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RESEARCH ARTICLE

Competitive markets with private information on both sides

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Abstract We consider competitive markets with asymmetric information. We define a notion of equilibrium that allows individuals to act strategically both as buyers and as sellers. In an example, the wage is common to all types of labor, and it does not reveal information concerning the skill levels of workers. However, at the solution we propose, an informed firm can take advantage of its superior information: it can choose the extent to which it concentrates its employment offers to workers of different types. The probabilities that offers to workers of different types produce a hire are treated parametrically by firms who have correct expectations about them, and firms forego the wage when they extend an offer whether the offer is successful or not. In a general framework, we prove that equilibria exist.

Keywords Competitive markets · Asymmetric information · Large games

JEL Classification D50 · D52 · D82

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1 Introduction

Can anonymous competitive markets function under asymmetric information: in particular, do competitive equilibria exist when individuals can act strategically on both sides of the market? This is the question that we address here, in a model that extends recent contributions. We prove that, for an appropriate specification, competitive equilibria indeed exist.

The problem of resource allocation was posed in full abstraction and generality in Hurwicz (1960), while Hurwicz (1972) introduced the notion of incentive compatibility appropriate for economies with asymmetric information. But the work that followed did not pay particular attention to competitive, anonymous markets. It focused either on bilateral interactions, as in agency theory, or on a grand contract between a planner/mechanism designer and a population of agents. This work provided a rich understanding of the role of nonlinear prices and quantity constraints in the provision of incentives. But importantly, in both cases, the mechanism designer maintained control of the trades of the (or every) agent; and the question that naturally arises is whether insights thus obtained are robust to settings in which individuals interact in large markets.

There are two main approaches to this question in the literature. The first approach goes back to Rothschild and Stiglitz (1976), and it emphasizes bilateral, exclusive interactions. Quantity constraints remain central in the characterization of equilibria, but the fact that each bilateral interaction takes place in the context of large markets provides new insights through its effect on reservation values and, more generally, on payoffs after possible deviations from the equilibrium contracts. Important contributions to this line of research are Gale (1992, 1996), Dubey et al. (2005), Dubey and Geanakoplos (2002) and, recently, Guerrieri et al. (2010).[1]

The second approach follows Akerlof (1970). Individuals interact in large anonymous markets, no exclusivity clause can be enforced, and, at equilibrium, the payoffs of the contracts traded reflect the information and strategic behavior of all individuals in the market.

Our work follows this second approach, and it is appropriate that we discuss more fully related contributions. Dubey et al. (2005), Dubey and Geanakoplos (2002), generalized the argument of Akerlof in an extension of the standard model of general equilibrium with incomplete markets (GEI). In the GEI model, an asset is a promise contingent on the realization of a state of the world. The realized state of the world is fully observable, and its realization unaffected by the choices of individuals. For any given asset, the obligations of sellers and the rights of buyers are fully specified and independent of their characteristics and choices: the asset structure is exogenous. Dubey et al. extended the model by allowing individuals to default on their promises. The default rate, and thus the actual payoff of the asset in a given state, depends on

[1] Zame (2007) also introduces a strategic element in a general equilibrium model to capture different types of asymmetric information and un-observability. He does not restrict to bilateral exclusive interactions, but maintains the focus on strategic interactions in small groups. Similarly to the literature following the Rothschild Stiglitz approach, general equilibrium aspects are relevant because they influence the outside options of different individuals. See also related work by Legros and Newman (1996), and Prescott and Townsend (2006).

the endowment and utility of the seller (a problem of adverse selection), as well as on the overall unobservable market position of the seller (a problem of moral hazard). The message of Dubey et al. is that large, anonymous markets remain viable if every buyer receives the *average* rate of return of each asset in each state, which he anticipates correctly. The payoffs to the buyers become endogenous variables that reflect, at equilibrium, the private choices and characteristics of the sellers. This is ingenious and important; but ss in Akerlof's lemons examples, superior information is effective only in one side of the market.

Others expanded on this theme. In particular, Minelli and Polemarchakis (2000) allowed the obligations of sellers to depend on their private characteristics, on their strategic choices and on the overall distribution of these choices in the economy. At an equilibrium, an individual chooses her trades in assets and her strategy as a seller taking as given prices, the strategies of others and (as a buyer) the payoffs of assets. The assumption of uniform payoffs for the buyers is, again, sufficient for the existence of competitive equilibria.[2]

Beyond the existence of competitive equilibria, Bisin et al. (2011) proved that equilibria are generically constrained suboptimal: even if restricted to uniform taxes on the observable trades in assets, a planner can improve on market outcomes. This formalizes the claim in Greenwald and Stiglitz (1986) that constrained suboptimality is a pervasive property of equilibria with asymmetric information.

One limitation is common to Akerlof's original work and to all further developments of his approach: the one-sided treatment of private information and strategic behavior. Each seller may deliver differently on the same promise, but deliveries are pooled and every buyer of a given asset receives exactly the same (average) payoff, conditional on the state of the world. And the implications of this one-sided treatment are twofold: sellers receive the same price for goods they supply even if they supply goods of different qualities, while buyers get the same mix of goods of different qualities even if they can distinguish between them.

In many markets, both buyers and sellers have some private information that they may try to exploit. For example, workers are usually better informed about their skills, but firms may know more about working conditions and may have different abilities to select the best workers. Also, in the prototypical examples of the used cars market, some buyers may be better than others in spotting "lemons," which the formulation of Akerlof does not allow.

Motivated by these examples, we extend the model by allowing both buyers and sellers to have different information and strategic abilities, and we prove the existence of equilibria. Importantly, the notion of equilibrium we propose allows informed buyers to take advantage of their superior information and get a better mix than other, uninformed buyers; the price of commodities, as in Akerlof, does not reflect their quality. In this sense, and not only, this is a first step in an attempt to model competitive markets with two-sided private information.[3]

[2] See also Bisin and Gottardi (1999).

[3] Meier and Minelli (2011) and Correia-da-Silva (2012) address the second limitation of the one-sided model and propose different trading mechanisms allowing sellers of different qualities to obtain different prices for their goods.

The rest of the paper is organized as follows. After illustrating the main idea in a simple job market example in Sect. 2, we consider, in Sect. 3, a class of economies with asymmetric information about commodities that includes as special cases Akerlof's original one-sided model and our two-sided labor market example. We also provide, as a second application, an example of a financial markets equilibrium with an endogenous asset structure; in Sect. 4, we present the abstract model of the economy that encompasses a variety of forms of asymmetric information, and we define our notion of Nash–Walras equilibrium with effective private information on both sides of the market; in Sect. 5, we prove the existence of equilibria.

2 A simple labor market example

A labor market with asymmetric information concerning the productivity of workers illustrates our argument.

Labor is the only input to production, and the technology is linear.

Workers are of two types: "L," with productivity 1, and "H," with productivity 9. The output, y, produced by n_L units of labor of low productivity and n_H units of labor of high productivity is

$$y = n_L + 9n_H.$$

There are N_L individual workers of type L and N_H of type H, each with one unit of working time and no utility for leisure.

In the formulation of Radner (1968), workers receive a real wage equal to their marginal product, $w_L = 1$ and $w_H = 9$, respectively. An informed firm employs a number of workers of either type as it chooses. An uninformed firm employs an equal number of workers of each type at an *average* wage equal to the *average* marginal product of $w = 5$. Importantly, a worker of type L or of type H employed by an uninformed firm receives the wage that corresponds to his productivity, $w_L = 1$ or $w_H = 9$, and not the average wage of $w = 5$. The mechanism by which the average wage paid by the firm is distributed to workers according to their productivity is not specified, which is the flaw of the argument.

In the standard one-side model, as in Akerlof (1970) or Minelli and Polemarchakis (2000), all the firms are uninformed, unable to distinguish the two types of workers. The supply of workers is pooled, and each firm obtains the same mixture of types per unit of labor employed. At equilibrium, the wage is equal to the average productivity, $w^* = (N_L + 9N_H)/(N_L + N_H)$, and all workers are fully employed.

The point here is to capture situations in which firms may differ in their ability to discriminate between types of workers, and this difference in abilities is reflected in the allocation of workers across firms at equilibrium.

There are 2 types of firms in the economy, of which one, type I, is informed and can perfectly discriminate among workers, while the other, type U, is uninformed, unable to tell one type of worker from the other.

The real wage, w, is common to all types of labor; as in Akerlof, this prevents uninformed firms from extracting information concerning the skill level of workers.

However, at the solution we propose, the informed firm can take advantage of its superior information: it ends up hiring only workers of type H, while the uninformed firm hires a mixture of workers of type H and of type L.

We postpone to Sects. 3 and 4 a formal discussion of our notion of equilibrium; here, we provide, in the context of the example, a detailed description of a possible scenario that sustains it.

An uninformed employment agency is operative. Workers may submit their CV's to the agency at no cost. The agency commits to pay a wage w to every worker that will be hired by a firm.

For a fee, firms obtain from the employment agency the right to access the pool of workers' files for a fixed amount of time.

During this time, a firm is allowed to see workers' CV's on its computer screen, each with some observable characteristics listed, and it has to decide for which workers to fill in a hiring form. Filling the form requires a fixed amount of time, so every firm can fill in a fixed number of forms for every access. A firm can buy any number $\theta^i \geq 0$ of accesses it wants, each time paying a fee.

