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Pareto improving taxes

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We dedicate this paper to Roko Aliprantis. His scholarship, energy, and enterprising spirit have dramatically improved economic theory.

Abstract

We show that in almost every economy with separable externalities, every competitive equilibrium can be Pareto improved by a package of anonymous commodity taxes that causes prices to adjust and markets to reclear at different levels of individual consumption. This constrained suboptimality of competitive allocations might provide a rationale for economic policy in economies with externalities. It shows that policy makers should look for good tax packages that help everybody, rather than thinking taxes must inevitably be bad for some lobby that will oppose them.

JEL Classification: D50; D60; D62; D82

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1. Introduction

It is a curious fact that most policy makers regard taxes as bad, while at the same time they recognize the existence of widespread externalities. In this paper we try to make the case that there is almost always a tax *package* that is good for everybody.

The classical theorems of welfare economics, formulated definitively in Arrow (1951) and Debreu (1951), established the equivalence between competitive equilibrium allocations and Pareto optimal allocations in economies without externalities. When there are externalities, private costs and social costs differ, and competitive equilibria are not likely to be Pareto efficient. Agents will typically make poor social choices, for example, smoking too much or driving too much, because they do not take into account the cost they impose on bystanders who must inhale their smoke or exhaust fumes, not to mention getting crowded out of highway space. Lindahl (1919) and Pigou (1920, 1932) famously argued that taxes could be an appropriate antidote to the socially false incentives provided by competitive prices, because if the taxes were set equal to the external cost imposed on third parties, then agents would effectively internalize the externality, taking into account the cost they imposed on others. Despite general familiarity with Pigouvian taxation, policy makers have not embraced the concept of "good taxes". We believe there are at least three reasons.

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In the first place, any one tax hurts some people while helping others. The bystander does not have to breathe as much noxious air, but the smoker must pay a tax and not enjoy as many cigarettes. Unless one is prepared to make interpersonal utility comparisons, valuing bystanders' utilities more than smokers' utilities, the Pigouvian observation about divergent private and social costs is not an argument by itself for taxation.

Second, if Pigouvian taxation is taken to its logical conclusion, then different individuals should face different tax rates for the same good. (A smoker who always lights up outdoors should pay less tax than his brother who only smokes in crowded restaurants, because he causes less damage.) This idea was elaborated in Lindahl (1919, 1928), Samuelson (1954), Coase (1960), and Arrow (1970). In "Lindahl equilibrium", Pareto efficiency is achieved by charging different taxes for the same good, depending on the buyer. Furthermore, combining these individual specific taxes with a carefully chosen program of individually targeted income redistribution (often exceeding the revenue raised by the taxes) can achieve allocations that Pareto dominate purely competitive equilibrium. But such detailed, and discriminatory, interventions seem hopelessly complicated, and possibly illegal.

Third, even if it were practical to implement a plan that taxed commodity purchases and redistributed income on a person specific basis, how would the tax authorities ever know which individuals to charge the higher taxes? As Arrow (1970) pointed out, Lindahl equilibrium does not satisfy the incentive compatibility constraints of Hurwicz (1972), since it is not in the interest of individuals to reveal the information necessary for the price mechanism to function.

In this paper we suppose that the social planner can discover the population distribution of household types (say a continuum of each of $I > 1$ types of households, where a type defines an agent's preferences, endowments, and the externalities his consumption generates). The planner does not need to know which agent is of which type. We also suppose there are more commodities L than household types, $L > I$. We then show that for almost all externalities, there is a way to make everybody better off than they would be under perfect competition by taxing or subsidizing commodities anonymously (everyone pays the same tax) and redistributing the tax revenue anonymously (each household gets the same rebate, independent of their income or how much they spent or what taxes they paid). It is not necessary to make interpersonal utility comparisons to see that this tax package is better than laissez faire, taxes do not need to be individual specific, and the central planner needs to know about population characteristics and not about individuals.

Techniques of differential topology became standard in general equilibrium theory after the pioneering work of Debreu (1970, 1972, 1976). Our proof of generic Pareto improving taxes introduces these techniques into public finance, where, though natural, they have been seldom applied. Guesnerie (1977) did provide sufficient conditions on the externalities, prices, excess demands, and the derivatives of demand at equilibrium for tax interventions to Pareto improve over the status quo. But he did not connect these to primitives on preferences and endowments, as we do by showing that they occur at every equilibrium for almost all preferences, endowments, and separable externalities.

Our theorem, on the other hand, does not address several important questions. It does not say how big the taxes could be, and thus how much revenue they could generate, and still Pareto improve on laissez faire. Taxes in modern economies are quite high, and have potentially large incentive effects. One message of our theorem is that it might be useful to consider how much of this revenue could be raised through a package of "good taxes" that raise welfare instead of simple income taxes that might discourage work.

The theorem compares welfare at competitive equilibrium (with no taxes) to equilibrium after the "good tax package" has been implemented. It does not examine whether starting from a situation with an income tax, it is always possible to find a tax package raising the same revenue but making everyone better off.

The theorem considers only separable externalities, which do not affect any agent's marginal rate of substitution between goods. Moreover, the theorem is proven for almost all externalities. It is possible that a deeper analysis might show that for any nontrivial (separable or nonseparable) externality, some tax package could be found to Pareto improve on competitive equilibrium. Economies that allow for strategic interactions generalize economies with externalities. It would then be an immediate extension of the deeper argument to show that, with strategic interactions, generically, there are commodity taxes that lead to Pareto improvement in welfare.

Our theorem also does not suggest how the social planner might come to know the distribution of agent types in the population. But that is a far smaller information burden than knowing the type of every individual.

The demonstration that competitive equilibria in economies with externalities are constrained suboptimal makes an important methodological point. Tax intervention is often said to be counterproductive because competitive equilibrium cannot be Pareto improved by anonymous taxes. Since externalities are ubiquitous, our theorem shows that such a view is untenable. Tax intervention may be counterproductive because the fiscal authority does not know enough about the population distribution of tastes and endowments to set the right taxes and subsidies, but not because there are no beneficial taxes and subsidies. The argument shows that such taxes exist; it does not indicate how to compute them.

An alternative approach would be to ask which allocations can be implemented as strategic equilibria, through the design of mechanisms and an explicit recognition of incentive compatibility constraints, as introduced by Hurwicz (1973, 1979) or, in an abstract setting, by Maskin (1999) and developed in the theory of contracts. We have eschewed this approach in order to focus on the functioning of competitive, anonymous markets. We have in effect severely constrained the kinds of interventions that a policy maker could use, and yet we still prove the existence of Pareto improving taxes.

