1}\right)\left(v\left(s^{t+1}\right) + b^*\left(s^{t+1}\right)\right) + \left(c\left(s^t\right) - e\left(s^t\right)\right) &\leq \\
\sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right)\left(v\left(s^{t+1}\right) + b\left(s^{t+1}\right)\right) + \left(c\left(s^t\right) - e\left(s^t\right)\right) &\leq \left(v\left(s^t\right) + b\left(s^t\right)\right) \\
&\leq \left(v\left(s^t\right) + b^*\left(s^t\right)\right).
\end{aligned}$$

If there is a date-event s^t in $\mathcal{S}$ such that

$$b\left(s^t\right) > \sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right) b\left(s^{t+1}\right),$$

then $v\left(s^t\right) + b\left(s^t\right) = 0$. Define the process w in $V \cap L^+$ as $w\left(s^t\right) = v\left(s^t\right) + b\left(s^t\right) = 0$ and $w\left(s^{t+r}\right) = v\left(s^{t+r}\right) + b^*\left(s^{t+r}\right) \geq 0$ at any strict successor s^{t+r} in $\mathcal{S}\left(s^t\right)$. This sustains the given consumption plan without exhausting the budget at the date event s^t in $\mathcal{S}$. That is, the consumption plan c in L^+ is not immune to default, a contradiction.

It follows that, at every date-event s^t in $\mathcal{S}$, we must have

$$b\left(s^t\right) = \sup_{p \in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right) b\left(s^{t+1}\right).$$

The last condition implies that, for every arbitrary $\eta > 0$, there exists a price process p in P such that, at every date-event s^t in $\mathcal{S}$,

$$b\left(s^t\right) \leq \frac{1}{p\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right) b\left(s^{t+1}\right) + \eta e\left(s^t\right).$$

Hence, recalling that $b \leq g$, we get

$$\begin{aligned} b\left(s^0\right) &\leq \frac{1}{p\left(s^0\right)} \sum_{s^{t+1} \in \mathcal{S}^{t+1}} p\left(s^{t+1}\right) b\left(s^{t+1}\right) + \eta \frac{1}{p\left(s^0\right)} \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right) \\ &\leq \frac{1}{p\left(s^0\right)} \sum_{s^{t+1} \in \mathcal{S}^{t+1}} p\left(s^{t+1}\right) g\left(s^{t+1}\right) + \eta \frac{1}{p\left(s^0\right)} \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right), \end{aligned}$$

where $\mathcal{S}^t$ (respectively, $\mathcal{S}^{0,t}$) contains all date-events in $\mathcal{S}$ at date t in $\mathbb{T}$ (respectively, from the initial date up to date t in $\mathbb{T}$). Taking the limit, by uniformly high implied interest rates (condition (H)), this implies that, for any arbitrary $\eta > 0$,

$$b\left(s^0\right) \leq \eta \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}} p\left(s^t\right) e\left(s^t\right),$$

which shows that $b\left(s^0\right) \leq 0$. Reproducing the argument beginning from any date-event proves that $b \leq 0$ and, as $v + b \geq 0$, $v \geq 0$, thus establishing the claim. □

6. EXAMPLES

We here present some examples of failure of Bulow and Rogoff (1989) under incomplete markets. The cause of this failure is that the incompleteness of markets does not allow for replication when debt is prohibited after default. Differently from Hellwig and Lorenzoni (2009), the valuation of future endowment is finite for some prices, *i.e.*, condition (E) is satisfied. However, condition (F) and, therefore, condition (H) are violated. Notice that the failure of replication itself does not imply that the country cannot benefit from default, because the cost might be compensated by the saving on debt repayments. We prove that incentives to default do indeed disappear in the examples.

Example 6.1. The first example is simple but it delivers the basic intuition underlying the failure of Bulow and Rogoff (1989)'s result. The economy is subject to binomial uncertainty over states $S = \{l, h\}$ occurring with equal probability. The initial state is h in S. To fix ideas, also assume that preferences over consumption are additively separable, *i.e.*,

$$U(c)\left(s^0\right) = \sum_{s^t \in \mathcal{S}} \beta^t \mu_t\left(s^t\right) u\left(c\left(s^t\right)\right),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a per-period utility function, $1 > \beta > 0$ is the discount factor and $\mu_t\left(s^t\right)$ is the unconditional probability of date-event s^t in $\mathcal{S}$. Here, as well as in the other examples, the set of date-events $\mathcal{S}$ consists of all partial histories of Markov states in S having strictly positive probability, given a predefined initial Markov state in S.

Markets are incomplete. Indeed, at every date-event s^t in $\mathcal{S}$, there is a single asset with payoffs $(R_l, R_h) = (1, -1)$ and price $q\left(s^t\right) = 0$. The endowment is $(e_l, e_h) = (0, 2)$. The economy begins at state h in S with an inherited liability $v\left(s^0\right) = -1$. Trivially, holding one unit of the security permits complete insurance at constant consumption $c\left(s^t\right) = 1$ at every date-event s^t in $\mathcal{S}$. However, whenever the economy is in state h in S, the country holds a liability. We verify whether default is profitable in such contingencies.

Upon default, liabilities are not allowed. Hence, in this simple economy, no asset can be traded and autarchy is the only budget-feasible consumption after default. In general, defaulting may produce no benefit, thus violating Bulow and Rogoff (1989)'s result. Indeed, a sufficient condition for this is that the instantaneous utility function satisfies the following inequality

$$(1-\beta)(u(2) - u(1)) < \beta\left(u(1) - \frac{u(2) + u(0)}{2}\right).$$

That is, the sovereign cannot benefit from defaulting if the current gain of not repaying the debt is compensated by the loss of smoothing future consumption.

Remark 6.1. Observe that in this example condition (F) and, hence, condition (H) is violated while condition (E) is satisfied. Indeed, the present value of the endowment is infinite (respectively, finite) for every implicit price process p^* in P such that, at every date-event s^t in $\mathcal{S}$, $p^*(s^t) = (1/2)^t \delta^t$ with $\delta > 1$ (respectively, $0 < \delta < 1$).

Example 6.2. The previous example does not satisfy our assumptions because markets do not permit a strictly positive transfer. We here develop a more complicated example under this additional restriction. Uncertainty is given by Markov states $S = \{l, m, h\}$, all occurring with the same probability. The economy begins in state h in S.