The cost of filling in the hiring form and the probability of actually being able to hire the chosen worker depend on the proportion of workers of type H and L and also on the proportion and chosen strategies of informed firms in the economy. On both of these parameters, the informed firm has to form an expectation, and we assume that these expectations are correct.

At the end of the process, the agency collects all the hiring forms submitted. If more than one firm has submitted a hiring form for the same worker, the working time of the worker is equally divided among these firms. Hiring has no additional cost for the firm, the wage being paid directly by the employment agency to the worker.

It is only a matter of normalization to set the access fee equal to the wage, w, and the number of forms that can be filed at each access at 1.

At equilibrium, the employment agency makes zero profit.

With these rules, uninformed firm cannot do better than choosing randomly the workers for whom to fill in a form. Informed firms, on the other hand, have a choice to make: how to allocate the time allowed by one access between filling demands for one type of worker or another.

The degree to which a form submitted for a given type of worker actually gives rise to a hiring depends on how many other firms are submitting forms for that type of worker, and this in turn depends on the strategies of the other informed firms present in the economy.

To calculate the actual outcome, an informed firm has to form expectations on the number of workers of a given type it may end up hiring if it was to focus all of its form submissions on that type: k_H and k_L, respectively. The share of time that, for any given access, an informed firm decides to allocate to filing demands for workers of type H is $e^I \in [0, 1]$.

At equilibrium, given the fee w and the (correct) expectations k_H and k_L, a firm choosing θ^I and e^I ends up hiring $e^I k_H \theta^I$ workers of type H and $(1 - e^I) k_L \theta^I$ workers of type L. When making its choices, the firm takes k_H and k_L as given, but, at equilibrium, these numbers reflect the distribution of workers' types and the aggregate behavior of firms.

For an uninformed firm, e^U, the relative share of forms directed to workers of type H, is, simply, the relative proportion of workers of type H present in the employment agency list. In the context of this example, with no utility for leisure, all workers are on the market, and $e^U = N_H/(N_H + N_L)$. The final outcome for an uninformed firm is $e^U k_H \theta^U$ workers of type H and $(1 - e^U)k_L\theta^U$ of type L.

At equilibrium,

$$k_H = \frac{N_H}{e^U\theta^U + e^I\theta^I}, \qquad k_L = \frac{N_L}{(1 - e^U)\theta^U + (1 - e^I)\theta^I}.$$

It easy to check that, as soon as the denominators in these expressions are nonzero, these values guarantee that all workers are employed.

To find the equilibrium, we should also determine the values of e^I, θ^I, θ^U and w. Let us fix for the moment $e^I = 1$; we will then have to verify at the end that to focus only on workers of type H is indeed an optimal choice for the informed firms.

Optimality in the choice of $\theta^i > 0$ for the two types of firms requires that

$$9k_H = w, \quad \text{and} \quad (1 - e^U)k_L + 9e^U k_H = w,$$

while the zero profit condition for the employment agency is

$$w(N_H + N_L) = w(\theta^I + \theta^U).$$

Given the technology and the number of workers of each type, these five equations determine the five unknowns:

$$\begin{aligned}
w^* &= \frac{9N_H + N_L}{N_H + N_L},\\
k_L^* &= w^*, \qquad k_H^* = \frac{w^*}{9},\\
\theta^{U*} &= \frac{N_H + N_L}{w^*}, \qquad \theta^{I*} = \frac{8N_H}{w^*}.
\end{aligned}$$

We already know that $e^U = N_H/(N_H + N_L)$. It remains to verify that $e^I = 1$ is indeed optimal for the informed firms. By insisting on picking only highly productive workers, an informed firm hires $k_H^* = w^*/9$ workers of type H each time it pays the fee. This is a rational strategy even if the firm understands that if it were to look for workers of type L, it could hire $k_L^* = w^*$ of them each time it pays the fee; indeed, in terms of expected productivity, the firm is indifferent between these two alternatives: $k_L^* = 9k_H^* = w^*$.

Remark In this linear example, the two types of firms make zero profits at equilibrium, exactly as in the one-sided model. The informed firm exploits its ability to distinguish the workers' types, and this leads to a different allocation with respect to the one-sided model, but with no effect either on profits or on wages at equilibrium.

In a more general framework, final payoffs are affected by the choices of informed buyers. We can illustrate this, even in our linear example, if we introduce a quantity constraint: the total fee to the employment agency must be paid in advance and each firm is endowed with $M = 144$ units of "money." We assume that there is only one firm of each type, and to save on notation, we us also assume that there is an equal number of workers of high and low productivity, $N_L = N_H = 36$.

In the one-sided model, with only uninformed firms, this quantity constraint would lead to an equilibrium with a wage $w^* = 4$ in which each firm employs 18 workers of each type and makes a profit of 36.

At our solution, we have instead: $w^* = 4$, $k_L^* = 2$, $k_H^* = 2/3$, $\theta^{I*} = \theta^{U*} = 36$, and the informed firm picks only type H workers, $e^{I*} = 1$.

For the firm U, the expected return to accessing the database is again equal to the fee, $1k_L^*(1/2)+9k_H^*(1/2) = 4$, while for the firm I, which picks only type H workers, it is higher $9k_H^* = 6 > 4$. Given the quantity constraint, firm I can pay the fee and access the database at most $\theta^{I*} = M/w^* = 36$ times. Firm I thus hires $k_H^*\theta^{I*} = 24$ workers of type H, while firm U, picking at random, ends up hiring all 36 workers of type L and the remaining 12 workers of type H.

The uninformed firm make zero profits, while the informed firm makes positive profit of $(9 - w^*/k_H^*)24 = 72$. As before, all the 72 workers are employed and the agency makes no profit: $w^*(\theta^{I*} + \theta^{U*}) - w^*72 = 0$.

3 Asymmetric information about commodities

In this section, we generalize the example of the previous section and analyze a class of economies in which individuals are perfectly informed about commodities that they own but may differ in their abilities to distinguish commodities that they buy.

Let there be finitely many types of individuals, $\mathcal{I} = \{1, 2, \ldots, I\}$, trading commodities $\mathcal{L} = \{1, 2, \ldots, L\}$. Each individual is characterized by her utility function $u^i : R_+^L \to R$, initial endowment $w^i \in R_+^L$ and a partition $\mathcal{P}^i$ on the set of commodities $\mathcal{L}$. For any commodity l, $P^i(l)$ is the cell of the partition to which l belongs. The interpretation is that while each individual is able to distinguish commodities when selling, the ability of individual i to distinguish commodities on the market is captured by the partition. Two commodities l and l' are in the same cell of $\mathcal{P}^i$ if and only if individual i cannot distinguish between them. The meet of the individual partitions is $\mathcal{Q}$, their finest common coarsening. We index the cells of the meet by $m = 1, 2, \ldots, M$ so that the set of commodities can be partitioned as $\mathcal{L} = \cup_{m=1}^M Q_m$. One could think of commodities l and l' as different "qualities" of the same good m. For each cell Q_m, there is a contract that we also index by m, which is a promise of one unit of good m.[4]

The assumption that only contracts indexed by the cells of $\mathcal{Q}$ are available for trade reflects a restriction on verifiability. One may think of a situation in which there is a time delay between the moment in which contracts are traded and the moment in which deliveries take place. We assume that courts can only enforce contracts contingent on

[4] In this class of examples, each commodity l can be delivered as a payoff of only one contract, which need not be the case in the general model of Sect. 4.

publicly verifiable, common knowledge information. At the moment when deliveries take place, both the buyer and the seller may be able to perfectly distinguish the qualities of the good delivered, but a contract contingent on this information is not enforceable because an eventual breach could not be proved in a court.

The final payoff of a contract in terms of commodities is influenced by the buyer's strategic choices and by aggregate variables. To be more specific, let us first define the strategy set of individual i.

When acting as a seller of contract m, individual i can choose the proportions of the different "qualities" of good m that she brings to the market. Her strategy set as a seller is thus an element of $\times_{m=1}^{M} \Delta^{Q_m}$ where

$$\Delta^{Q_m} = \left\{ s_m : Q_m \to [0,1] \mid \sum_{l \in Q_m} s_m(l) = 1 \right\}$$

If she sells ϕ_m^i units of contract m and chooses a selling strategy s^i, the individual delivers to the market a quantity $s_m^i(l)\phi_m^i$ of good $l \in Q_m$.

When acting as a buyer, she cannot distinguish qualities which are in the same cell of her partition $\mathcal{P}^i$, so she "picks" the different qualities in the relative proportions in which they are brought to the market. Her strategy as a buyer is to choose with which intensity to concentrate picking on the different cells of her partition, i.e., she has to decide, for each contract m, which weights to put on the elements of the set $\mathcal{P}_m^i = \{P^i \in \mathcal{P}^i \mid P^i \subset Q_m\}$. The strategy set of i as a buyer is thus $\times_{m=1}^{M} \Delta^{\mathcal{P}_m^i}$ where:

$$\Delta^{\mathcal{P}_m^i} = \left\{ b_m^i : \mathcal{P}_m^i \to [0,1] \mid \sum_{P^i \subset Q_m} b_m^i(P^i) = 1 \right\}$$

For each unit of contract m bought, individual i choosing strategy b^i ends up "picking" commodity l with intensity:

$$e_{lm}^i((s^j, \phi^j)_{j \in I}, b^i) = \frac{S_{lm}}{\sum_{l' \in P^i(l)} S_{l'm}} b_m^i(P^i(l)), \tag{1}$$

where if m indexes the cell of the meet to which l belongs,

$$S_{lm} = \sum_j s_m^j(l)\phi_m^j,$$

the total amount of commodity l offered on the market.