Work in public finance, starting with Ramsey (1927), and developed in Diamond and Mirrlees (1971) and Diamond (1975) characterized second-best commodity taxes in the absence of lump-sum transfers. Corrective uniform taxation was considered for simple economies in Diamond (1973).

While our question of the existence of Pareto improving taxes, constrained to be anonymous, does not seem to have been posed in precisely our form for externalities, the analogous question when externalities are replaced by incomplete asset markets or asymmetric information has been analyzed repeatedly.

With uncertainty and an incomplete asset market, Geanakoplos and Polemarchakis (1986) proved the constrained suboptimality of competitive equilibrium allocations: generically, there is a reallocation of assets that leads to a Pareto superior allocation of goods after prices in commodity spot markets adjust and markets clear. This phenomenon had been illustrated (but not proved) in Stiglitz (1982), while Citanna et al. (1998) refined the proof. Citanna et al. (2006) showed that taxation, which is anonymous, could induce a Pareto improving reallocation of assets.

The Pareto improving possibilities generated by the taxation of exchanges in economies with asymmetric information was introduced by Greenwald and Stiglitz (1986). Dubey et al. (1997), and later Bisin et al. (2001), showed that many adverse selection and moral hazard problems, including the adverse selection problem described in Akerlof (1970) or Rothschild and Stiglitz (1976), and the moral hazard phenomena described in Mirrlees (1999), could be recast in a more standard general equilibrium context with two changes: promises by different agents are pooled together, and the deliveries each agent makes are an option for him. Bisin et al. (2001) showed that, generically, the anonymous taxation of these contracts can effect a Pareto improvement.

In this prior work on constrained inefficiency it was possible to confront directly the question of how the central planner could discover enough about the characteristics of the agents to find the right tax rates. Even when the asset market is incomplete, Geanakoplos and Polemarchakis (1990) showed that the utility function of an individual can be identified from his demand function for commodities and assets; recently, Kübler et al. (2002) extended the argument to show that every individual utility can be obtained from aggregate demand or the graph of the equilibrium correspondence as the allocation of endowments varies. For economies with a public good, Snyder (1999) obtained results concerning restrictions on the market behavior of optimizing individuals, but Carvajal (2002) showed that the results do not generalize.

Unfortunately the separable externalities we assume in this paper do not have observable consequences for agent demands, so we must leave open the question of how the central planner could discover the size of the externalities one man's consumption inflicts on others.

2. The economy

Household types are represented by individuals $i \in I = \{1, \dots, I\}$, and commodities by $\ell \in L = \{1, \dots, L\}$. We imagine a continuum of individuals of each type i.

For any individual $i \in I$, we denote a non-negative consumption bundle by $x^i = (x^i_1, \dots, x^i_l, \dots, x^i_L) \in \mathbb{R}^L_+$; across individuals, we denote an allocation of commodities by $\vec{x} = (x^1, \dots, x^i, \dots, x^I) \in \mathbb{R}^{LI}_+$. When we wish to emphasize the consumption of some individual i, we write $\vec{x} = (x^i, x^{-i})$, where $x^{-i} = (x^h : h = 1, \dots, I, h \neq i)$ is the complementary allocation.

An individual is described by his utility function and endowment. His utility function $u^i : \mathbb{R}^{LI}_+ \to \mathbb{R}$ has domain the set of allocations. The dependence of u^i on x^{-i} is what we mean by an externality, since in competitive equilibrium individual i has no control over x^{-i} and yet his utility depends on it. His endowment is a vector of goods $e^i \in \mathbb{R}^L_{++}$, a consumption bundle.

Across individuals, the profile of utility functions is $\vec{u} = (u^1, \dots, u^i, \dots, u^I)$, and the allocation of endowments is $\vec{e} = (e^1 \dots, e^i, \dots, e^I)$. The pair $(\vec{u}, \vec{e})$ defines an economy.

The aggregate endowment is $e = \sum_i e^i$. At an allocation, aggregate consumption is $x = \sum_i x^i$, and the allocation is feasible if $x = e$.

The profile of utilities at an allocation is $\vec{u}(\vec{x}) = (u^1(\vec{x}), \ldots, u^i(\vec{x}), \ldots, u^I(\vec{x}))$. An allocation, $\vec{x}_1$, is Pareto superior to another, $\vec{x}_2$, if $\vec{u}(\vec{x}_1) > \vec{u}(\vec{x}_2)$; a feasible allocation is Pareto optimal if a Pareto superior feasible allocation does not exist.

Prices of commodities are denoted by $p = (p_1, \ldots, p_l, \ldots p_L) \in \mathbb{R}^L_+$, and commodity tax rates by $(t_1, \ldots, t_l, \ldots t_L) \in \mathbb{R}^L$; prices are positive, but tax rates may be negative— $t_l < 0$ is a subsidy. We shall always regard a tax as levied on the buyers, so that a commodity with price p_ℓ and tax rate t_ℓ costs any buyer $p_\ell + t_\ell$, but brings revenue of only p_ℓ to the seller. Lump-sum transfers of revenue are determined by a single scalar $\tau \in \mathbb{R}$ representing the transfer to each individual. The transfer can be positive or negative. Taxes and transfers are anonymous.

An individual regards the transfer of revenue that he receives as independent of the commodity taxes that he pays.

At prices p and tax rates t and revenue τ and complementary allocation x^{-i}, the optimization problem of an individual $i \in I$ is

$$\begin{aligned} \max_{x \in \mathbb{R}^L_+} \quad & u^i(x, x^{-i}) \\ \text{s.t.} \quad & (p+t)\cdot(x-e^i)_+ - p\cdot(x-e^i)_- \leq \tau \end{aligned}$$

The solution to the optimization problem is $x^i(p, t, \tau, x^{-i}, e^i)$.

Given an economy $(\vec{u}, \vec{e})$ and tax rates t, a competitive t-equilibrium consists of prices and a feasible allocation, $(p, \vec{x})$, such that $x^i \in x^i(p, t, \tau, x^{-i}, e^i)$, where $\tau = (1/I)\sum_i t\cdot(x^i - e^i)_+$ is the per capita share of the tax revenue $\sum_i t\cdot(x^i - e^i)_+$.

A competitive equilibrium is a competitive t-equilibrium at tax rates $t = 0$.

A feasible allocation is constrained Pareto suboptimal if there exists a tax package t and a Pareto superior competitive t-equilibrium allocation.

The purpose of the paper is to prove the following theorem, whose terms will be made formally precise in the next sections.