As in the previous example, preferences are assumed to be additively separable with a per-period utility function of the form

$$u(c) = \frac{c^{1-(1/\gamma)} - 1}{1 - (1/\gamma)},$$

where $\gamma > 0$ is the elasticity of intertemporal substitution. Notice that, for any γ in the interval $(0, 1/2)$, this utility function is uniformly bounded from above, as

$$\text{(*)} \qquad u(c) \leq \frac{-1}{1 - (1/\gamma)} \leq 1.$$

Furthermore, as it can be verified by direct computation, for every $1 > \eta > 0$,

$$\text{(**)} \qquad \lim_{\gamma \to 0} u(1 - \eta) = \lim_{\gamma \to 0} \frac{(1 - \eta)^{1-(1/\gamma)} - 1}{1 - (1/\gamma)} = -\infty.$$

The important implication is that any small drop in consumption induces an arbitrarily large loss in utility when $\gamma > 0$ is sufficiently small.

At every date-event s^t in $\mathcal{S}$, there are only two securities paying off, for some sufficiently small $1 > \epsilon > 0$, $(R_l, R_m, R_h) = (1, \epsilon, 0)$ and $(R_l, R_m, R_h) = (0, \epsilon, 1)$. The price of each security is $q = (1/3)\,\beta(1 + \epsilon)$, where $1 > \beta > 0$ is the discount factor. The endowment is $(e_l, e_m, e_h) = (0, 1, 2)$. Notice that strictly positive holdings of both securities permit a strictly positive transfer, as long as $\epsilon > 0$, thus satisfying our general assumptions.

Notice that a balanced portfolio delivers $(v_l, v_m, v_h) = (1, 0, -1)$. This sustains full-insurance with constant consumption $c(s^t) = 1$ at every date-event s^t in $\mathcal{S}$. Furthermore, it is optimal for all sufficiently large bounded debt limits. Does the country benefit from defaulting when holding a liability (and, hence, at the initial date-event s^0 in $\mathcal{S}$ when the economy is in state h in S)?

Preliminarily notice that, at every date-event s^t in $\mathcal{S}$, by budget feasibility without liabilities, consumption is bounded by a process ξ in L^+ such that, at every date-event s^t in $\mathcal{S}$,

$$\xi\left(s^{t+1}\right) \geq \frac{1}{q}\xi\left(s^t\right) + e\left(s^{t+1}\right),$$

for some sufficiently large initial value $\xi\left(s^0\right)$ in $\mathbb{R}_+$. Such bounds overestimate the payoffs of available assets (because $1 > \epsilon > 0$). Moreover, they hold true independently of any sufficiently small $\epsilon > 0$. We assume that the following inequality is satisfied:

$$(\dagger) \qquad \frac{q}{\epsilon} > \frac{1}{3}\frac{\beta}{\epsilon} > 1.$$

Under this condition, we evaluate default incentives along a sequence of monotonically vanishing $\gamma > 0$.

Suppose that there exists a sequence of consumption plans $\left(c^\gamma\right)_{\gamma>0}$ such that each plan is supported by a trading strategy involving no liabilities and guarantees an overall utility after defaulting at least equal to the overall utility from full insurance. That is, given any $\gamma > 0$, assume that

$$U\left(c^\gamma\right)\left(s^0\right) \geq U\left(c\right)\left(s^0\right) = 0.$$

At no loss of generality, it can be assumed that the sequence of consumption plans converges, that is, at every date-event s^t in $\mathcal{S}$, $c^0\left(s^t\right) = \lim_{\gamma\to 0} c^\gamma\left(s^t\right) \leq \xi\left(s^t\right)$. We first argue that this limit guarantees at least the full-insurance consumption.

To verify this, assume that, at some date-event s^t in $\mathcal{S}$, $c^0\left(s^t\right) < 1 - \eta$ for some $1 > \eta > 0$. By condition (**), this implies an infinite loss, which cannot be compensated by bounded gains in other periods, because of (*). Hence, at every date-event s^t in $\mathcal{S}$, the consumption in the limit exceeds the full-insurance consumption, that is, $c^0\left(s^t\right) \geq 1$. We shall now argue by contradiction.

The example is constructed in such a way that, as long as the economy remains in state m in S, the net return on securities is negative (by condition (†), because the price of the security is q, while its payoff is ϵ). Hence, it is costly to roll over resources in order to provide insurance when the averse state l in S occurs again. Indeed, consider a date-event s^t in $\mathcal{S}$ in which the economy is in state m in S. Suppose that the economy remains in state m in S for r in $\mathbb{N}$ consecutive dates and, after this phase, enters the averse state l in S. This happens at the date-event s^{t+r} in $\mathcal{S}$. A direct computation yields

$$c^0\left(s^{t+r}\right) \leq \left(\frac{1}{q}\right)\left(\frac{\epsilon}{q}\right)^{r-1}\xi\left(s^t\right).$$

For the computation of this bound, we use the fact, in the limit as $\gamma > 0$ vanishes, the endowment is completely consumed in state m in S. Initial resources are rolled over so as to overestimate their contribution to consumption when the economy enters in state l in S. For a sufficiently large r in $\mathbb{N}$, by condition (†), full-insurance consumption cannot be guaranteed.

Remark 6.2. We can verify that, for Example 6.2, condition (F) and, hence, condition (H) is violated, while condition (E) is satisfied. Indeed, as the economy is stationary, state prices (π_l, π_m, π_h) in $\mathbb{R}^S_+$ are only restricted by the two pricing equations

$$\pi_h + \epsilon\pi_m = q = \frac{1}{3}\beta\left(1+\epsilon\right) \text{ and } \pi_l + \epsilon\pi_m = q = \frac{1}{3}\beta\left(1+\epsilon\right).$$

State prices correspond to the ratios of implicit present value prices in our general analysis. For any choice of (stationary) state prices π in $\mathbb{R}^S_+$, the present value of the endowment, if finite, is determined by the system of equations

$$\begin{aligned} g_l\left(\pi\right) &= e_l + \pi_l g_l\left(\pi\right) + \pi_m g_m\left(\pi\right) + \pi_h g_h(\pi), \\ g_m\left(\pi\right) &= e_m + \pi_l g_l\left(\pi\right) + \pi_m g_m\left(\pi\right) + \pi_h g_h\left(\pi\right), \\ g_h\left(\pi\right) &= e_h + \pi_l g_l\left(\pi\right) + \pi_m g_m\left(\pi\right) + \pi_h g_h\left(\pi\right). \end{aligned}$$

We shall now argue that the value is finite for some state prices and infinite for other state prices.