To determine the final bundle of goods obtained by individual i, we should take into account the fact that many other individuals may be picking different qualities of good m at the same time. This is captured by the introduction of a matrix K which the individuals take as given when choosing their strategies but that, at equilibrium, reflects the strategic choices of buyers and sellers. Individual i, when she buys a quantity θ_m^i of contract m and chooses a buying strategy b^i anticipates that she will obtain a quantity of commodity l equal to $k_{lm}e^i_{lm}\theta^i_m$, where the (l, m)-term of the K matrix is

$$k_{lm} = \frac{\sum_j s^j_m(l)\phi^j_m}{\sum_j e^j_{lm}\theta^j_m}.$$

A possible interpretation of this expression is the following. Consider the market for good m. For each commodity $l \in Q_m$, there is a total quantity $S_{lm} = \sum_j s^j_m(l)\phi^j_m$ offered. All buyers pick simultaneously up to the exhaustion of each commodity. A buyer who holds θ^i_m units of the contract has the right to pick for θ^i_m units of time. If she chooses a buying strategy b^i, she obtains a fraction of the total offer of commodity l equal to

$$\frac{e^i_{lm}\theta^i_m}{\sum_j e^j_{lm}\theta^j_m},$$

where e^i_{lm} is as defined in Eq. (1).

When choosing her trades in contracts and her selling and buying strategies, the individual takes the prices q and the matrix K as given, but at equilibrium, those are such that we obtain market clearing for contracts and commodities. To calculate the effect of her buying strategy on the her final allocation of commodities, she also has to anticipate the distribution of the total offers across the different qualities on which she lacks information. More precisely, given the prices of contracts q, the matrix K and the other individuals' choices, individual i chooses $(s^i, b^i, \phi^i, \theta^i, z^i)$ to maximize u^i over the budget:[5]

$$q(\theta^i - \phi^i) = 0,$$
$$z^i_l = k_{lm}e^i_{lm}((s^j, \phi^j)_{j\in I}, b^i)\theta^i_m - s^i_m(l)\phi^i_m,\ l = 1, \ldots, L,$$

where again, for each l, m is the cell of the meet to which l belongs.

At equilibrium, markets for contracts and for commodities clear:

$$\sum_i \theta^i_m = \sum_i \phi^i_m,\ m = 1, \ldots, M,$$
$$\sum_i z^i_l = 0,\ l = 1, \ldots, L.$$

To prove that an equilibrium exists, we impose an upper bound on the sales of contracts. In the particular setting of this section, a natural upper bound is

$$s^i_m(l)\phi^i_m \leq w^i_l,\ l = 1, \ldots, L,$$

5 From the budget equations, one see that, at equilibrium, q_m/k_{lm} may be interpreted as the "effective price of commodity l" for an informed buyer.

individual i cannot promise to deliver qualities that she does not have in her endowment.

Remark 1 In the special case of one-sided information, no buyer of contract m can distinguish commodities in Q_m and each picks with uniform intensity $b^i_m(Q_m) = 1$, so that for every i $e^i_{lm} = e_{lm} = \frac{S_{lm}}{\sum_{l' \in P^i(l)} S_{l'm}}$. Then,

$$k_{lm} = \frac{\sum_{l' \in Q_m} S_{l'm}}{\sum_j \theta^j_m},$$

and each buyer i receives a fraction $\theta^i_m / \sum_j \theta^j_m$ of the total offer of commodity l:

$$k_{lm} e_{lm} \theta^i_m = \frac{\sum_j s^j_m(l) \phi^j_m}{\sum_j \theta^j_m} \theta^i_m.$$

This is the case originally studied by Akerlof (1970) and, in a more general setting, by Minelli and Polemarchakis (2000).

Remark 2 The choice of the meet of the individual partitions as the representation of the information verifiable in a court is only for convenience. What matters to distinguish our model from the standard general equilibrium Arrow–Debreu model is that, at least for some commodities, issues of verifiability put *some* restrictions on the trades that are contractually feasible. See the next section for a general formulation.

For concreteness, let us now consider a simple numerical specification.

Cherry-picking There are three individuals, $I = \{1, 2, 3\}$, and commodities are $L = \{0, r, g\}$. We can think of $l = 0$ as "money," a good that everybody can distinguish, and of $l = r$ and $l = g$ as two qualities of a commodity, let us say red and green cherries. Money is traded directly. Equivalently, there is an asset, $m = 1$, such that every seller delivers $(1, 0, 0)$ for each unit sold and each buyer receives $(1, 0, 0)$ for each unit bought, independently of anybody's actions. The only other asset, $m = 2$, is the "cherry" asset. Mr. 1 is color-blind, $\mathcal{P}^1 = \{\{0\}, \{r, g\}\}$ while Mr. 2 and Mr. 3 are able to distinguish the two qualities of cherries and can act strategically both as buyers and as sellers of asset $m = 2$, $\mathcal{P}^2 = \mathcal{P}^3 = \{\{0\}, \{r\}, \{g\}\}$.

An action is $a = (s, b)$, where s is the proportion of red cherries that an individual delivers per unit of cherries sold, and b the relative intensity with which an informed buyer picks red cherries.

Utilities are as follows:

$$u^i = \ln x^i_0 + \ln x^i_r + 2 \ln x^i_g,$$

for $i = 1, 3$, and

$$u^2 = \ln x^2_0 + 2 \ln x^2_g,$$

while the initial endowments of commodities are $w^1 = (12, 0, 0)$, $w^2 = (0, 12, 0)$ and $w^3 = (0, 0, 12)$.

The budget constraint is

$$\theta_1^i + q\theta_2^i \leq \phi_1^i + q\phi_2^i.$$

If we write the K matrix as

$$K = \begin{pmatrix} 1 & 0 \\ 0 & \beta \\ 0 & \gamma \end{pmatrix},$$

and we use the notation S_l for the total quantity of good l offered on the market, the consumption of commodities is

$$x_0^1 = 12 + \theta_1^1 - \phi_1^1,$$

$$x_r^1 = \tfrac{S_r}{S_r+S_g}\beta\theta_2^1 - s^1\phi_2^1,$$

$$x_g^1 = \tfrac{S_g}{S_r+S_g}\gamma\theta_2^1 - (1-s^1)\phi_2^1;$$

$$x_0^2 = \theta_1^2 - \phi_1^2,$$

$$x_r^2 = 12 + b^2\beta\theta_2^2 - s^2\phi_2^2,$$

$$x_g^2 = (1-b^2)\gamma\theta_2^2 - (1-s^2)\phi_2^2;$$

$$x_0^3 = \theta_1^3 - \phi_1^3,$$

$$x_r^3 = b^3\beta\theta_2^3 - s^3\phi_2^3,$$

$$x_g^3 = 12 + (1-b^3)\gamma\theta_2^3 - (1-s^3)\phi_2^3.$$

Given the bounds on sales, $s_m^i(l)\phi_m^i \leq w_l^i$, the selling strategies of 2 and 3 are determined: $s^2 = 1$ and $s^3 = 0$. Given that $i = 2$ has no utility for x_r: $b^2 = 0$. We are thus left with the task of finding asset trades $(\theta^i, \phi^i)_{i=1,2,3}$, prices $q = (1, q)$, the buying action b^3 and the coefficients β and γ.

The solution is

$$\theta_1^1 = 0, \quad \phi_1^1 = 9,$$
$$\theta_2^1 = \frac{9}{q}, \quad \phi_2^1 = 0;$$
$$\theta_1^2 = 4q, \quad \phi_1^2 = 0,$$
$$\theta_2^2 = 8, \quad \phi_2^2 = 12;$$

$$\theta_1^3 = 3q, \quad \phi_1^3 = 0,$$
$$\theta_2^3 = 3, \quad \phi_2^3 = 6;$$
$$q = \frac{9}{7}, \quad \beta = \frac{36}{23}, \quad \gamma = \frac{18}{31};$$
$$b^3 = 1$$

At equilibrium, the uninformed individual 1 buys $\theta_2^1 = 7$ units of the cherry-asset and ends up with the following bundle of cherries

$$x_r^1 = \left(\tfrac{2}{3}\beta\right) 7 = 7.3 \quad \text{units of r-cherries},$$

$$x_g^1 = \left(\tfrac{1}{3}\gamma\right) 7 = 1.4 \quad \text{units of g-cherries}.$$

Individuals 2 and 3 sell the cherry-asset and buy it back while doing ‘cherry-picking’. Individual 2 only picks g-cherries, while 3 only picks r-cherries:

$$\theta_2^2 = 8, \quad \phi_2^2 = 12,$$

$$x_r^2 = (0\beta)\, 8 = 0,$$

$$x_g^2 = (1\gamma)\, 8 = 4.6;$$

$$\theta_2^3 = 3, \; \phi_2^3 = 6,$$

$$x_r^3 = (1\beta)\, 3 = 4.7,$$

$$x_g^3 = 12 - \phi_2^3 = 6.$$

The cherry-asset sells on an anonymous market for a price $q = (9/7)$, but, at equilibrium, each buyer is effectively facing a different matrix of asset payoffs. □

4 An abstract economy and equilibrium

We generalize the model to encompass different interpretations of the strategic choices of individuals, which may refer to effort, deliveries, declarations (see the discussion at the end of Minelli and Polemarchakis 2000). For some applications (e.g., production, or insurance), it is important to allow the payoff of contracts to depend not only on the sellers choice but also on the distribution of choices in the economy.