Theorem. *For almost all economies with separable externalities and $L > I$, every competitive equilibrium is constrained Pareto suboptimal*; *that is*, *for each competitive equilibrium*, *there exists an anonymous tax package t and a competitive t-equilibrium allocation which Pareto dominates it.*

3. Walrasian equilibria

An economy is Walrasian if there are no external effects: for every individual, the utility function, u^i, is independent of the complementary allocation, x^{-i}, and there are no taxes or transfers: $t = 0$ and $\tau = 0$. For such economies it is notationally easier to take the domain of u^i to be simply $\mathbb{R}^L_+$.

A Walrasian economy is smooth if, for every individual $i \in I$,

1. the utility function, u^i, is continuous, strictly monotonically increasing and strictly quasi-concave;
2. in the interior of its domain of definition, the utility function is twice continuously differentiable, differentiably strictly monotonically increasing: $Du^i \gg 0$, and differentiably strictly quasi concave: $y'Du^i = 0 \Rightarrow y'(Du^i)y < 0$, for $y \neq 0$; along any sequence $(x^i_n : n = 1, \ldots)$, with $\lim_{n\to\infty} x^i_n = \bar{x}^i$ a consumption bundle on the boundary, $\lim_{n\to\infty} = \|Du^i(x^i_n)\|^{-1} Du^i(x^i_n)\cdot x^i_n = 0$. In particular, if $x \gg 0$ and $u^i(y) \geq u^i(x)$, then $y \gg 0$;
3. the endowment is strictly positive: $e^i \gg 0$.

From now on we fix $L > I > 1$, and $\vec{u}^* = (u^{*1}, \ldots, u^{*i}, \ldots, u^{*I})$ satisfying 1–2 above. The set of economies, $\mathcal{E}$, is then parameterized by the allocation of endowments, $\vec{e}$; $\mathcal{E} = \mathbb{R}^{LI}_{++}$ is an open set in Euclidean space of dimension LI. A property holds for "almost all" economies or "generically" if it holds for an open set $\mathcal{E}' \subset \mathcal{E}$ of economies whose complement $\mathcal{E} \setminus \mathcal{E}'$ has Lebesgue measure zero. We call the set $\mathcal{E}'$ generic or say it has full Lebesgue measure.

We now quickly review the main properties of competitive equilibrium for Walrasian economies. These were established by Debreu in 1970.

With no taxes or income transfers or externalities, the optimization problem of the individual $i \in I$ is

$$\max_{x^i \in \mathbb{R}^L_+} \quad u^{*i}(x^i)$$
$$\text{s.t.} \qquad p \cdot (x^i - e^i) \leq 0.$$

The solution to the optimization problem of the individual exists and is unique, and is denoted by $x^i(p, e^i)$. From our assumptions 1–3 we know that $x^i(p, e^i) \gg 0$ whenever $p \gg 0$. The excess demand function of the individual, z^i, is defined by $z^i(p, e^i) = x^i(p, e^i) - e^i$.

The individual excess demand function is continuously differentiable, and it satisfies homogeneity of degree 0 in prices: $z^i(kp, e^i) = z^i(p, e^i)$, for $k > 0$, and Walras' law: $p \cdot z^i(p, e^i) = 0$; also, along any sequence $(p_n : n = 1, \ldots)$, with $\lim_{n\to\infty} = \bar{p}$ on the boundary of the strictly positive domain, $\lim_{n\to\infty} = \|z^i(p_n, e^i)\| = \infty$.[1]

A Walrasian equilibrium consists of prices and a feasible allocation, $(p, \vec{x})$, such that $x^i \in x^i(p, e^i)$, for every individual.

Since the utility functions of individuals are strictly monotonically increasing, it is sufficient to restrict attention to strictly positive prices: $p \gg 0$. By homogeneity it also suffices to restrict attention to equilibrium prices $p = (p_1, \ldots, p_{L-1}, 1)$ for which commodity $l = L$ is numéraire.

The aggregate excess demand function, z, is defined by $z(p, \vec{e}) = \sum_i z^i(p, e^i)$; it inherits the continuous differentiability of the excess demand functions of individuals, and it satisfies homogeneity of degree 0 and Walras' law; along any sequence $(p_n : n = 1, \ldots)$, with $\lim_{n\to\infty} = p$ on the boundary of the strictly positive domain, $\lim_{n\to\infty} = \|z(p_n, \vec{e})\| = \infty$.

The truncated excess demand of an individual is $\hat{z}^i = (z^i_1, \ldots, z^i_{L-1})$; it is the demand for commodities other than the numéraire. Similarly, the truncated excess demand for the economy is $\hat{z} = (z_1, \ldots, z_{L-1})$.

For an economy, $\vec{e}$, we write $\hat{z}_{\vec{e}}(p) = \hat{z}(p, \vec{e})$. Walrasian equilibrium prices satisfy $\hat{z}_{\vec{e}}(p) = 0$. Conversely, by Walras Law, any price vector p satisfying $\hat{z}_{\vec{e}}(p) = 0$ is part of a Walrasian equilibrium.

An equilibrium $(p, \vec{x})$ for the economy $\vec{e}$ is called regular if $\dim[D\hat{z}_{\vec{e}}(p)] = L - 1$. By the implicit function theorem, a regular equilibrium is locally unique and varies smoothly as the endowment varies on a small open set around $\vec{e}$. The economy $\vec{e}$ itself is called regular if all its equilibria are regular, that is, if $\hat{z}_{\vec{e}}(p) = 0 \Rightarrow \dim[D\hat{z}_{\vec{e}}(p)] = L - 1$. We sometimes denote this situation by $\hat{z}_{\vec{e}} \pitchfork 0$.

Let $K \subset \mathcal{E} = \mathbb{R}^{LI}_{++}$ be a compact set of endowment vectors. The set of equilibria $(p, \vec{x})$ corresponding to endowments $\vec{e} \in K$ must be compact. It is obviously closed. To see that it lies in a bounded set, note first that there is some $\bar{e}\sum_i \vec{e}^i$ for all $\vec{e} \in K$. Hence in any equilibrium, all individual consumptions lie in the compact set $0 \leq x^i \leq \bar{e}$. Furthermore, the minimum of the continuous function u^{*1} on K must be attained at some $\bar{x}^1 \gg 0$. Thus we can further conclude that any equilibrium consumption x^1 must satisfy $u^1(x^1) \geq u^1(\bar{x}^1)$. From assumption 2 that means $x^1 \gg 0$. From this interiority and the first order conditions in the above optimization problem, we deduce that equilibrium prices must be $p = Du^{*1}(x^1)/(\partial u^{*1}(x^1)/\partial x_L)$ for some x^1 satisfying the aforementioned inequalities. Since u^{*1} is continuously differentiable and since x^1 must lie in a compact set, so must the prices p.