Setting $\pi_h = \pi_l = 0$, we obtain $\pi_m = (q/\epsilon)$. For these state prices, the present value of the endowment satisfies

$$g_m\left(\pi\right) = e_m + \pi_l g_l\left(\pi\right) + \pi_m g_m\left(\pi\right) + \pi_h g_h\left(\pi\right) \geq e_m + \frac{q}{\epsilon} g_m\left(\pi\right).$$

No positive solution exists whenever $\epsilon > 0$ fulfils condition (†) in Example 6.2, thus showing that condition (F) and, hence, condition (H) is violated.

To show that condition (E) holds true, it suffices to exhibit alternative state prices π in $\mathbb{R}^S_+$ for which the value of the endowment is finite. For instance, setting $\pi_h = \pi_m = \pi_l = (1/3)\,\beta$, a positive solution exists and is bounded by the value of receiving surely the largest endowment forever, $e_h/\left(1-\beta\right)$.

7. Conclusion

We have shown, by means of examples, that market incompleteness may induce incentives for repayment when liabilities are prohibited after default.

A sovereign may not benefit from defaulting on its debt and positive borrowing can be sustainable by reputation. But the Bulow and Rogoff (1989)'s result does extend to economies with uninsurable risks if the pricing functional satisfies stronger restrictions. In particular, repayment incentives disappear when the value of the most optimistic valuation of future endowment eventually vanishes in the long-run.

APPENDIX A. CONDITIONS ON PRICES

This is a rather technical appendix that clarifies the nature of the different hypotheses on prices. We begin with showing that, as the terminology suggests, uniformly high implied interest rates (condition (H)) are more demanding than simply high implied interest rates (condition (F)).

Proposition A.1 (High interest rates)**.** *Condition (H) implies condition (F).*

Proof. Indeed, assuming that (F) is violated, we can show that, for any $\epsilon > 0$, at every date t in $\mathbb{T}$,

$$\sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) \geq \epsilon,$$

thus violating restriction (H). To this purpose, it suffices to argue that, for every t in $\mathbb{T}$,

$$\sup_{p \in P} \frac{1}{p(s^0)} \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right) \text{ is finite,}$$

where $\mathcal{S}^{0,t}$ contains all contingencies in $\mathcal{S}$ from the initial date up to date t in $\mathbb{T}$. This is what we accomplish in the following, by exploiting that some strictly positive claim u is in the tradable space V.

Fixing any t in $\mathbb{T}$, suppose that $\left(w\left(s^{t+1}\right)\right)_{s^{t+1} \in \mathcal{S}^{t+1}}$ is a tradable claim in $V \cap L^+$. It is immediate to verify that there exists a tradable claim $\left(w\left(s^t\right)\right)_{s^t \in \mathcal{S}^t}$ in $V \cap L^+$ such that, for every price p in P, at every date-event s^t in $\mathcal{S}^t$

$$\sum_{s^{t+1} \succ s^t} p\left(s^{t+1}\right) w\left(s^{t+1}\right) + p\left(s^t\right) e\left(s^t\right) \leq p\left(s^t\right) w\left(s^t\right).$$

This is true because, for some sufficiently large $\lambda > 0$, the expansion $\lambda \left(u\left(s^t\right)\right)_{s^t \in \mathcal{S}^t}$ is an arbitrarily large strictly positive tradable claim in $V \cap L^+$, where u is the strictly positive claim in V. Therefore, by backward induction, beginning with $\left(w\left(s^{t+1}\right)\right)_{s^{t+1} \in \mathcal{S}^{t+1}} = 0$, there exists a sufficiently large $w(s^0)$ in $\mathbb{R}_+$ such that, for every p in P,

$$\sum_{s^{t-r}\in\mathcal{S}^{0,t}} p\left(s^{t-r}\right) e\left(s^{t-r}\right) \leq p\left(s^0\right) w\left(s^0\right),$$

thus proving the claim. □

We now present an example in which (H) is certainly satisfied: when interest rate is uniformly positive in an economy with bounded endowment. We also provide an example in which condition (H) fails when the natural debt limit is finite, that is, condition (E) holds true. This happens when the endowment evolves according to a random-walk, the risk-free bond is the only asset and interest rate is constant, provided that the variance of the endowment is sufficiently large.

Example A.1. Consider an economy with *bounded* endowment and a tradable risk-free bond yielding unitary payoff. The pricing kernel is assumed to satisfy, for some sufficiently large $1 > \beta > 0$, at every date-event s^t in $\mathcal{S}$,

$$\beta \geq \sup_{p\in P} \frac{1}{p\left(s^t\right)} \sum_{s^{t+1}\succ s^t} p\left(s^{t+1}\right).$$

As the right-hand side is the price of the bond, this restriction imposes a sort of lower bound on the interest rates uniformly across all contingencies. When this uniform lower bound exists, the hypothesis of uniformly high implied interest rates is satisfied. Indeed, given any price p in P, at every t in $\mathbb{T}$, it follows that

$$\beta \sum_{s^t\in\mathcal{S}^t} p\left(s^t\right) \geq \sum_{s^{t+1}\in\mathcal{S}^{t+1}} p\left(s^{t+1}\right),$$

where $\mathcal{S}^t$ contains all date-events in $\mathcal{S}$ at date t in $\mathbb{T}$. Therefore, at every t in $\mathbb{T}$,

$$\frac{1}{p\left(s^0\right)} \sum_{s^t\in\mathcal{S}^t} \sum_{s^{t+r}\in\mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) \leq \eta \frac{\beta^t}{1-\beta}$$

where $\eta > 0$ is such that $e\left(s^t\right) \leq \eta$ for every s^t in $\mathcal{S}$.