In our model, individuals jointly choose trades in contracts and strategic actions influencing the payoff of contracts. The equilibrium notion we use then has both a competitive and a game-theoretic dimension. As in Minelli and Polemarchakis (2000), we do not describe the equilibrium in terms of functions from individual names to choices, but in terms of restrictions on the joint distribution of individual characteristics and choices. This approach has been pioneered by Hart et al. (1974) for large exchange economies and by Mas-Colell (1984) for large games; it has recently been extended and generalized by Noguchi and Zame (2006), and Balder (2008).

Notice also that we model asymmetric information on commodities and actions but we do not explicitly model asymmetric information on individual characteristics. For recent studies of large games with asymmetric information, see Yannelis (2009) and Noguchi (2010), which extends the distributional approach to a bayesian setting.

Actions are $a \in \mathcal{A}$, where $\mathcal{A}$ is a non-empty, compact, separable, metric space. Distributions of actions are $\nu \in \Delta(\mathcal{A})$.[6]

Commodities are $l = 1, \ldots, L$.

Trades in commodities are $z = (\ldots, z_l, \ldots) \in \mathbb{R}^L$.

Contracts for the delivery of commodities are $m = 1, \ldots, M$. The fact that commodities cannot be traded directly, but only through contracts, is a way to capture restrictions on trades due to issues of asymmetric information and verifiability. Sales of contracts, portfolios of short positions, are $\phi = (\ldots, \phi_m, \ldots) \in \Phi = \{\phi : 0 \leq \phi \leq \overline{\phi}\}$, where $\overline{\phi}$ is a vector of strictly positive real numbers, while purchases of contracts, portfolios of long positions, are $\theta = (\ldots, \theta_m, \ldots) \in \Theta = \{\theta : 0 \leq \theta\}$.

An individual is described by a real-valued utility function, $u : \mathbb{R}^L \times \mathcal{A} \times \Delta(\mathcal{A}) \to \mathbb{R}$, a vector of initial endowments $w \in \mathbb{R}^L_{++}$, and maps D and E, with domain $\mathcal{A} \times \Delta(\mathcal{A})$ and $\mathcal{A} \times \Delta(\mathcal{A} \times \Phi)$, respectively, and range $\mathcal{R}$, where $\mathcal{R}$ is a compact, convex subset of positive[7] matrices of dimension $L \times M$.

The utility of the individual varies with (z, a, ν) : the net trade in commodities, the action of the individual and the distribution of actions.

The matrix of deliveries on contracts sold by the individual is

$$D(a, \nu) = \{d_{l,m}\}_{m=1,\ldots,M}^{l=1,\ldots,L} \in \mathcal{R};$$

it varies with the action of the individual and the distribution of actions, $\nu \in \Delta(\mathcal{A})$.

The schedule of payoffs of contracts purchased by an individual is

$$E(a, \sigma) = \{e_{l,m}\}_{m=1,\ldots,M}^{l=1,\ldots,L} \in \mathcal{R};$$

it accordingly varies with the action of the individual and the joint distribution of actions and sales of contracts, $\sigma \in \Delta(\mathcal{A} \times \Phi)$.[8]

Notice that we always have $\sigma_{\mathcal{A}} = \nu$, that is, that marginal of σ on $\mathcal{A}$ coincides with ν. Sometimes, we prefer to write ν instead of $\sigma_{\mathcal{A}}$ for notational convenience.

The (actual) net trade in commodities of an individual is

$$z = K \boxtimes E\,(a, \sigma)\,\theta - D(a, \sigma_{\mathcal{A}})\phi,$$

[6] For $\mathcal{X}$, a non-empty, compact, separable metric space, $\Delta(\mathcal{X})$ denotes the set of Borel probability distributions on $\mathcal{X}$, which, when endowed with the topology of weak convergence, is itself a non-empty, compact, separable metrizable space. For n, a positive integer, Δ^n denotes the simplex of dimension $(n - 1)$. When not stated otherwise, all the mathematical definitions and results can be found in Hildenbrand (1974).

[7] A positive matrix has all entries non-negative and at least one different from zero.

[8] A particular case occurs when $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, D only depends on the delivery action $a_1 \in \mathcal{A}_1$ (and the marginal of ν on $\mathcal{A}_1$), and E only depends on the purchase action $a_2 \in \mathcal{A}_2$ (and the marginal of σ on $(\mathcal{A}_2 \times \Phi)$).

where K is an additional equilibrating variable, a matrix of dimension $L \times M$ of adjustment factors common to all the individuals, who take it as given, but which will be endogenously determined at equilibrium. The need for adjustment comes from the fact that only assets are priced in the market while, through their heterogeneous strategic choices, individuals express demands and offers for individual commodities.

Individual i's "buying matrix" $R^i\left(a^i, \sigma\right) := K \boxtimes E^i\left(a^i, \sigma\right)$ consists of a common part K and a private part $E^i\left(a^i, \sigma\right)$. The product $\boxtimes$ is defined entry-wise; that is, $r_{l,m} := k_{l,m} \cdot e_{l,m}$, where $e_{l,m}$ is the number of units of commodity l that the individual would obtain by purchasing one unit of contract m if no adjustment was necessary. When choosing a^i, the individual takes $k_{l,m}$ as given and understands that the actual number of units of good l she obtains for each unit of contract m she bought is $k_{l,m} \cdot e_{l,m}(a^i, \sigma)$.

Let $\mathcal{U}$ be the set of bounded, continuous utility functions which are strictly monotone in consumption: $z' > z \Rightarrow u(z', a, \nu) > u(z, a, \nu)$. Let $\mathcal{D}$ be the set of continuous functions of deliveries on contracts, and $\mathcal{E}$, the set of continuous functions of choice options on contracts. We endow these sets with the topology of uniform convergence on compacta (compact-open topology, see Mas Colell 1985, p. 50).[9] The set of characteristics of individuals is thus $\mathcal{U} \times \mathbb{R}^L_{++} \times \mathcal{D} \times \mathcal{E}$, a complete, separable, metric space.

An economy is $\mu \in \Delta(\mathcal{I})$, where $\mathcal{I} \subset \mathcal{U} \times \mathbb{R}^L_{++} \times \mathcal{D} \times \mathcal{E}$.

Our assumptions on the characteristics of individuals are summarized in:

A1. The set $\mathcal{I} \subset \mathcal{U} \times \mathbb{R}^L_{++} \times \mathcal{D} \times \mathcal{E}$ is compact.

Prices of contracts are $q = (\ldots, q_m, \ldots)$, with domain Δ^M.

The budget set of an individual varies with (q, K, σ), the prices of contracts, the adjustment matrix and the joint distribution of actions and sales of contracts (and its marginal distribution on actions); the budget set of an individual with characteristics (u, w, D, E) is

$$\begin{aligned} &\beta(u, w, D, E, q, K, \sigma) \\ &\quad = \{(a, \theta, \phi, z) \mid q(\theta - \phi) \leq 0,\ \ z = K \boxtimes E\,(a, \sigma)\,\theta - D(a, \sigma_A)\phi,\ \ z \geq -w\}. \end{aligned}$$

(Recall that $\nu = \sigma_{\mathcal{A}}$).

An individual chooses (a, θ, ϕ, z), an action, a purchase of contracts, a sale of contracts and a net trade in commodities, so as to

$$\max\ u(z, a, \sigma_A), \quad \text{s.t. } (a, \theta, \phi, z) \in \beta(u, w, D, E, q, K, \sigma).$$

The set of solutions to the optimization problem of an individual is $\psi(u, w, D, E, q, K, \sigma)$.

[9] The choice of the compact-open topology is motivated by the following Lemma, a reformulation of Theorem K.1.2 in Mas Colell (1985):
Lemma: *Let $\mathcal{F}$ be a set of real-valued continuous functions defined on some separable metrizable space X. If a sequence $(f_n : n, \ldots) \subset \mathcal{F}$ converges to $f \in \mathcal{F}$ in the compact-open topology, and $(x_n : n, \ldots) \subset X$ converges to $x \in X$, then $f_n(x_n)$ converges to $f(x)$.*

For individuals in $\mathcal{I}$, initial endowments lie in a compact set bounded from above by $\bar{w}$, and hence, we can restrict trades to the set $\mathcal{Z}$=$\{z : z \geq -\bar{w}\}$. The set of choices of individuals is then $\mathcal{C} = \mathcal{A} \times \Theta \times \Phi \times \mathcal{Z}$.

Note that all dimensions of choice except $\mathcal{A}$ are Euclidean spaces. $\mathcal{A}$ is compact, and so is Φ. Since the range of E and D is $\mathcal{R}$, and $\mathcal{R}$ is compact, hence bounded, then restricting the purchases to some bounded subset of Θ implies also boundedness of net trades. If we restrict choices of purchases to some compact subset $\hat{\Theta}$ of Θ, then, since D and E are continuous, also the set of feasible net trades is inside a compact set. In short, if we restrict choices of θ to be in some compact subset $\hat{\Theta}$, then the resulting set of feasible choices is also inside a compact subset $\mathcal{C}_{\hat{\Theta}}$, because this set is a product of compact spaces.

A joint distribution of characteristics and choices of individuals is $\tau \in \Delta(\mathcal{I} \times \mathcal{C})$.