It follows that the set of Walrasian equilibria for a single regular economy is finite.[2]

A theorem of Arrow and Debreu (1954) assures us that equilibrium prices exist for every economy. The following theorem is essentially due to Debreu (1970).

Debreu's Theorem. *The set of regular economies is generic.*

Proof. Let us first see that $\dim[D\hat{z}_{e^1}(p, \vec{e})] = L - 1$. For any commodity $1 \leq \ell < L$, consider the infinitesimal variation in endowments δ^1_ℓ defined by decreasing e^1_l by 1 unit, increasing e^1_L by (p_l/p_L) units and leaving all other endowments unchanged. Evidently, the income and demand of individual 1 is unaffected by the perturbation δ^1_ℓ, and, as a consequence, $\partial z^1_\ell/\partial\delta^1_\ell = 1$; the excess demand of individual 1 for all commodities other than l or the numéraire, as well as the excess demand of every other individual, are unaffected by the perturbation. It follows that $\partial\hat{z}_\ell/\partial\delta^1_\ell = 1$ and $\partial\hat{z}_k/\partial\delta^1_\ell = 0$ for all $k \neq \ell, L$. As a consequence, $\dim[D\hat{z}(p, \vec{e})] = L - 1$ or, equivalently, $\hat{z} \pitchfork 0$. By the transversal

[1] Debreu (1972, 1976).
[2] Debreu (1970).

density theorem,[3] there exists a set of full Lebesgue measure of regular economies $\mathcal{E}_0 \subset \mathcal{E}$ such that, $\hat{z}_{\vec{e}} \pitchfork 0$ whenever $\vec{e} \in \mathcal{E}_0$.

Finally we show that $\mathcal{E}_0$ is open. If not, then there is a sequence of non-regular economies (sometimes called critical) $\vec{e}(n)$ converging to a regular economy $\vec{e} \in \mathcal{E}_0$. Each economy $\vec{e}(n)$ has a non-regular (critical) equilibrium $(p(n), \vec{x}(n))$. Any convergent sequence $\{\vec{e}(n)\}$ together with its limit $\vec{e}$ is a compact set, hence by the argument given just before the theorem, the set of all corresponding equilibria must be compact. Hence there is a convergent subsequence $(p(n_k), \vec{x}(n_k))$ which must converge to a critical equilibrium $(p, \vec{x})$ of $\vec{e}$, since $\vec{e}(n_k) \to \vec{e}$ and any limit of critical equilibria is a critical equilibrium of the limiting economy . But that is impossible since $\vec{e}$ is regular and therefore has no critical equilibrium. This contradiction proves $\mathcal{E}_0$ is open. $\square$

At a Walrasian equilibrium, the profile of marginal utilities of revenue, $\vec{\mu} = (\mu^1, \ldots, \mu^i, \ldots, \mu^I)$, is determined by $\mu^i = (\partial u^i / \partial x^i_L)(x^i)$.

Using these marginal utilities of income we can define equilibrium for an economy $\vec{e}$ as a triple $(\vec{x}, \vec{\mu}, p)$ satisfying the equations $F_{\vec{e}}(\vec{x}, \vec{\mu}, p) = 0$ where

$$F_{\vec{e}} : (\mathbb{R}^L_{++} \times \mathbb{R}_{++})^I \times \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{(L+1)I} \times \mathbb{R}^{L-1}$$

is defined by

$$F_{\vec{e}}(\vec{x}, \vec{\mu}, p) = \begin{pmatrix} \vdots \\ Du^{*i}(x^i) - \mu^i p \\ p \cdot (x^i - e^i) \\ \vdots \\ \sum_i (\hat{x}^i - \hat{e}^i) \end{pmatrix},$$

It can be shown that $\hat{z}_{\vec{e}} \pitchfork 0$ if and only if $F_{\vec{e}} \pitchfork 0$.

3.1. Trade at equilibrium

At prices of commodities, p, an individual, i, trades a commodity, l, if $x^i_l(p, e^i) \neq e^i_l$.

Lemma 1. *Suppose we restrict attention to heterogenous economies with $I \geq 2$, and $L \geq 2$. Then, generically, at every Walrasian equilibrium, every individual trades every commodity.*

Proof. Fix an individual $h \in I$ and a commodity $k \in L$; by renumbering commodities we can take $k \neq L$, and by renumbering individuals we can take $h \neq 1$.

By definition, the truncated individual excess demand function consists of the excess demand for the $(L-1)$ commodities other than the numéraire, $l = L$, and the price of the numéraire commodity is set at $p_L = 1$.

The function

$$F^h_k : \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \to \mathbb{R}^{L-1} \times \mathbb{R},$$

is defined by

$$F^h_k(p, \vec{e}) = \begin{pmatrix} \hat{z}(p, \vec{e}) \\ z^h_k(p, e^i) \end{pmatrix}.$$

[3] Abraham and Robin (1967, thm. 19.1).

The Jacobian matrix $D_{e^1}\hat{z}(p, \vec{e})$ has rank $L-1$, (as we saw in the proof of Debreu's theorem), while $D_{e^1}\hat{z}^h_k(p, \vec{e}) = 0$, since $h \neq 1$. But by exactly that same argument in Debreu's theorem, there is an infinitesimal variation, δ^h_k in the endowments of agent h such that $D_{\delta^h_k}\hat{z}^h_k(p, \vec{e}) = 1$. It follows that

$$(D_{e^1}F^h_k, D_{\delta^h_k}F^h_k) = \begin{pmatrix} D_{e^1}\hat{z} & D_{\delta^h_k}\hat{z} \\ 0 & 1 \end{pmatrix}$$

has rank L. Since all the columns of this matrix are linear combinations of columns of the matrix $D_{\vec{e}}F^h_k$, this Jacobian matrix, evaluated at any point $(p, \vec{e})$ with $\hat{z}(p, \vec{e}) = 0$, has rank L, and, as a consequence, $F^h_k \pitchfork 0$.

By the transversal density theorem and the boundary behavior of the excess demand function, there is an open set $\mathcal{E}^h_k \subset \mathcal{E}$ of endowments of full Lebesgue measure such that for allocations of endowments, $\vec{e} \in \mathcal{E}^h_k$, $(F^h_k)_{\vec{e}} \pitchfork 0$. But $(F^h_k)_{\vec{e}} \pitchfork 0$ only if $(F^h_k)^{-1}_{\vec{e}}(0) = \emptyset$, since the domain of $(F^h_k)_{\vec{e}}$ has dimension $L-1$, while the range has dimension L.