Example A.2. Consider an economy in which the endowment process evolves as a random walk, that is, at every date-event s^t in $\mathcal{S}$,

$$e\left(s^t\right) = \sum_{s^{t+1}\succ s^t} \mu_{t+1}\left(s^{t+1}|s^t\right) e\left(s^{t+1}\right),$$

where $\mu_{t+1}\left(s^{t+1}|s^t\right)$ is the probability conditional on date-event s^t in $\mathcal{S}$. There is only a risk-free bond (with unitary deliveries) having a price that is constantly equal to $1 > \beta > 0$ at all contingencies. We further assume that the conditional variance of the endowment process satisfies

$$1 \leq \beta \sum_{s^{t+1} \succ s^t} \mu_{t+1} \left(s^{t+1} | s^t\right) \left(\frac{e\left(s^{t+1}\right)}{e\left(s^t\right)}\right)^2 .$$

In such an environment, we verify that condition (E) holds true, while condition (F) fails.

To verify condition (E), notice that a particular price p in P is given by $p\left(s^t\right) = \beta^t \mu_t\left(s^t\right)$ at every date-event s^t in $\mathcal{S}$, where $\mu_t\left(s^t\right)$ is the unconditional probability of date-event s^t in $\mathcal{S}$. By the random walk property of the endowment, its present value is finite at this price, that is,

$$\sum_{s^t \in \mathcal{S}} p\left(s^t\right) e\left(s^t\right) = \sum_{s^t \in \mathcal{S}} \beta^t \mu_t\left(s^t\right) e\left(s^t\right) = \left(\frac{1}{1-\beta}\right) e\left(s^0\right).$$

To uncover the violation of condition (F), notice that another particular price p^* in P is given by $p^*\left(s^t\right) = \beta^t \mu_t\left(s^t\right) e\left(s^t\right)$ at every date-event s^t in $\mathcal{S}$. Indeed, observe that, by the random-walk hypothesis for the endowment,

$$\frac{1}{p^*\left(s^t\right)} \sum_{s^{t+1} \succ s^t} p^*\left(s^{t+1}\right) = \beta.$$

Given any date-event s^t in $\mathcal{S}$, simple computations deliver

$$p^*\left(s^t\right) e\left(s^t\right) \leq \sum_{s^{t+1} \succ s^t} p^*\left(s^{t+1}\right) e\left(s^{t+1}\right),$$

where we have exploited the hypothesis on the conditional variance of the endowment process. Thus, high implied interest rates (condition (F)) are violated, as there is a price p^* in P such that the value of the endowment is inflating over time.

To conclude, we provide a (rather technical) characterisation of uniformly high implied interest rates. The violation of uniformly high implied interest rates occurs if and only if some price in the closure of admissible prices contains a bubble, that is, a non-negligible value at infinity. Under complete markets, such a circumstance is ruled out by assumption, as prices are supposed to have a sequential representation. When markets are incomplete, a non-sequential price, as the limit of admissible prices, cannot be ruled out, unless some uniformity of valuation is imposed on the pricing kernel. As a complement to this characterisation, we provide an example in which condition (F) is satisfied and condition (H) is violated.

Consider the linear space

$$L(e) = \{x \in L : |x| \leq \lambda e \text{ for some } \lambda > 0\},$$

which is a Banach lattice when endowed with the norm $\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda e\}$. As usual, let $L^*(e)$ be its norm dual. Notice that, by Alaoglu's Theorem (see Aliprantis and Border (1999), Theorem 6.21), the closed unit ball in $L^*(e)$ is weak-$*$ compact.

Under high implied interest rates, the space of (normalized) state prices,

$$P_0 = \left\{p \in P : p\left(s^0\right) = 1\right\},$$

can be regarded as a set in the positive cone of $L^*(e)$, where the duality operation is given by

$$p(x) = \sum_{s^t \in \mathcal{S}} p\left(s^t\right) x\left(s^t\right).$$

Notice that, in general, the set P_0 is not weak-$*$ closed in $L^*(e)$, though, by Alaoglu's Theorem, it is contained in a weak-$*$ compact set.

Proposition A.2 (Value at infinity)**.** *Under high implied interest rates, the condition of uniformly high implied interest rates is satisfied if and only if, for every p in the weak-$*$ closure of P_0 in $L^*(e)$,*

$$\lim_{t \in \mathbb{T}} p\left(e^t\right) = p(e),$$

where e^t in L is the truncation of e in L at date t in $\mathbb{T}$.

Proof. For necessity, suppose that the equivalent condition is violated. By positivity, this means that there exists $\epsilon > 0$ such that, for some p^* in the weak-$*$ closure of P_0, at every t in $\mathbb{T}$,

$$p^*\left(e - e^t\right) = p^*(e) - p^*\left(e^t\right) > \epsilon.$$

As p^* lies in the weak-$*$ closure of P_0, for every t in $\mathbb{T}$, it can be approximated by some p in P_0 such that

$$p\left(e - e^t\right) > \epsilon.$$

Thus, for every t in $\mathbb{T}$,

$$\sup_{p \in P_0} p\left(e - e^t\right) > \epsilon,$$

which contradicts the fact that condition (H) is satisfied.

For sufficiency, suppose that the equivalent condition is satisfied and assume a violation of uniformly high implied interest rates. By monotonicity, this implies that there exists $\epsilon > 0$ such that, for every t in $\mathbb{T}$,

$$\sup_{p \in P_0} p\left(e - e^t\right) \geq \epsilon.$$

At every t in $\mathbb{T}$, consider the restricted set

$$\bar{P}_{0,t} = \left(\left\{p \in \bar{P}_0 : p\left(e - e^t\right) \geq \epsilon\right\}\right),$$

where $\bar{P}_0$ is the weak-$*$ closure of P_0 in the positive cone of $L^*(e)$. Observe that $\bar{P}_{0,t}$ is a non-empty closed subset of the compact set $\bar{P}_0$ and that, by monotonicity, $\bar{P}_{0,t+1} \subset \bar{P}_{0,t}$. By the Finite Intersection Property, as $\bar{P}_0$ is compact,

$$p^* \in \bigcap_{t \in \mathbb{T}} \bar{P}_{0,t} \subset \bar{P}_0.$$

For such an element p^* of $\bar{P}_0$, $p^*(e) \geq p^*(e^t) + \epsilon$ for every t in $\mathbb{T}$, thus delivering a contradiction. □

Example A.3. The Markov states are $S = \left\{(l_t)_{t \in \mathbb{T}}, (h_t)_{t \in \mathbb{T}}\right\}$, with l_0 being the initial state and each h_t being an absorbing state. In period t in $\mathbb{T}$, when the economy is in state l_t, with equal probability, it will move to the absorbing state h_{t+1} or to state l_{t+1}. The endowment is $e = (1 - \beta)$ in state l_t and $e = \beta^{-t}(1 - \beta)$ in state h_t. The only asset is an uncontingent bond, delivering a unitary payoff, with constant price $1 > \beta > 0$.