For (q, K, τ), prices of contracts, adjustment matrix and a joint distribution of characteristics and choices of individuals, the best response set is[10]

$$B(q, K, \tau) = \{(u, w, D, E, a, \theta, \phi, z) \mid (a, \theta, \phi, z) \in \psi(u, w, D, E, q, K, \tau_{\mathcal{A}\times\Phi})\} \subset \mathcal{I} \times \mathcal{C}$$

Given the way in which the matrix K will be determined at equilibrium, we have to make sure that no individual would receive infinite quantities of a commodity l if it were the first (the only one) who demands commodity l as a payoff from contract m. To insure this, we impose the following condition. Define the set of couples (l, m) such that commodity l might be delivered by contract m:

$M^* := \{(l, m) \mid$ there is $S \subseteq \mathcal{I}$, with $\mu(S) > 0$, such that for every $i \in S$ there is an action a and a distribution over actions, ν such that $D^i_{l,m}(a, \nu) > 0\}$.

This set is non-empty because delivery matrices are positive, and we only have finitely many contracts and commodities. We assume:

A2. For all commodities and contracts $(l, m) \in M^*$ there is a small but positive $\eta_{l,m} > 0$ such that, for all actions a and all distributions over actions and sales of contracts σ, $E^i_{l,m}(a, \sigma) \geq \eta_{l,m}$, for almost all $i \in \mathcal{I}$.[11]

One possible interpretation is that a positive amount of every commodity which might be delivered as payoff of contract m is indeed delivered and that, once in a while, even the most skilled buyers mistakenly pick commodity l when they did not intend to do so. This assumption is violated in our examples, but it plays a technical role in the proof because it allows us to put an upper bound on the entries of the adjustment matrix K. Indeed, for all commodities $l \in L$ and contracts $m \in M$, we define:

[10] If τ is a distribution on a product set, $\ldots \times \mathcal{B} \times \ldots$, then $\tau_{\mathcal{B}}$ denotes the marginal distribution on $\mathcal{B}$.

[11] Note however that it is not enough that the condition imposed on the buyers' side holds for some positive mass < 1 of buyers, because those buyers could just chose not to buy contract m. Then, a positive mass of the other buyers could buy a positive amount of contract m without demanding commodity l, and some sellers could chose to deliver commodity l as a payoff for contract m sold. Then again, the first deviator on the buyers' side who would now chose to demand a positive amount of commodity l as a payoff from contract m would obtain an infinite amount of commodity l.

$$\overline{k}_{l,m} := \frac{\max_{i \in I, a \in \mathcal{A}, \nu \in \Delta(\mathcal{A})} D^i_{l,m}(a, \nu)}{\eta_{l,m}},$$

if $(l, m) \in M^*$ and otherwise we set $\overline{k}_{l,m} := 0$.

Let $\mathcal{K} = \times_{l,m}[0, \overline{k}_{l,m}]$ be the space of possible adjustment matrices.

Definition 1 A Nash–Walras equilibrium for an economy $\mu \in \Delta(\mathcal{I})$ is a joint distribution on the set of characteristics and the set of choices of individuals, $\tau^* \in \Delta(\mathcal{I} \times C)$, such that

1. the marginal distribution of characteristics of individuals coincides with the distribution in the economy:
$$\tau^*_{\mathcal{I}} = \mu;$$
2. there exist prices of contracts, q^*, and an adjustment matrix, K^*, such that individuals optimize:
$$\tau^*(B(q^*, K^*, \tau^*)) = 1;$$
3. the markets for contracts clear:
$$\int_{\mathcal{I} \times \mathcal{C}} (\theta - \phi)\, \mathrm{d}\tau^* = 0;$$
4. the adjustment matrix rations the deliveries on contracts:
$$\int_{\mathcal{I} \times \mathcal{C}} e_{l,m}(a, \tau^*_{\mathcal{A} \times \Phi}) \theta_m \mathrm{d}\tau^* > 0 \quad \Rightarrow \quad k^*_{l,m} = \frac{\int_{\mathcal{I} \times \mathcal{C}} d_{l,m}(a, \tau^*_{\mathcal{A}}) \phi_m \mathrm{d}\tau^*}{\int_{\mathcal{I} \times \mathcal{C}} e_{l,m}(a, \tau^*_{\mathcal{A} \times \Phi}) \theta_m \mathrm{d}\tau^*},$$
5. the markets for commodities clear:
$$\int_{\mathcal{I} \times \mathcal{C}} z \mathrm{d}\tau^* = 0.$$

This is an extension of the notions of a competitive equilibrium for an economy and of a Nash equilibrium for a game to a large set of individuals.

Remark In Sect. 3, we allowed for individualized action sets for the buyers. This discrepancy between the two sections is only apparent. Indeed, in Sect. 3, we could have used the set Δ^{Q_m} also for the buying strategies and use individual matrices of scheduled payoffs defined as follows.

For a buying strategy $b^i \in \times_{m=1}^{M} \Delta^{Q_m}$,

$$e^i_{lm}((s^j, \phi^j)_{j \in J}, b^i) = \frac{S_{lm}}{\sum_{l' \in P^i(l)} S_{l'm}} \sum_{k \in P^i(l)} b^i_m(k).$$

5 Existence

Proposition 1 *Under A1 and A2, Nash–Walras equilibria exist.*

Before going into the details of the proof, we first explain its general strategy:

Fix a small $\epsilon > 0$, and force prices for each contract m to be at least ϵ and let Δ_ϵ^M denote the set of possible prices for contracts, given this restriction. Given $\epsilon > 0$, we get a compact set that contains (as subsets) the possible budgets (that is, choice sets) the individuals could have: $\mathcal{C}_\epsilon = \bigcup_{(i,q,K,\sigma)\in\mathcal{I}\times\Delta_\epsilon^M\times\mathcal{K}\times\Delta(\mathcal{A}\times\Phi)} \beta(i,q,K,\sigma)$. So, given $\epsilon > 0$, the relevant set of joint distributions on individuals and choices is the set $\mathcal{T}_\epsilon \subset \Delta(\mathcal{I}\times\mathcal{C}_\epsilon)$, such that $\tau_\mathcal{I} = \mu$, for all $\tau \in \mathcal{T}_\epsilon$.

We will then (after the steps to come) obtain, given these "trimmed" prices, certain fixed points. Then, we will let ϵ go to 0 and obtain a convergent subsequence of fixed points whose limit will be an equilibrium as defined above. In all the following steps except the last two, let $\epsilon > 0$ be a small fixed number.

1. The budget correspondence β is non-empty, compact valued and continuous.
2. The best reply correspondence B is non-empty, upper hemi-continuous and compact valued.
3. The correspondence $\Phi_{1,\epsilon} : \Delta_\epsilon^M \times \mathcal{K} \times \mathcal{T}_\epsilon \to \mathcal{T}_\epsilon$, defined by
$$\Phi_{1,\epsilon}(q,K,\tau) = \{\tau' \in \mathcal{T}_\epsilon : \tau'(B(q,K,\tau)) = 1\},$$
is non-empty, convex compact valued and upper hemi-continuous.
4. The function $\Phi_{2,\epsilon} : \mathcal{T}_\epsilon \to \mathcal{K}$ is defined by
$$\Phi_{2,\epsilon,l,m}(\tau) := \min\left\{\overline{k}_{l,m}, \frac{\epsilon\overline{k}_{l,m} + \int_{\mathcal{I}\times\mathcal{C}_\epsilon} d_{l,m}(a,\tau_\mathcal{A})\phi_m \mathrm{d}\tau}{\epsilon + \int_{\mathcal{I}\times\mathcal{C}_\epsilon} e_{l,m}(a,\tau_{\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau}\right\},$$
where $\overline{K} \in \mathcal{K}$ is the matrix of payoffs of contracts introduced before Definition 1. $\Phi_{2,\epsilon}$ is continuous.
5. The correspondence
$$\Phi_{3,\epsilon} : \mathcal{T}_\epsilon \to \Delta_\epsilon^M$$
defined by $\Phi_{3,\epsilon}(\tau) = \arg\max_{\Delta_\epsilon^M} q \int_{\mathcal{I}\times\mathcal{C}_\epsilon}(\theta-\phi)\mathrm{d}\tau$ is non-empty and convex and compact valued and upper hemi-continuous.
6. The correspondence Φ_ϵ that is defined as the product of $\Phi_{1,\epsilon}$, $\Phi_{2,\epsilon}$ and $\Phi_{3,\epsilon}$, is non-empty, convex and compact valued and upper hemi-continuous; therefore, it has a fixed point, $(q_\epsilon, K_\epsilon, \tau_\epsilon)$.
7. For $\epsilon = 1/n$, let the sequence of fixed points be $((q_n, K_n, \tau_n) : n = M+1, M+2, \ldots)$. In this step, we show that we can extract from the sequence of fixed points $((q_n, K_n, \tau_n) : n = M+1, \ldots)$ a subsequence, converging to a point $(q^*, K^*, \tau^*) \in \Delta^M \times \mathcal{K} \times \Delta(\mathcal{I}\times\mathcal{C})$.
8. (q^*, K^*, τ^*) is a Nash–Walras equilibrium.

Proof For $0 < \epsilon < 1/M$, let $\Delta_\epsilon^M = \{q : \sum_{m=1}^M q_m = 1, q_m \geq \epsilon, m = 1, \ldots M\}$.