For $\vec{e} \in \mathcal{E}^h_k$, there are no prices of commodities such that markets clear, $\hat{z}(p, \vec{e}) = 0$, while individual h does not trade in commodity k, $z^h_k(p, \vec{e}) = 0$.

The set of economies $\mathcal{E}_1 = \cap_h \cap_k \mathcal{E}^h_k$ has the desired property: for any allocation of endowments $\vec{e} \in \mathcal{E}_1$, at every Walrasian equilibrium, every individual trades every commodity. □

Since the set of regular economies, $\mathcal{E}_0$, is open and of full Lebesgue measure, the set

$$\mathcal{E}^* = \mathcal{E}_0 \cap \mathcal{E}_1$$

of regular economies with the property that, at every Walrasian equilibrium, every individual trades every commodity, is open and of full Lebesgue measure.

We will restrict attention to this generic set $\mathcal{E}^*$ of regular economies with full trade.

4. Taxes and transfers

Now we return to the case where there are taxes, but we retain our hypothesis that there are no external effects: for every individual, the utility function, u^i, is independent of the complementary allocation, x^{-i}. Hence we think of utility as fixed exactly as in the Walrasian case as a function $u^{*i} : \mathbb{R}^L_+ \to \mathbb{R}$.

Recall that individuals are taxed t_l for purchasing a unit of commodity l; the tax t_l is a mark-up or a subsidy to a buyer of the commodity, but it leaves a seller unaffected. Fiscal revenue is returned to individuals in equal amounts, τ. Thus taxes and transfers are anonymous. Also, every individual regards the transfer of revenue that he receives as independent of the commodity taxes he pays.

At prices p and tax rates t and revenue τ, the optimization problem of an individual is

$$\begin{array}{ll} \max\limits_{x^i \in \mathbb{R}^L_+} & u^{*i}(x^i) \\ \text{s.t.} & (p+t)\cdot(x^i - e^i)_+ - p\cdot(x^i - e^i)_- \leq \tau. \end{array}$$

The solution $x^i(p, t, \tau, e^i)$, to the optimization problem, defines the demand correspondence of the individual. The budget set is kinked at the endowment point, and when there is a subsidy instead of a tax, so some $t_l < 0$, the budget set is not even convex. Thus in general we cannot be sure that $x^i(p, t, \tau, e^i)$ is differentiable, or even single-valued. But in fact it is, for small taxes and prices at which there is trade in every commodity.

Suppose that at prices p^*, and taxes $t = 0$, and revenue $\tau = 0$, the individual trades every commodity; that is, suppose his Walrasian demand at p^* differs from his endowment, $x^i_l(p^*, e^i) \neq e^i_l$ for all $\ell \in L$. For any tax package $t = (t_1, \ldots, t_l, \ldots, t_L)$, define $t^i = (t^i_1, \ldots, t^i_l, \ldots, t^i_L)$, where

$$t^i_l(p^*, t) = \begin{cases} t_l & \text{if} \quad x^i_l(p^*, e^i) > e^i_l \\ 0 & \text{if} \quad x^i_l(p^*, e^i) < e^i_l \end{cases}$$

Since the utility function is strictly quasi-concave, it follows that for all p near enough to p^*, and t near enough to 0, and τ near enough to 0, the following two budget sets lead to the same choices

$$\begin{aligned} x^i \quad &\in \arg\max\{u^{*i}(x^i) : (p+t)\cdot(x^i-e^i)_+ - p\cdot(x^i-e^i)_- \le \tau\} \\ &\Leftrightarrow x^i \in \arg\max\{u^{*i}(x^i) : (p+t^i)\cdot(x^i-e^i) \le \tau\}. \end{aligned}$$

The reason is that the non-overlapping parts of the budget sets contain only points x for which $u^{*i}(x) < u^{*i}(x^i(p^*, e^i))$ by a big gap, and hence are irrelevant to the maximization problem, as long as commodity taxes, t, and transfers, τ, are close enough to 0, and p is near to p^*. The advantage of the second budget set is that it is linear in x^i, and hence behaves like a Walrasian budget set. Furthermore, for fixed p^*, $t^i_l(p^*, t)$ is obviously a smooth function of t.

Since the Walrasian demand function, $x^i(p, e^i)$, is smooth in (p, e^i), the demand function with commodity taxes and transfers of revenue, $x^i(p, t, \tau, e^i)$, is smooth for p near enough to p^* and t and τ^i small enough; in particular it is smooth at $(p, t, \tau) = (p^*, 0, 0)$.

Recall that in a competitive t-equilibrium, the fiscal authority must calculate the fiscal revenue that must be redistributed. The tax rebate itself affects spending across commodities and, consequently, fiscal revenue. The fiscal authority must foresee the change in spending and announce anonymous lump-sum transfers that coincide with the fiscal revenue that will ensue. So it is not obvious that a competitive t-equilibrium exists. The following Lemma shows that in an economy with a regular Walrasian equilibrium with full trade, tax equilibrium also exists for small enough taxes.

Lemma 2. *Let $\vec{e}$ be a regular Walrasian economy (with no externalities).* If $(\vec{x}^*, \vec{\mu}^*, p^*)$ *is a regular Walrasian equilibrium with full trade, then there exists an open set, $\mathcal{U} \subset \mathbb{R}^L$, with $0 \in \mathcal{U}$, and smooth functions $\tilde{x}^i : \mathcal{U} \to \mathbb{R}^L_{++}$, $\tilde{p} : \mathcal{U} \to \mathbb{R}^{L-1}_{++}$, and $\tilde{\tau} : \mathcal{U} \to \mathbb{R}$, such that, $(\tilde{x}(0), \tilde{p}(0), \tilde{\tau}(0)) = (\vec{x}^*, p^*, 0)$ and for all $t \in \mathcal{U}$,*

$$\begin{aligned} &\sum_i \tilde{x}^i(t) = e, \\ &\tilde{\tau}(t) = \frac{1}{I}\sum_i t\cdot(\tilde{x}^i(t) - e^i)_+, \\ &\tilde{x}^i(t) = x^i(p(t), t, \tilde{\tau}(t), e^i). \end{aligned}$$