To verify that condition (F) holds true, notice the most optimistic valuation of the endowment satisfies the recursive equation

$$g(l_t) = (1 - \beta) + \sup_{(\pi_l, \pi_h) \in \mathbb{R}_+ \times \mathbb{R}_+} \pi_l g(l_{t+1}) + \pi_h g(h_{t+1})$$

subject to

$$\pi_l + \pi_h = \beta.$$

Furthermore, by direct computation, $g(h_t) = \beta^{-t}$. It also immediate to prove that this recursive equation is solved by

$$g(l_t) = 1 + \beta^{-t}.$$

We now shall argue that condition (H) is instead violated.

For fixed non-initial t in $\mathbb{T}$, consider the following feasible sequence of Markov states in S:

$$(l_0, \ldots, l_{t-1}, h_t, h_t, \ldots).$$

Let $\mathcal{D}$ be the path of date-events in $\mathcal{S}$ corresponding to the selected sequence of Markov states and construct a price p in the closure of P by means of $p\left(s^{\hat{t}}\right) = \beta^{\hat{t}}$, if $s^{\hat{t}}$ lies in $\mathcal{D}$, and $p\left(s^{\hat{t}}\right) = 0$, otherwise. For such a price p

in P, direct computation shows that

$$\frac{1}{p\left(s^0\right)}\sum_{s^t\in\mathcal{S}^t}\sum_{s^{t+r}\in\mathcal{S}(s^t)}p\left(s^{t+r}\right)e\left(s^{t+r}\right)=\sum_{r\in\mathbb{T}}\beta^{t+r}\beta^{-t}\left(1-\beta\right)=1.$$

Thus, the residual does not vanish, which shows that condition (H) cannot be satisfied.

APPENDIX B. COMPETITIVE EQUILIBRIUM

We here show that, at a competitive equilibrium, condition (H) is necessarily enforced under some assumptions on fundamentals. More precisely, we consider an infinite-horizon economy with sequential incomplete markets as in Santos and Woodford (1997). Their framework is particularly suitable for our analysis because they do not introduce any specific hypothesis on the nature of debt limits (apart from ruling out mandatory savings). For the sake of completeness, we illustrate how to adapt their arguments to our purposes.

For each individual i in a finite set I, the budget constraint, at date-event s^t in $\mathcal{S}$, takes the form

$$q\left(s^t\right)\cdot z^i\left(s^t\right)+\left(c^i\left(s^t\right)-e^i\left(s^t\right)\right)\leq v^i\left(s^t\right),$$

where the evolution of wealth is given by

$$v^i\left(s^{t+1}\right)=R\left(s^{t+1}\right)\cdot z^i\left(s^t\right).$$

Here the process z^i in L^J describes the portfolio of securities (in a finite set J), with prices q in L^J, and the process R in L^J represents their (one-period) payoffs. Market clearing for securities requires, at every date-event s^t in $\mathcal{S}$,

$$\sum_{i\in I}z^i\left(s^t\right)=\bar{z}\left(s^t\right)=\bar{z}\left(s^0\right),$$

where the constant process $\bar{z}$ in L^J captures the *net supply* of securities. Market clearing for consumption, at every date-event s^t in $\mathcal{S}$, takes the form

$$\sum_{i\in I}c^i\left(s^t\right)=\sum_{i\in I}e^i\left(s^t\right)+\bar{y}\left(s^t\right)=\bar{e}\left(s^t\right),$$

where $\bar{y}$ in L^+ is the aggregate dividend of securities and $\bar{e}$ in L^+ is the aggregate overall endowment. We assume that the aggregate divided is delivered by some infinite-maturity securities in positive net supply (*i.e.*, such that $\bar{z}_j\left(s^0\right)>0$). In addition, no security is in negative net supply (*i.e.*, $\bar{z}_j\left(s^0\right)\geq 0$ for every j in J).

We now describe the assumptions on fundamentals that are crucial for uniformly high implied interest rates at a competitive equilibrium. These restrictions are introduced and discussed in Santos and Woodford (1997) (as well as in Levine and Zame (1996) and Magill and Quinzii (1994)). For brevity, and to simplify notation, we only present the derived implications that are essential for our condition (H).

First, we assume that the aggregate dividend is at least a fraction of the aggregate endowment, that is, for some sufficiently small $\epsilon > 0$, at every date-event s^t in $\mathcal{S}$,

$$\epsilon \bar{e}\left(s^t\right) \leq \bar{y}\left(s^t\right). \tag{*}$$

In the terminology of Santos and Woodford (1997), this is the hypothesis of the *aggregate endowment is bounded by a trading portfolio* (in positive net supply). This restriction on fundamentals necessarily implies condition (F) at a competitive equilibrium (see Lemma 2.4 in Santos and Woodford (1997)): the market value of securities bounds the present value of dividends (the fundamental value) for all prices consistent with no arbitrage; as the aggregate endowment is at least a fraction of the aggregate dividend, its present value is also finite for all prices.

Second, we assume that preferences satisfy *uniform impatience* as in Santos and Woodford (1997) (see also Levine and Zame (1996) and Magill and Quinzii (1994)). This imposes a uniform bound on the rate of substitution between current and permanent future consumption. The hypothesis is more restrictive than simply impatience, though it is satisfied by additively-separable (bounded) utility functions with constant discounting. We refer to Santos and Woodford (1997) for a precise definition (Assumption (A.3) and the discussion thereafter) and only illustrate its relevant implication at a competitive equilibrium.

Santos and Woodford (1997) (Equation (6.17) in the proof of Lemma 3.8) show that, for every individual i in I, at every date-event s^t in $\mathcal{S}$,

$$(1-\gamma)\, q\left(s^t\right) \cdot z^i\left(s^t\right) \leq \bar{e}\left(s^t\right), \tag{**}$$

where $1 > \gamma > 1$ is given by the hypothesis of uniform impatience. This restriction holds true because otherwise the individual would benefit from an expansion of current consumption balanced by a permanent γ-contraction of future consumption, thus contradicting optimality.