Step 1. The correspondence $\beta : \mathcal{I} \times \Delta_\epsilon^M \times \mathcal{K} \times \Delta(\mathcal{A} \times \Phi) \to \mathcal{C}$ is non-empty, compact valued and continuous. Non-emptiness is clear, since $(a, 0, 0, 0)$ is always a feasible choice, for any action $a \in \mathcal{A}$. Let us argue that the image of every compact set is bounded. Indeed, there is a compact set which contains as subsets all the images of β: $\mathcal{A}$ and Φ are compact sets. For every compact set of prices in Δ_ϵ^M, the θ must lie in the same compact set for every individual. The matrices E and D characterizing the individuals lie in a compact set, as do the initial endowments. Thus, there exists a compact set in which all z must lie.

The set $\beta(u, w, D, E, q, K, \sigma)$ is then compact, because it is closed in the compact set mentioned above.

To prove upper hemi-continuity, it is enough to show that the graph of β is closed. The only problem may be to show that if $E_n \to E$ and $(a_n, \sigma_n) \to (a, \sigma)$, then $E_n(a_n, \sigma_n) \to E(a, \sigma)$. To show this, fix $\delta > 0$. By continuity of E, there is n_1 such that for $n \geq n_1$ $\mid E(a_n, \sigma_n) - E(a, \sigma) \mid < \frac{\delta}{2}$; because $E_n \to E$ in the topology of uniform continuity, there is n_2 such that for $n \geq n_2$ $\mid E_n(a_n, \sigma_n) - E(a_n, \sigma_n) \mid < \frac{\delta}{2}$. Then, using the triangle inequality, $\mid E_n(a_n, \sigma_n) - E(a, \sigma) \mid < \delta$ for $n \geq \max[n_1, n_2]$. Analogously for D.

We now prove lower hemi-continuity. Take a sequence $((u_n, w_n, D_n, E_n, q_n, K_n, \sigma_n) : n = 1, \ldots)$ converging to $(u, w, D, E, q, K, \sigma)$, and a point $(a, \theta, \phi, z) \in \beta(u, w, D, E, q, K, \sigma)$, such that $q(\theta - \phi) = 0$, and $z = K \boxtimes E\,(a, \sigma)\,\theta - D(a, \sigma_A)\phi$. There exists $\eta > 0$, such that[12] $a' = a, \theta' = 0, \phi' = \eta \mathbf{1}_M, z' = K \boxtimes E\left(a', \sigma\right)\theta' - D(a', \sigma_A)\phi' = -\eta D(a, \sigma_A)\mathbf{1}_M$ satisfies $q(\theta' - \phi') = -\eta < 0$, and $-w < z'$. A sequence converging to $c = (a, \theta, \phi, z)$ is constructed by taking convex combinations. Since the action, a, remains fixed along the sequence, the possible non-convexity of the budget set with respect to actions does not interfere with the argument. Let

$$c_m = \left(a, \frac{1}{m}\theta' + \left(1 - \frac{1}{m}\right)\theta, \frac{1}{m}\phi' + \left(1 - \frac{1}{m}\right)\phi, \frac{1}{m}z' + \left(1 - \frac{1}{m}\right)z\right).$$

For given m, there exists $n(m)$ such that, for all $n \geq n(m)$, $c_m \in \beta(u_n, w_n, D_n, E_n, q_n, K_n, \sigma_n)$. We construct the converging sequence by taking $c_n = c_m$ for all $n(m) \leq n < n(m+1)$, and c_n any point in $\beta(u_n, w_n, D_n, E_n, q_n, K_n, \sigma_n)$ if $n < n(1)$.

Step 2. The set $\mathcal{C}_\epsilon = \bigcup_{(i,q,K,\sigma) \in \mathcal{I} \times \Delta_\epsilon^M \times \mathcal{K} \times \Delta(\mathcal{A} \times \Phi)} \beta(i, q, K, \sigma)$ is compact. Since the domain of β is compact and β is u.h.c., the graph of β is a compact set. $\mathcal{C}_\epsilon$ is the projection of the graph on $\mathcal{C}$ and hence compact.

The correspondence $B : \Delta_\epsilon^M \times \mathcal{K} \times \Delta(\mathcal{I} \times \mathcal{C}_\epsilon) \to \mathcal{I} \times \mathcal{C}_\epsilon$ is non-empty, upper hemi-continuous and compact valued.

It is non-empty because each individual is maximizing a continuous function on a non-empty compact set.

We now show that B is upper hemi-continuous. Because the range of B is compact, we just need to prove that it has a closed graph. Take a sequence, $((q_n, K_n, \tau_n) :$

[12] "$\mathbf{1}_K$" denotes the column vector of 1's of dimension K.

$n = 1, \ldots$), converging to (q, K, τ), and a sequence $((u_n, w_n, D_n, E_n, a_n, \theta_n, \phi_n, z_n) : n = 1, \ldots)$ converging to $(u, w, D, E, a, \theta, \phi, z)$ with $(a_n, \theta_n, \phi_n, z_n) \in \psi(u_n, w_n, D_n, E_n, q_n, K_n, \tau_{\mathcal{A}\times\Phi,n})$, for $n = 1, \ldots$. Because β is u.h.c., $(a, \theta, \phi, z) \in \beta(u, w, D, E, q, K, \tau_{\mathcal{A}\times\Phi})$. If $(a, \theta, \phi, z) \notin \psi(u, w, D, E, q, K, \tau_{\mathcal{A}\times\Phi})$, by the l.h.c. of β, there exists a sequence $(a'_n, \theta'_n, \phi'_n, z'_n) \in \beta(u_n, w_n, D_n, E_n, q_n, K_n, \tau_{\mathcal{A}\times\Phi,n})$ that converges to a point (a', θ', ϕ', z'), with $u(z', a', \tau_{\mathcal{A}}) > u(z, a, \tau_{\mathcal{A}})$. Since $u_n(z_n, a_n, \tau_{\mathcal{A},n}) \geq u_n(z'_n, a'_n, \tau_{\mathcal{A},n})$, this contradicts the convergence of the sequence $(u_n : 1, \ldots)$ to u in the compact-open topology.

Step 3. The set $\mathcal{T}_\epsilon \subset \Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$, such that, if $\tau \in \mathcal{T}_\epsilon$, then $\tau_{\mathcal{I}} = \mu$, is obviously convex and it is compact because it is a closed subset (Hildenbrand (1974), (27) on p. 48: if a sequence of measures converges weakly, then so do the marginals) of a compact set.

The correspondence $\Phi_{1,\epsilon} : \Delta^M_\epsilon \times \mathcal{K} \times \mathcal{T}_\epsilon \to \mathcal{T}_\epsilon$, defined by

$$\Phi_{1,\epsilon}(q, K, \tau) = \{\tau' \in \mathcal{T}_\epsilon : \tau'(B(q, K, \tau)) = 1\},$$

is non-empty, convex and compact valued and upper hemi-continuous.

Convex valuedness is clear.

Let us prove non-emptiness. For any $(q, K, \tau) \in \Delta^M_\epsilon \times \mathcal{K} \times \mathcal{T}_\epsilon$, $\Phi_{1,\epsilon}(q, K, \tau) \neq \emptyset$. For any (q, K, τ), the correspondence $\psi(\cdot, q, K, \tau_{\mathcal{A}\times\Phi}) : \mathcal{I} \to \mathcal{C}_\epsilon$ is non-empty valued and upper hemi-continuous. Therefore, it admits a measurable selection $s : \mathcal{I} \to \mathcal{C}_\epsilon$ (Aliprantis and Border (1999), Theorem 17.13, p. 567). We construct a measure $\tau_s \in \Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$ such that the marginal of τ_s on $\mathcal{I}$ is μ. To do this, define, for any measurable rectangle $A \times B \subset \mathcal{I} \times \mathcal{C}_\epsilon$, $\tau_s(A \times B) := \mu(A \cap s^{-1}(B))$. This determines uniquely a σ-additive probability measure in $\Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$ which obviously has the property that the marginal on $\mathcal{I}$ is μ.

We now prove that $\Phi_{1,\epsilon}$ is upper hemi-continuous. Because the range of $\Phi_{1,\epsilon}$ is compact, we just need to prove that it has a closed graph. If a sequence, $((q_n, K_n, \tau_n) : n = 1, \ldots)$, converges to (q, K, τ), and a sequence, $(\tau'_n : n = 1, \ldots)$, such that $\tau'_n \in \Phi_{1,\epsilon}(q_n, K_n, \tau_n)$, converges to τ', then $\tau' \in \Phi_{1,\epsilon}(q, K, \tau)$. If not, $\tau'(B(q, K, \tau)) < 1$. $B(q, K, \tau)$ is closed in the metrizable space $\mathcal{I} \times \mathcal{C}_\epsilon$, and therefore, there exist open sets U and V such that $B(q, K, \tau) \subset V \subset \overline{V} \subset U$ and $\tau'(U) < 1$. Since the correspondence B is upper hemi-continuous, there exists $\overline{n}$, such that $B(q_n, K_n, \tau_n) \subset V$,, for $n = \overline{n}, \ldots$. Since $\tau'_n(B(q_n, K_n, \tau_n)) = 1$, $\tau'_n(\overline{V}) = 1$ for $n = \overline{n}, \ldots$. Since the sequence $(\tau'_n : n = 1, \ldots)$ converges weakly to τ, $\limsup_n \tau'_n(\overline{V}) \leq \tau'(\overline{V})$ (Hildenbrand (1974), iii) of (26), p. 48). Then, $\tau'(\overline{V}) = 1$ and $\tau'(U) = 1$, a contradiction.