Proof. The first order, market clearing, and fiscal balance conditions

$$F_{\vec{e}} : (\mathbb{R}^L_{++} \times \mathbb{R}_{++})^I \times \mathbb{R}^{L-1}_{++} \times \mathbb{R} \times \mathbb{R}^L \to \mathbb{R}^{(L+1)I} \times \mathbb{R}^{L-1}$$

and

$$R_{\vec{e}} : (\mathbb{R}^L_{++} \times \mathbb{R}_{++})^I \times \mathbb{R}^{L-1}_{++} \times \mathbb{R} \times \mathbb{R}^L \to \mathbb{R}$$

are defined by

$$F_{\vec{e}}(\vec{x}, \vec{\mu}, p, \tau, t) = \begin{pmatrix} \vdots \\ Du^{*i}(x^i) - \mu^i(p + t^i(p^*, t)) \\ (p + t^i(p^*, t))\cdot(x^i - e^i) - \tau \\ \vdots \\ \sum_i (\hat{x}^i - \hat{e}^i) \end{pmatrix},$$

and

$$R_{\vec{e}}(\vec{x}, \vec{\mu}, p, \tau, t) = \tau - \frac{1}{I}\sum_i t^i(p^*, t)\cdot(x^i - e^i).$$

By hypothesis there is full trade, $x^i_l(p^*, e^i) \neq e^i_l$ for all i, ℓ, and, as a consequence, t^i is a smooth function of t. It follows that $F_{\vec{e}}$ is smooth. Furthermore, for x^i near x^{*i}, $t^i(p^*, t)\cdot(x^i - e^i) = t\cdot(x^i - e^i)_+$

Clearly $F_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) = 0$, since with $t = 0$ and $\tau = 0$, the function $F_{\vec{e}}$ simply recapitulates the standard Walrasian equilibrium conditions. Similarly, a solution $(\vec{x}, \vec{\mu}, p, \tau)$ to $F_{\vec{e}}(\vec{x}, \vec{\mu}, p, \tau, t) = 0$ and to $R_{\vec{e}}(\vec{x}, \vec{\mu}, p, \tau, t) = 0$ is a competitive t-equilibrium, if t is small enough. The endogenous Walrasian variables are $\eta = (\vec{x}, \vec{\mu}, p) \in \mathbb{R}^{(L+1)I}_{++} \times \mathbb{R}^{L-1}_{++}$. At the regular Walrasian equilibrium $(\vec{x}^*, \vec{\mu}^*, p^*)$,

$$D_\eta F_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0)$$

is, by the definition of regularity, a square matrix with full rank, $(L + 1)I + L - 1$. When tax rates are zero, changing consumption or marginal utilities or prices does not affect tax revenue, hence

$$D_\eta R_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) = 0, \quad \text{and} \quad D_\tau R_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) = 1.$$

It follows that the Jacobian matrix

$$\begin{pmatrix} D_\eta F_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) & D_\tau F_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) \\ D_\eta R_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) & D_\tau R_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) \end{pmatrix} = \begin{pmatrix} D_\eta F_{\vec{e}}(\vec{x}^*, \vec{\mu}^*, p^*, 0, 0) & ? \\ 0 & 1 \end{pmatrix}$$

has full rank, $(L + 1)I + (L - 1) + 1$.

Using (η, τ) as endogenous variables and t as exogenous variables, the implicit function theorem guarantees the existence of competitive t-equilibrium in a neighborhood of $t = 0$, as a function of the tax rates on commodities. □

We are now ready to establish the crucial step in our proof. When there are no externalities, taxes help some people and hurt others. But with externalities, a tax can change the choices of some individual h, and thereby improve the utility of another individual i. We first establish that by changing taxes, the fiscal authority can produce a rich array of changes in consumption choices. Those changes will later be used to effect beneficial externalities.

We will not impose taxes on the last, numeraire, commodity: $t_L = 0$. Thus we confine attention to infinitesimal tax rates $\mathrm{d}t = (\mathrm{d}t_1, \ldots, \mathrm{d}t_{L-1})$.

The reason is that, in general, taxation of all the commodities, including the numéraire commodity, may not achieve anything more than taxing just the first $L - 1$ commodities. That is the reason we needed to assume $L > I$.

Corollary 1. *At a regular Walrasian equilibrium with full trade, the matrix*

$$D_t\vec{\tilde{x}} = \begin{pmatrix} \dfrac{\mathrm{d}\tilde{x}^1_1}{\mathrm{d}t_1} & \cdots & \dfrac{\mathrm{d}\tilde{x}^1_1}{\mathrm{d}t_{L-1}} \\ \vdots & & \vdots \\ \dfrac{\mathrm{d}\tilde{x}^1_L}{\mathrm{d}t_1} & \cdots & \dfrac{\mathrm{d}\tilde{x}^1_L}{\mathrm{d}t_{L-1}} \\ \vdots & & \vdots \\ \dfrac{\mathrm{d}\tilde{x}^I_1}{\mathrm{d}t_1} & \cdots & \dfrac{\mathrm{d}\tilde{x}^I_1}{\mathrm{d}t_{L-1}} \\ \vdots & & \vdots \\ \dfrac{\mathrm{d}\tilde{x}^I_L}{\mathrm{d}t_1} & \cdots & \dfrac{\mathrm{d}\tilde{x}^I_L}{\mathrm{d}t_{L-1}} \end{pmatrix}$$

has full column rank, $L - 1$.

Proof. If not, there would exist $\mathrm{d}t \neq 0$, such that

$$\mathrm{d}\tilde{x}^i = D_t\tilde{x}^i \mathrm{d}t = 0, \quad \text{for all } i.$$

If consumption does not change, and p_L is fixed at 1, and t_L is fixed at 0, it follows from the first-order conditions for each individual for commodity $l = L$ that μ^i does not change either: $d\mu^i = D_t\mu^i \mathrm{d}t = 0$. From the first coordinates of the function $F_{\vec{e}}$, it follows, then, that for any commodity, $l \neq L$, for which some individual i is a buyer, that

$\mathrm{d}p_l + \mathrm{d}t^i_\ell = D_t p_l \mathrm{d}t + \mathrm{d}t_l = 0$. Similarly, for any commodity, $l \neq L$, for which some individual j is a seller, it follows that $\mathrm{d}p_l = D_t p_l \mathrm{d}t = 0$. Since every commodity has both a buyer and a seller, $\mathrm{d}t_l = 0$ for all $l \neq L$, i.e. $\mathrm{d}t = 0$, a contradiction. □

In the next Corollary, we show that for any commodity $\ell \neq L$ and any agent i we could imagine externalities on i such that increasing the tax on good ℓ would lead to changes in consumption by agents $h \neq i$ that would help agent i, while changes in any other tax would generate no external effects on i's utility.