We now expand the analysis in Santos and Woodford (1997) in order to derive our condition (H) under the stated primitive assumptions. Adding across individuals, and using market clearing, condition (**) implies, at every date-event s^t in $\mathcal{S}$,

$$(\dagger) \qquad \left(\frac{1-\gamma}{\#I}\right) q\left(s^t\right) \cdot \bar{z}\left(s^0\right) \leq \bar{e}\left(s^t\right),$$

where $\#I$ in $\mathbb{N}$ is the number of individuals. For any price p in P, the fundamental value is bounded by the market value of the security (Proposition 2.1 in Santos and Woodford (1997)). Thus, by (†),

$$\left(\frac{1-\gamma}{\#I}\right) \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) \bar{y}\left(t+r\right) \leq p\left(s^t\right) \bar{e}\left(s^t\right)$$

and, using condition (*),

$$\epsilon \left(\frac{1-\gamma}{\#I}\right) \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) \bar{e}\left(t+r\right) \leq p\left(s^t\right) \bar{e}\left(s^t\right)$$

Therefore, for some sufficiently small $\eta > 0$, for every price p in P, at every date-event s^t in $\mathcal{S}$,

$$(\dagger\dagger) \qquad \eta \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) \bar{e}\left(t+r\right) \leq p\left(s^t\right) \bar{e}\left(s^t\right)$$

Fixing a period t in $\mathbb{T}$, we add up terms in inequality (††) so as to obtain

$$(t+1)\,\eta \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) \bar{e}\left(t+r\right) \leq \sum_{s^{t-r} \in \mathcal{S}^{0,t}} p\left(s^{t-r}\right) \bar{e}\left(s^{t-r}\right),$$

where $\mathcal{S}^t$ (respectively, $\mathcal{S}^{0,t}$) contains all date-events at period t in $\mathbb{T}$ (respectively, from the initial period up to period t in $\mathbb{T}$). In computing the series on the left-had side, we drop all positive terms corresponding to date-events occurring before period t in $\mathbb{T}$. The above implies

$$\sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) \bar{e}\left(t+r\right) \leq \frac{1}{\eta} \left(\frac{1}{t+1}\right) \sup_{p \in P} \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}} p\left(s^t\right) \bar{e}\left(s^t\right).$$

As condition (F) is satisfied, the right-hand side vanishes in the limit, so proving that condition (H) holds true.

Appendix C. Markov pricing

We here consider a Markov economy with finite state space. When the pricing kernel is Markovian, we provide a complete characterization of uniformly high implied interest rates (*i.e.*, condition (H)). We accomplish this by extending the dominant root (Perron-Frobenius) approach to incomplete markets (see Aiyagari and Peled (1991) for a well-established economic application to complete markets). This analysis can be of independent interest, as a way of introducing time-varying interest rate in widely adopted frameworks (*e.g*, Eaton and Gersovitz (1981)).

Fundamentals are governed by a Markov process on the finite state space S, with strictly positive transition probabilities. We assume that the endowment satisfies $e(s) > 0$ for every state s in S. Markov state prices π in $\mathbb{R}^{S\times S}$ are represented as a matrix, with $\pi(s, s') \geq 0$ being the implicit price in state s in S of one unit of consumption to be delivered in state s' in S. As markets are incomplete (and a strictly positive claim is tradable), there is a compact set $\Pi \subset \mathbb{R}^{S\times S}$ of such state prices. Under these premises, condition (F) amounts to verify the existence of positive solution to the system of equations, for every s in S,

$$\text{(M)} \qquad g(s) = e(s) + \Pi(g)(s),$$

where, with some abuse of notation,

$$\Pi(g)(s) = \sup_{\pi\in\Pi} \sum_{s'\in S} \pi(s, s') g(s').$$

Notice that, by no arbitrage (as far as all states can be reached with strictly positive probability), for every g in $\mathbb{R}^S_+$, at every state s in S,

$$\text{(*)} \qquad \Pi(g)(s) = 0 \text{ only if } g = 0.$$

This is a sort of indecomposability property.

The pricing kernel Π is a sublinear operator, that is, it is subadditive, $\Pi(g' + g'') \leq \Pi(g') + \Pi(g'')$, and positive homogeneous, $\Pi(\lambda g) = \lambda\Pi(g)$ for every λ in $\mathbb{R}_+$. This is the relevant implication of market incompleteness. To deal with this case, we provide an extension of Perron-Frobenius theorem to sublinear (rather than linear) operators.

The *dominant root* $\rho(\Pi)$ of the pricing kernel Π is defined as the (strictly positive) eigenvalue such that, for some (non-zero) eigenvector d in $\mathbb{R}^S_+$,

$$\rho(\Pi) d = \Pi(d).$$

Notice that, necessarily, this eigenvector is strictly positive, *i.e.*, $d(s) > 0$ for every s in S, because of property (*). The dominant root exists by an adaptation of Perron-Frobenius Theorem to sublinear operators.

Proposition C.1 (Dominant root)**.** *A unique dominant root $\rho(\Pi)$ of the pricing kernel Π exists.*

Proof. Consider the canonical unitary simplex $\Delta = \left\{d \in \mathbb{R}^S_+ : \sum_{s\in S} d(s) = 1\right\}$ and endow $\mathbb{R}^S$ with the norm $\|d\|_1 = \sum_{s\in S} |d(s)|$. An eigenvector d in Δ is the fixed point of the continuous map

$$d \mapsto \frac{\Pi(d)}{\|\Pi(d)\|_1},$$

with the associated eigenvalue being $\rho(\Pi) = \|\Pi(d)\|_1$. It exists by Brouwer Fixed Point Theorem (see (Aliprantis and Border 1999, Corollary 17.56)). To verify that the dominant root is unique, suppose that $\rho' > 0$ and $\rho'' > 0$ are both dominant roots with $\rho' > \rho''$. As corresponding eigenvectors are strictly positive, consider the largest $\lambda > 0$ such that $\lambda d' \leq d''$ and, at no loss of generality, assume that $\lambda = 1$. Hence, by monotonicity,