Step 4. The function $\Phi_{2,\epsilon} : \mathcal{T}_\epsilon \to \mathcal{K}$ is defined by

$$\Phi_{2,\epsilon,l,m}(\tau) := \min\left\{\overline{k}_{l,m}, \frac{\epsilon\overline{k}_{l,m} + \int_{\mathcal{I}\times\mathcal{C}_\epsilon} d_{l,m}(a, \tau_{\mathcal{A}})\phi_m \mathrm{d}\tau}{\epsilon + \int_{\mathcal{I}\times\mathcal{C}_\epsilon} e_{l,m}(a, \tau_{\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau}\right\},$$

where $\overline{K} \in \mathcal{K}$ is the matrix of payoffs of contracts introduced before Definition 1. Given that $\epsilon > 0$, to prove continuity, it is enough to show that the two integrals are continuous functions of the distribution τ. Consider $\int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau$ (the same argument holds for $\int_{\mathcal{I}\times\mathcal{C}} e_{l,m}(a,\tau_{\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau$). Take a sequence $\tau_n (n=1,\ldots) \in \mathcal{T}_\epsilon$ converging to $\tau \in \mathcal{T}_\epsilon$. We want to show that, fixing any $\delta > 0$, there is $\overline{n}$ such that for all $n \geq \overline{n}$

$$\left| \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m \mathrm{d}\tau_n \right| < \delta$$

The previous expression can be written as

$$\left| \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau_n \right.$$
$$\left. + \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau_n - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m \mathrm{d}\tau_n \right| < \delta$$

Because $d_{l,m}(a,\tau_{\mathcal{A}})\phi_m$ is a bounded continuous function of the integrating variables (i,c) on $\mathcal{I}\times\mathcal{C}_\epsilon$ (the restriction to $\mathcal{C}_\epsilon$ is important to bound θ when considering $\int_{\mathcal{I}\times\mathcal{C}} e_{l,m}(a,\tau_{\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau$), and τ_n converges to τ in the weak topology, there exists n_1 such that, for $n \geq n_1$,

$$\left| \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m \mathrm{d}\tau_n \right| < \frac{\delta}{2}.$$

As a function from the compact set $\mathcal{I}\times\mathcal{C}_\epsilon\times\mathcal{T}_\epsilon$ to R_+^L, the expression $d_{l,m}(a,\tau_{\mathcal{A}})\phi_m$ is uniformly continuous. Then, there exists n_2 such that, for $n \geq n_2$,

$$\mid d_{l,m}(a,\tau_{\mathcal{A}})\phi_m - d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m \mid < \frac{\delta}{2}$$

for all (i,c).

Then, for any $n \geq n_2$

$$\int \mid d_{l,m}(a,\tau_{\mathcal{A}})\phi_m - d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m \mid < \frac{\delta}{2} \int \mathrm{d}\tau_n = \frac{\delta}{2}$$

Let then $\overline{n} = \max[n_1, n_2]$.

Step 5. Define the correspondence

$$\Phi_{3,\epsilon} : \mathcal{T}_\epsilon \to \Delta_\epsilon^M$$

by $\Phi_{3,\epsilon}(\tau) = \arg\max_{\Delta_\epsilon^M} q \int_{\mathcal{I}\times\mathcal{C}_\epsilon}(\theta - \phi)\mathrm{d}\tau$. Clearly, it is non-empty and convex valued. To see that it is upper hemi-continuous, first notice that, as a function of τ, $\int_{\mathcal{I}\times\mathcal{C}_\epsilon}(\theta - \phi)\mathrm{d}\tau$ is continuous (weak convergence) and it has a compact range in $\mathbb{R}^M$. Then, by a standard argument, the graph of $\Phi_{3,\epsilon}$ is closed.

Step 6. The correspondence $\Phi_\epsilon = \Phi_{1,\epsilon}\times\Phi_{2,\epsilon}\times\Phi_{3,\epsilon} : \Delta_\epsilon^M\times\mathcal{K}\times\mathcal{T}_\epsilon \to \Delta_\epsilon^M\times\mathcal{K}\times\mathcal{T}_\epsilon$ is non-empty, convex, compact valued and upper hemi-continuous; therefore, it has a fixed point, $(q_\epsilon, K_\epsilon, \tau_\epsilon)$.

Step 7. For $\epsilon = 1/n$, the sequence of fixed points is $((q_n, K_n, \tau_n) : n = M+1, \ldots,)$.

Aggregating the budget constraints and using the definition of $\Phi_{3,\epsilon}$, $q\int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)\mathrm{d}\tau_n \le q_n\int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)\mathrm{d}\tau_n \le 0$, for all $q \in \Delta_{1/n}^M$. Take $q = \mathbf{1}_M(1/M)$, then $\sum_{m=1}^M \int_{\mathcal{I}\times\mathcal{C}} \theta_m\mathrm{d}\tau_n \le \sum_{m=1}^M \int_{\mathcal{I}\times\mathcal{C}} \phi_m\mathrm{d}\tau_n \le \sum_{m=1}^M \overline{\phi}_m = \overline{\theta}$. Let $\mathcal{C}_{\overline{\theta}}$ be the compact subset of $\mathcal{C}$ where we impose the restriction that, for all m, $\theta_m \le \overline{\theta}$ (remember that $z = K \boxtimes E\left(a, \tau_{\mathcal{A}\times\Phi}\right)\theta - D(a, \tau_{\mathcal{A}})\phi$ and that E, D and ϕ are bounded, so this provides also a bound on z).

We want to show that the sequence τ_n for $n = M+1, \ldots$ is a tight family of measures (see Hildenbrand 1974, p. 49). If not, there exists $\delta > 0$ such that, for any compact $F \subset \mathcal{I}\times\mathcal{C}$, there exists n with $\tau_n(F) \le 1-\delta$. Fix $\delta > 0$ and consider $F = \mathcal{I}\times\mathcal{C}_{\hat{\theta}}$ where $\hat{\theta} = \frac{2\overline{\theta}M}{\delta}$. Then $\tau_n(\mathcal{I}\times(\mathcal{C}\setminus\mathcal{C}_{\hat{\theta}})) > \delta$. Given that $\theta \ge 0$, this would imply that, for some m, $\int_{\mathcal{I}\times\mathcal{C}}\theta_m\mathrm{d}\tau_n > \overline{\theta}$, a contradiction.

Using the fact that Δ^M and $\mathcal{K}$ are compact, the property of tight family of measures stated in Hildenbrand (1974) (31) on p. 49, and the fact that $\Delta(\mathcal{I}\times\mathcal{C})$[13] is complete, we can extract from the sequence of fixed points $((q_n, K_n, \tau_n) : n = M+1, \ldots)$ a subsequence, which we denote again $((q_n, K_n, \tau_n) : n = M+1, \ldots)$, converging to a point $(q^*, K^*, \tau^*) \in \Delta^M \times \mathcal{K}\times\Delta(\mathcal{I}\times\mathcal{C})$.

Step 8. We want to show that the limit point (q^*, K^*, τ^*) is an equilibrium.

Let us first argue that markets for contracts clear. Since $0 \notin \mathcal{R}$, and M^* is non-empty, at each fixed point, there exists at least one (l, m) such that $\Phi_{2,\epsilon,l,m}(\tau_\epsilon) > 0$ and such that for almost all $i \in \mathcal{I}$ and every $a \in \mathcal{A}$, $E_{l,m}^i(a, \tau_{\epsilon,\mathcal{A}\times\Phi}) > 0$. E is a positive matrix and the utility function strictly monotone in consumption, so that at each fixed point, almost all individuals are spending all their revenue from the sales of contracts. Taking limits, $q^*\int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)\mathrm{d}\tau^* = 0$, and, using the definition of $\Phi_{3,\epsilon}$, $q\int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)\mathrm{d}\tau^* \le 0$, for all $q \in \cup_\epsilon\Delta_\epsilon^M = \Delta_{++}^M$. This implies that $\int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)\mathrm{d}\tau^* \le 0$. If, for some m, $\int_{\mathcal{I}\times\mathcal{C}}(\theta_m - \phi_m)\mathrm{d}\tau^* < 0$, then $q_m^* = 0$. Consider the following modification $\hat{\tau}^*$ of τ^*. Write $\mathcal{C}$ as $\mathcal{C} = \mathcal{A}\times\Theta^{-m}\times\mathbb{R}_+\times\Phi\times\mathcal{Z}$. Let the per capita excess supply of asset m be $\hat{\theta} = -\int_{\mathcal{I}\times\mathcal{C}}(\theta_m - \phi_m)\mathrm{d}\tau^*$. For any measurable rectangle $A \times B \in \mathcal{I}\times\mathcal{C}$ with $A \subset \mathcal{I}\times\mathcal{A}\times\Theta^{-m}\times\Phi\times\mathcal{Z}$ and $B \subset \mathbb{R}_+$, let $\hat{\tau}^*(A\times B) = \tau(A\times(B-\hat{\theta})\cap\mathbb{R}_+)$. This modification assures market clearing for contracts without changing prices nor the other choices of the individuals.