Corollary 2. *At a regular Walrasian equilibrium with full trade, for every individual, i, and every commodity $l \neq L$, there exist real numbers $\lambda^i_{h,k}$, for $h \neq i$, such that*

$$\sum_{h \neq i} \sum_k \lambda^i_{h,k} \frac{\partial \tilde{x}^h_k}{\partial t_l} = 1$$

$$\sum_{h \neq i} \sum_k \lambda^i_{h,k} \frac{\partial \tilde{x}^h_k}{\partial t_m} = 0, \quad m \neq l, L.$$

Proof. From the last corollary, the matrix $D_t \vec{\tilde{x}}$ has rank $L-1$. But $\sum_i D_t \tilde{x}^i = 0$. It follows that if all the rows corresponding to any individual were removed, the resulting sub-matrix M^{-i} would also have rank $L-1$.

The matrix M^{-i} has full column rank if and only if, for each column M^{-i}_ℓ, there is a vector, λ^i_l, such that $\lambda^i_l \cdot M^{-i}_l = 1$, while $\lambda^i_l \cdot M^{-i}_m = 0$ for every other column $m \neq l$. □

5. Separable externalities

Externalities are separable if the consumption of others, x^{-i}, does not affect the marginal utility of an individual's own consumption.

With separable externalities, the utility function of an individual is defined by

$$u^i(x^i, x^{-i}) = u^{*i}(x^i) + \sum_{h \neq i} \sum_l \lambda^i_{h,l} x^h_l,$$

where u^{*i} is the private utility function of the individual defined over his own consumption, x^i, while the vector $\lambda^i = (\ldots, \lambda^i_{h,l}, \ldots)$, $h \neq i$, is the vector of external effects on i.

If $\lambda^i_{h,l} > 0$, the consumption of commodity l by individual h has a positive external effect on individual i; if $\lambda^i_{h,l} < 0$, the external effect is negative; if $\lambda^i_{h,l} = 0$, there is no external effect.

The profile of external effects is $\vec{\lambda} = (\lambda^1, \ldots, \lambda^i, \ldots, \lambda^I)$.

An economy with separable externalities is described by the vector $(\vec{u}^*, \vec{e}, \vec{\lambda})$. The private utility functions $\vec{u}^*$ of individuals will be held fixed as in previous Sections. (We assume the utilities satisfy the same smoothness conditions 1–2 defined in Section 2). Economies are then indexed by $(\vec{e}, \vec{\lambda})$, the allocation of endowments and the profile of external effects, and the set of economies is $\mathbb{R}^{LI}_{++} \times \mathbb{R}^{L(I-1)I}$, an open set in Euclidean space of dimension $IL + IL(I-1)$.

With separable externalities, there is an economy without external effects associated unambiguously with an economy with externalities; it obtains by setting $\vec{\lambda} = 0$.

Competitive t-equilibria in the externalities economy $(\vec{u}^*, \vec{e}, \vec{\lambda})$ coincide with competitive t-equilibria for the associated economy $(\vec{u}^*, \vec{e})$ without the external effects because the separable externalities do not affect the choices of individuals. Hence the definition of regular economy and regular equilibrium need not change.

Lemma 3. *At a regular competitive equilibrium with full trade (but without commodity taxes), the infinitesimal change in the utility of each individual due to an infinitesimal change in taxes is smooth and*

$$\mathrm{d}u^i = D_t u^i \mathrm{d}t = (-\tilde{\mu}^i (\tilde{x}^i - e^i)' D_t \tilde{p} - \tilde{\mu}^i (\tilde{x}^i - e^i)'_+ + \tilde{\mu}^i D_t \tilde{\tau} + \lambda^i D_t \tilde{x}^{-i})\, \mathrm{d}t.$$

Proof. This is a simple extension of Roy's identity. The first term is the effect of the induced change in prices on wealth, multiplied by the marginal utility of revenue; the second and third terms are the direct effect of the change in taxes on wealth, multiplied by the marginal utility of wealth; the last term is the external effect caused by the induced change in consumption of individuals $h \neq i$. By the envelope theorem, the changes in utility caused via the reoptimization of own consumption, x^i, and the change in the marginal utility of revenue, μ^i, are of second order and can be dropped. □

5.1. Pareto improving taxes

When there are no externalities, it can be shown that after normalizing utilities properly, no tax package can raise the sum of utilities. We now show that with separable externalities, generically, every competitive equilibrium can be Pareto improved upon by commodity taxes and anonymous transfers. A key step in the proof is to show that for generic endowments and externalities, no matter what weights $\pi = (\pi_1, \ldots, \pi_I) \in S^{I-1}$ are used on the utilities, there will be some tax package that does increase the π-weighted sum of utilities.

Proposition 1. *In the class of smooth economies with separable externalities and $I < L$, generically, every competitive equilibrium can be Pareto improved by anonymous commodity taxes and transfers.*

Proof. We saw in Lemma 1 that for fixed smooth utilities $\vec{u}^*$, there is a generic set $\mathcal{E}^*$ of allocations of endowments such that every Walrasian economy with $\vec{e} \in \mathcal{E}^*$ is regular, and every Walrasian equilibrium of $\vec{e}$ has full trade (every individual trades every commodity). In fact the argument we gave after Debreu's theorem shows that the set

$$\mathcal{M} = \{(\vec{x}, \vec{\mu}, p, \vec{e}) : (\vec{x}, \vec{\mu}, p) \text{ is a Walrasian equilibrium for the economy } \vec{e} \in \mathcal{E}^*\}$$

is a smooth manifold of dimension *IL*.

Define the function

$$G : \mathcal{M} \times \mathbb{R}^{IL(I-1)} \times S^{I-1} \to \mathbb{R}^{L-1}$$

by

$$G(\vec{x}, \vec{\mu}, p, \vec{e}, \vec{\lambda}, \pi) = \pi D_t \vec{u} = \sum_i \pi^i D_t u^i,$$

where $D_t \vec{u} = (D_t u^1, \ldots, D_t u^i, \ldots, D_t u^I)$ is a matrix of dimensions $I \times (L-1)$, and $\pi = (\pi^1, \ldots, \pi^i, \ldots \pi^I)$ is an element of the sphere of dimension $I-1$. G denotes the change in the π-weighted sum of utilities that can be wrought with each tax, taking into account all the changes in consumption caused by the tax and the re-equilibration of prices.