$$\rho' d' \leq \Pi(d') \leq \Pi(d'') \leq \rho'' d'',$$

a contradiction because $\rho' > \rho''$. □

Remark C.1. An interesting interpretation of the dominant root is related to the yield to maturity of long-term discount bounds, or the long-term interest rate. Indeed, assume that safe bonds of any maturity are traded in every state s in S. The yield to maturity of a one-period discount bound issued in state s in S is given by

$$r_1(s) = \frac{1}{\Pi(\mathbf{1})(s)} - 1.$$

Analogously, given any n in $\mathbb{N}$, the yield to maturity of an n-period discount bond issued in state s in S can be computed as

$$r_n(s) = \frac{1}{\sqrt[n]{\Pi^n(\mathbf{1})(s)}} - 1.$$

Notice that, by monotonicity, for every state s in S,

$$\lambda_* \rho(\Pi)^n d(s) \leq \Pi^n(\mathbf{1})(s) \leq \lambda^* \rho(\Pi)^n d(s),$$

where d in $\mathbb{R}^S_+$ is an eigenvector associated with the dominant root and $\lambda^* > 0$ and $\lambda^* > 0$ are chosen so as to satisfy $\lambda_* d \leq \mathbf{1} \leq \lambda^* d$. Hence,

$$\rho(\Pi) \sqrt[n]{\lambda_* d(s)} \leq \sqrt[n]{\Pi^n(\mathbf{1})(s)} \leq \rho(\Pi) \sqrt[n]{\lambda^* d(s)}.$$

It follows that the dominant root corresponds to the long-run yield to maturity, or long-run interest rate, that is, for every s in S,

$$\lim_{n\to\infty} r_n(s) = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\Pi^n(\mathbf{1})(s)}} - 1 = \frac{1}{\rho(\Pi)} - 1.$$

Remark C.2. It is worth noticing that the dominant root can be less than unity even when interest rate is negative in some state of nature. To verify this, consider the following example. Markets are complete and, hence, there are unique state prices π in $\mathbb{R}^{S\times S}$. These state prices are given by

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{7}{4} \\ \frac{1}{28} & \frac{7}{28} \end{pmatrix}$$

The price of the safe bond in the first state is $\pi_{11} + \pi_{12} = 2$, thus implying a negative interest rate. However, the dominant root is $\rho(\Pi) = 1/2$, with associated eigenvector $(d_1, d_2) = (1, 1/7)$.

Proposition C.2 (High implied interest rates)**.** *A solution g in $\mathbb{R}^S_+$ to equations (M) exists if and only if $\rho(\Pi) < 1$.*

Proof. Arguing by contradiction, suppose that $\rho(\Pi) \geq 1$ and that, for some g in $\mathbb{R}^S_+$,

$$g = e + \Pi(g).$$

Choose $\lambda \geq 0$ as the *largest* value satisfying $g \geq \lambda d$, where d in $\mathbb{R}^S_+$ is an eigenvector associated with the dominant root. Also, $e \geq \epsilon d$ for some sufficiently small $\epsilon > 0$. By monotonicity,

$$g \geq e + \Pi(\lambda d) \geq \epsilon d + \Pi(\lambda d) \geq (\epsilon + \rho(\Pi)\lambda) d \geq (\epsilon + \lambda) d,$$

a contradiction. To prove the reverse implication, consider the operator $T : \mathbb{R}^S_+ \to \mathbb{R}^S_+$ given by

$$T(g) = e + \Pi(g).$$

Endow the linear space $\mathbb{R}^S$ with the (equivalent) norm

$$\|g' - g''\| = \inf\{\lambda \geq 0 : |g' - g''| \leq \lambda d\}.$$

Monotone sublinearity implies

$$T(g'') \leq T(g' + \|g' - g''\| d) \leq T(g') + \rho(\Pi)\|g' - g''\| d$$

and

$$T(g') \leq T(g'' + \|g' - g''\| d) \leq T(g'') + \rho(\Pi)\|g' - g''\| d.$$

Thus,

$$\|T(g') - T(g'')\| \leq \rho(\Pi)\|g' - g''\|.$$

The Contraction Mapping Theorem (Aliprantis and Border 1999, Theorem 3.51) guarantees existence, and uniqueness, of the solution to equations (M). □

As condition (F) is satisfied if and only if the system of equations (M) admits a positive solution, the above proposition provides a complete characterization in a Markov setting. We now turn to condition (H) and show that, under Markovian pricing, it is not more restrictive than condition (F).

Proposition C.3 (Characterization)**.** *Condition (H) is satisfied if and only if* $\rho(\Pi) < 1$.

Proof. We only have to verify that condition (H) is satisfied when $\rho(\Pi) < 1$. Notice that, for every price p in P, at every date-event s^t in $\mathcal{S}$, there is π in Π such that

$$p\left(s^{t+1}\right) = \pi\left(s_t, s_{t+1}\right) p\left(s^t\right),$$

where we use the fact that a date-event is a sequence of Markov states, that is, $s^t = (s_0, s_1, \ldots, s_t)$ in S^{t+1}. Assuming that $g \leq d$ at no loss of generality, for every t in $\mathbb{T}$, given a price p in P,

$$\begin{aligned}\frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} \sum_{s^{t+r} \in \mathcal{S}(s^t)} p\left(s^{t+r}\right) e\left(s^{t+r}\right) &\leq \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) g\left(s_t\right) \\ &\leq \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) d\left(s_t\right) \\ &\leq \rho\left(\Pi\right)^t d\left(s_0\right).\end{aligned}$$

To verify the upper bound in the extreme right-hand side, just argue by induction. Clearly, $g\left(s^0\right) \leq d\left(s_0\right)$. Furthermore, at every t in $\mathbb{T}$,

$$\begin{aligned}\frac{1}{p\left(s^0\right)} \sum_{s^{t+1} \in \mathcal{S}^{t+1}} p\left(s^{t+1}\right) d\left(s_{t+1}\right) &\leq \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) \left(\sum_{s_{t+1} \in S} \pi\left(s_t, s_{t+1}\right) d\left(s_{t+1}\right)\right) \\ &\leq \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) \Pi\left(d\right)\left(s_t\right) \\ &\leq \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) \rho\left(\Pi\right) d\left(s_t\right) \\ &\leq \rho\left(\Pi\right) \frac{1}{p\left(s^0\right)} \sum_{s^t \in \mathcal{S}^t} p\left(s^t\right) d\left(s_t\right).\end{aligned}$$

This proves the claim as $\rho(\Pi) < 1$. □

Appendix D. Arbitrage-free pricing

We here collect some basic facts about arbitrage-free asset pricing which are used in the body of the text. These are well-known implications of duality. We provide simple proofs for convenience, independently of their applications in this paper.