Let us now show that, at the limit, there is also market clearing for commodities. From the budget constraints of individuals and the definition of $\Phi_{2,\epsilon}$, we have, for any $l \in L$

[13] Note that $\mathcal{C}$ is not compact. Therefore, we need the tightness argument for the family τ_n $(n = M+1, \ldots)$.

$$\int\limits_{\mathcal{I}\times\mathcal{C}} z_l \mathrm{d}\tau_\epsilon = \sum_m [\Phi_{2,\epsilon,l,m}(\tau_\epsilon) \int\limits_{\mathcal{I}\times\mathcal{C}} e_{lm}(a, \tau_{\epsilon,\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau_\epsilon - \int\limits_{\mathcal{I}\times\mathcal{C}} d_{lm}(a, \tau_{\mathcal{A},\epsilon})\phi_m \mathrm{d}\tau_\epsilon]$$

To simplify notation, let $\int_{\mathcal{I}\times\mathcal{C}} e_{lm}(a, \tau_{\epsilon,\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau_\epsilon = D^\epsilon_{lm}$ and $\int_{\mathcal{I}\times\mathcal{C}} d_{lm}(a, \tau_{\mathcal{A},\epsilon}) \phi_m \mathrm{d}\tau_\epsilon = S^\epsilon_{lm}$, both non-negative numbers.

We claim that, for any $\epsilon > 0$, at a fixed point, $\overline{k}_{lm} D^\epsilon_{lm} - S^\epsilon_{lm} \geq 0$. If $S^\epsilon_{lm} = 0$, this is clear. If $S^\epsilon_{lm} > 0$, it must be that $(l, m) \in M^*$ and $\int \phi_m(\epsilon) > 0$. By market clearing in contracts, $\int \theta_m(\epsilon) = \int \phi_m(\epsilon) > 0$, and because $(l, m) \in M^*$, $\int_{\mathcal{I}\times\mathcal{C}} e_{lm}(a, \tau_{\epsilon,\mathcal{A}\times\Phi})\theta_m \mathrm{d}\tau_\epsilon > 0$. Thus, if $\overline{k}_{lm} D^\epsilon_{lm} < S^\epsilon_{lm}$, we could divide both sides by D^ϵ_{lm} and obtain

$$\overline{k}_{lm} < \frac{S^\epsilon_{lm}}{D^\epsilon_{lm}} \leq \frac{\max_{i\in I, a\in\mathcal{A}, \nu\in\Delta(\mathcal{A})} D^i_{l,m}(a, \nu) \int \phi_m(\epsilon)}{\eta_{l,m} \int \theta_m(\epsilon)} = \overline{k}_{lm}$$

a contradiction.

Because $\overline{k}_{lm} D^\epsilon_{lm} - S^\epsilon_{lm} \geq 0$, for every $\epsilon > 0$, at a fixed point:

$$\Phi_{2,\epsilon,l,m}(\tau_\epsilon) = \frac{\epsilon \overline{k}_{lm} + S^\epsilon_{lm}}{\epsilon + D^\epsilon_{lm}}.$$

Thus,

$$\begin{aligned} 0 &\leq \int\limits_{\mathcal{I}\times\mathcal{C}} z_l \mathrm{d}\tau_\epsilon \\ &= \sum_m \left[\frac{\epsilon \overline{k}_{lm} + S^\epsilon_{lm}}{\epsilon + D^\epsilon_{lm}} D^\epsilon_{lm} - S^\epsilon_{lm} \right] \\ &= \sum_m \left[\frac{\epsilon}{\epsilon + D^\epsilon_{lm}} (\overline{k}_{lm} D^\epsilon_{lm} - S^\epsilon_{lm}) \right] \\ &\leq \sum_m \left[\frac{\epsilon D^\epsilon_{lm}}{\epsilon + D^\epsilon_{lm}} \overline{k}_{lm} \right] \\ &\leq \epsilon \sum_m \overline{k}_{lm}, \end{aligned}$$

and, at the limit, $\int_{\mathcal{I}\times\mathcal{C}} z \mathrm{d}\tau^* = 0$.

It remains to show that, at the limit, individuals are indeed optimizing, $\tau^*(B(q^*, K^*, \tau^*)) = 1$.

To simplify notation, let us define $B_n = B(q_n, K_n, \tau_n)$ and $B^* = B(q^*, K^*, \tau^*)$.

The proof proceeds in two steps. In the first one, we fix $\epsilon > 0$ and we show that there exists a compact set $\overline{V}$ such that, if we define $\overline{B}_n = B_n \cap \overline{V}$, there is a subsequence converging to some $\overline{B}$ such that $\tau^*(\overline{B}) > 1 - \epsilon$.

In the second step, we show that $\overline{B} \subset B^*$.

$\mathcal{I}, \mathcal{A}$ and Φ are compact, $\Theta = \{\theta \in \mathbb{R}^M \mid \theta \geq 0\}$ and $\mathcal{Z} = \mathbb{R}^L$ are Euclidean. For $N \in \mathbb{N}$, we have that $\Theta_N = \{\theta \in \mathbb{R}^M \mid 0 \leq \theta_m < N$, for $m = 1, \ldots, M\}$ and $\mathcal{Z}_N = \{z \in \mathbb{R}^L \mid -N < z_l < N$, for $l = 1, \ldots, M\}$ are open in the respective relative topologies. Their closures are $\overline{\Theta}_N = \{\theta \in \mathbb{R}^M \mid 0 \leq \theta_m \leq N$, for $m = 1, \ldots, M\}$ and $\overline{\mathcal{Z}}_N = \{z \in \mathbb{R}^L \mid -N \leq z_l \leq N$, for $l = 1, \ldots, M\}$.

Let $\mathcal{C}_N := \mathcal{A} \times \Theta_N \times \Phi \times \mathcal{Z}_N$. Then, $\mathcal{C} = \bigcup_{N \in \mathbb{N}} \mathcal{C}_N$, and accordingly $\mathcal{I} \times \mathcal{C} = \bigcup_{N \in \mathbb{N}} \mathcal{I} \times \mathcal{C}_N$. Since $\mathcal{I} \times \mathcal{C}_N \subseteq \mathcal{I} \times \mathcal{C}_{N'}$, for $N \leq N'$, and since $\tau^*(\mathcal{I} \times \mathcal{C}) = 1$, σ-additivity of τ^* implies that for every $\delta > 0$ there exists a $N \in \mathbb{N}$ such that $\tau^*(\mathcal{I} \times \mathcal{C}_N) > 1 - \delta$. Set $V := \mathcal{I} \times \mathcal{C}_N$ and $U := \mathcal{I} \times \mathcal{C}_{N+1}$. We have that $\overline{V} \subseteq U$ and that $\overline{V}$ and $\overline{U}$ are compact.

Because τ_n converges (weakly) to τ^*, this implies $\liminf_n \tau_n(V) \geq \tau^*(V) > 1 - \delta$. That is, there exists $\overline{n}$ such that, for all $n \geq \overline{n}$, we have $\tau_n(V) > 1 - \delta$. All along the sequence of fixed points, we have $\tau_n(B_n) = 1$, so it must be that, for $n \geq \overline{n}$, $\tau_n(\overline{B}_n) = \tau_n(B_n \cap \overline{V}) > 1 - \delta$. Since $\overline{U}$ is a compact metric space, its non-empty compact subsets, endowed with the Hausdorff metric, form a compact metric space. Because each B_n is non-empty and closed, the $\overline{B}_n$s are non-empty compact subsets of $\overline{U}$. We can therefore find a subsequence, which we denote again $(\overline{B}_n : n = 1, \ldots)$, converging to some compact $\overline{B}_\delta \subset \overline{U}$.

For given $\epsilon > 0$, choose $\delta < \epsilon$ and assume by contradiction that $\tau^*(\overline{B}_\delta) < 1 - \epsilon < 1 - \delta$. Let us define $\overline{B} = \overline{B}_\delta$ from now on. Since $\overline{B}$ is compact, there exists an open set W such that $\overline{B} \subset W$ and $\tau^*(\overline{W}) < 1 - \epsilon$. Then, we also have $\overline{B} \subset (W \cap U)$ and $\tau^*(\overline{W} \cap \overline{U}) < 1 - \epsilon$. Because $\overline{B}_n$ converges to $\overline{B}$, there exists $\overline{n}$ such that, for all $n \geq \overline{n}$, $\overline{B}_n \subset (W \cap U)$. But then, for all $n \geq \overline{n}$, $1 - \delta < \tau_n(\overline{B}_n) \leq \tau_n(\overline{W} \cap \overline{U})$. Using again the weak convergence of τ_n to τ^*, we conclude $1 - \delta < \limsup_n \tau_n(\overline{W} \cap \overline{U}) \leq \tau^*(\overline{W} \cap \overline{U}) < 1 - \epsilon$, a contradiction.

Let us now show that $\overline{B} \subset B^*$. Because $(\overline{B}_n : n = 1, \ldots)$ converges to $\overline{B}$ in the Hausdorff metric, for any given point $(i, c) \in \overline{B}$ we can construct a sequence (i_n, c_n) converging to (i, c) with $(i_n, c_n) \in B_n$. We need to show that $(i, c) \in B^*$. That is, we have to show the closed graph property of the correspondence $B : \Delta^M \times \mathcal{K} \times \Delta(\mathcal{I} \times \mathcal{C}) \to \mathcal{I} \times \mathcal{C}$. We already proved the closed graph property on the restricted domain $\Delta^M_\epsilon \times \mathcal{K} \times \Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$. It is then enough to observe that the same argument works here, because the budget correspondence β has a closed graph and is lower hemi-continuous even on the extended domain $\Delta^M \times \mathcal{K} \times \Delta(\mathcal{A} \times \Phi)$. □

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