The space of tradable claims Y is a linear subspace of some (finite-dimensional) linear space X, endowed with its canonical ordering. The pricing of tradable claims is given by a linear map $\varphi : Y \to \mathbb{R}$. This map is arbitrage free, in the sense that, for any claim y in Y, $y > 0$ only if $\varphi(y) > 0$. We assume that there exists a strictly positive tradable claim u in Y with $\varphi(u) = 1$. This needs not be the safe asset, though a safe asset would be sufficient for this property to be satisfied. The internal product on X is denoted by $x \cdot y$.

Let Π be the convex set of positive linear functionals π in X such that, for every y in Y,

$$\varphi(y) = \pi \cdot y.$$

Here is the Fundamental Theorem of Finance.

Fundamental Theorem of Finance. *The set Π is compact and contains a strictly positive linear functional π on X.*

Proof. Notice that the convex set $K = \{x \in X_+ : x \cdot u = 1\}$ does not interest the linear subspace $Z = \{y \in Y : \varphi(y) = 0\}$. By the Strong Separation Theorem (see for instance Aliprantis and Border (1999), Theorem 5.58), there exists a non-null π in X such that, for every k in K and for every z in Z,

$$\pi \cdot k > \pi \cdot z.$$

As Z is a linear space, $\pi \cdot z = 0$ for every z in Z. If $\pi \cdot x \leq 0$ for some non-null x in X_+, then

$$0 \geq \frac{1}{x \cdot u} \pi \cdot x \geq \pi \cdot \left(\frac{1}{x \cdot u} x \right) > 0,$$

a contradiction. Hence, π is a strictly positive positive linear functional on X. We next show that π is in Π.

At no loss of generality, it can be assumed that $\pi \cdot \bar{y} = \varphi(\bar{y}) > 0$ for some $\bar{y}$ in Y. Given any y in Y, suppose that $\varphi(y) > \pi \cdot y$. Hence,

$$\varphi \left(y - \frac{\varphi(y)}{\varphi(\bar{y})} \bar{y} \right) = 0 \text{ and } \pi \cdot \left(y - \frac{\varphi(y)}{\varphi(\bar{y})} \bar{y} \right) < 0,$$

a contradiction. The set Π is compact as it is contained in $\{\pi \in X_+ : \pi \cdot u = \varphi(u)\}$. $\square$

When markets are incomplete, Π contains multiple value kernels. Nevertheless, values are restricted by upper and lower bounds.

Theorem of Duality. *For every x in X,*

$$\max_{\pi\in\Pi}\pi\cdot x=\min_{y\in Y}\{\varphi(y):x\leq y\}$$

and

$$\min_{\pi\in\Pi}\pi\cdot x=\max_{y\in Y}\{\varphi(y):y\leq x\}.$$

Proof. We prove the first statement only, as the argument is specular for the other statement. We first show that there exists $\bar{y}$ in Y such that $x\leq\bar{y}$ and

$$\varphi(\bar{y})=\min_{y\in Y}\{\varphi(y):x\leq y\}.$$

Observe that, for some sufficiently large $\lambda>0$,

$$-\lambda u\leq x\leq\lambda u,$$

where u is the strictly positive claim in Y. Thus, by no arbitrage,

$$-\lambda\varphi(u)\leq\inf_{y\in Y}\{\varphi(y):x\leq y\}\leq\lambda\varphi(u).$$

This shows that the infimum is finite. For every n in $\mathbb{N}$, there exists a claim y^n in $\{y\in Y:x\leq y\}$ such that

$$\varphi(y^n)\leq\inf_{y\in Y}\{\varphi(y):x\leq y\}+\frac{1}{n}.$$

If the sequence $(y^n)_{n\in\mathbb{N}}$ is bounded, then the claim follows. Otherwise, observe that $\hat{y}^n=y^n/\|y^n\|$ is also a tradable claim in Y satisfying

$$\frac{x}{\|y^n\|}\leq\hat{y}^n$$

and

$$\varphi(\hat{y}^n)\leq\frac{\inf_{y\in Y}\{\varphi(y):x\leq y\}}{\|y^n\|}+\frac{1}{n\|y^n\|}.$$

Taking a subsequence of $(\hat{y}^n)_{n\in\mathbb{N}}$ in Y converging to $\hat{y}$ in Y, we obtain that $\hat{y}>0$ and $\varphi(\hat{y})\leq 0$, contradicting no arbitrage.

Clearly, $\pi\cdot(x-\bar{y})\leq 0$ for every π in Π. To prove that the opposite inequality is satisfied by some π in Π, consider the convex set C in $\mathbb{R}\times X$ defined by

$$\{(\varphi(\bar{y}-y),y-x)\in\mathbb{R}\times X:y\in Y\}.$$

This set does not intersect $\mathbb{R}_+\times X_{++}$. Hence, by the Separating Hyperplane Theorem, there exists a non-null (μ,π) in $\mathbb{R}_+\times X_+$ such that, for every y

in Y,

$$\mu\varphi(\bar{y}-y) \leq \pi\cdot(x-y).$$

It can be verified that $\mu > 0$ and, hence, $\mu = 1$ at no loss of generality. Also,

$$0 \leq \varphi(\bar{y}-\bar{y}) \leq \pi\cdot(x-\bar{y}) \leq 0,$$

thus proving that $\pi\cdot(x-\bar{y}) = 0$. Finally, notice that, when y lies in Y, also $(\bar{y}-y)$ is in Y. It follows that

$$\varphi(y) \leq \varphi(\bar{y}-(\bar{y}-y)) \leq \pi\cdot(x-(\bar{y}-y)) \leq \pi\cdot y.$$

As Y is a linear space, it is also true that $\varphi(-y) \leq \pi\cdot(-y)$. We conclude that, for every y in Y,

$$\varphi(y) = \pi\cdot y,$$

which reveals that π is an element of Π. □

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