According to Corollary 2 and Lemma 3, by appropriately perturbing λ, one can perturb each entry $\mathrm{d}u^h/\mathrm{d}t_k$, leaving $\mathrm{d}u^i/\mathrm{d}t_l = 0$, for all $i \neq h$and/or $l \neq k$. Since some $\pi^i \neq 0$, $D_\lambda G$ has full row rank $L-1$. Hence $G \pitchfork 0$.

It follows that $G^{-1}(0)$ is a manifold of dimension $IL + IL(I-1) + (I-1) - (L-1)$: if $I < L$, then this is less than $IL + IL(I-1)$.

Define the projection

$$G^{-1}(0) \rightarrow_{\text{proj}} \mathcal{E}^* \times \mathbb{R}^{IL(I-1)}$$

by

$$(\vec{x}, \vec{\mu}, p, \vec{e}, \vec{\lambda}) \rightarrow_{\text{proj}} (\vec{e}, \vec{\lambda}).$$

By Sard's theorem,[4] the set Λ of regular values of the projection has full measure in $\mathcal{E}^* \times \mathbb{R}^{IL(I-1)}$. Since $\vec{\lambda}$ does not affect equilibrium, we can argue exactly as in Debreu's theorem that the set Λ of regular values is open. In other words, Λ is generic. Since the domain of the projection has a lower dimension than the range, an economy, $(\vec{e}, \vec{\lambda})$ is a regular value of the projection if and only if it is not in the image of the projection, i.e. $(\vec{e}, \vec{\lambda}) \notin G^{-1}(0)$. Thus for any economy

[4] Abraham and Robin (1967, thm. 5.1).

in the generic set Λ, at every one of its Walrasian equilibria, there is no solution to the system of equations $\pi D_t\vec{u} = 0$; equivalently, the matrix $D_t\vec{u}$ has full row rank, I. Thus there is an infinitesimal tax package dt with $D_t\vec{u}\,\mathrm{d}t \gg 0$. It follows that every Walrasian equilibrium can be Pareto improved upon by some commodity tax package, dt (and the associated anonymous transfer of revenue). □

6. Example

Individuals are $i = 1, 2$, and commodities are $l = 1, 2, 3$.

The utility function of an individual is

$$u^i(x^i, x^{-i}) = u^{*i}(x^i) + \lambda^i_{-i,1}x_1^{-i} + \lambda^i_{-i,2}x_2^{-i} + \lambda^i_{-i,3}x_3^{-i},$$

where $\lambda^i = (\lambda^i_{-i,1}, \lambda^i_{-i,2}, \lambda^i_{-i,3})$ are the coefficients of external effects and

$$u^{*i}(x^i) = x_1^i - \frac{1}{2}\alpha_1^i(x_1^i)^2 + x_2^i - \frac{1}{2}\alpha_2^i(x_2^i)^2 + x_3^i, \quad 0 < \alpha_1^i, \alpha_2^i < 1$$

is the private utility function over own consumption.

Externalities are separable.

The endowments of individuals are

$$e^1 = (1, 0, e_3^1) \quad \text{and} \quad e^2 = (0, 1, e_3^2),$$

respectively, with the endowment in commodity $l = 3$ sufficiently large.

Prices of commodities are $p = (p_1, p_2, 1)$, and tax rates on commodities are $t = (t_1, t_2, 0)$; commodity $l = 3$ is numéraire and not subject to taxation.

Assuming external effects that are separable from the marginal utility of each individual's own consumption allows for competitive equilibrium prices and allocations, with or without taxes, that are independent of the coefficients of external effects.

The quasi-linearity of the utility functions in the numéraire commodity eliminates income effects and facilitates computations.

Competitive equilibrium prices are easily calculated to be

$$p_1(t) = 1 - \frac{1}{(1/\alpha_1^1) + (1/\alpha_1^2)}\left(1 + \frac{1}{\alpha_1^2}t_1\right),$$
$$p_2(t) = 1 - \frac{1}{(1/\alpha_2^1) + (1/\alpha_2^2)}\left(1 + \frac{1}{\alpha_2^1}t_2\right),$$

and equilibrium allocations are

$$x_1^1(t) = \frac{1}{\alpha_1^1 + \alpha_1^2}(\alpha_1^2 + t_1),$$
$$x_2^1(t) = \frac{1}{\alpha_2^1 + \alpha_2^2}(\alpha_2^2 - t_2),$$
$$x_3^1(t) = e_3^1 + p_1(t)x_1^2(t) - (p_2(t) + t_2)x_2^1(t) + \tfrac{1}{2}(t_1x_1^2(t) + t_2x_2^1(t)),$$
$$x_1^2(t) = \frac{1}{\alpha_1^1 + \alpha_1^2}(\alpha_1^1 - t_1),$$
$$x_2^2(t) = \frac{1}{\alpha_2^1 + \alpha_2^2}(\alpha_2^1 + t_2),$$
$$x_3^2(t) = e_3^2 + p_2(t)x_2^1(t) - (p_1(t) + t_1)x_1^2(t) + \tfrac{1}{2}(t_1x_1^2(t) + t_2x_2^1(t)).$$

The derivative of utilities with respect to taxes at the point of zero taxes is obtained first by computing the effect on consumption and prices:

$$\frac{\mathrm{d}p_1}{\mathrm{d}t_1} = -\frac{1}{(1/\alpha_1^1) + (1/\alpha_1^2)}\frac{1}{\alpha_1^2}$$

$$\frac{\mathrm{d}x_2^1}{\mathrm{d}t_1} = \frac{\mathrm{d}x_2^2}{\mathrm{d}t_1} = 0$$

$$\frac{\mathrm{d}x_1^2}{\mathrm{d}t_1} = -\frac{\mathrm{d}x_1^1}{\mathrm{d}t_1} = -\frac{1}{a_1^1 + a_1^2}$$

$$e_1^1 - x_1^1 = \frac{a_1^1}{a_1^1 + a_1^2}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t_1} = \frac{1}{2}x_1^2 = \frac{1}{2}\frac{a_1^1}{a_1^1 + a_1^2}$$

$$\frac{\mathrm{d}x_3^1}{\mathrm{d}t_1} = (e_1^1 - x_1^1)\frac{\mathrm{d}p_1}{\mathrm{d}t_1} + \frac{\mathrm{d}\tau}{\mathrm{d}t_1} = \frac{\alpha_1^1(\alpha_1^2 - \alpha_1^1)}{2(\alpha_1^1 + \alpha_1^2)^2}$$

$$\frac{\mathrm{d}x_3^2}{\mathrm{d}t_1} = -x_1^2\left(\frac{\mathrm{d}p_1}{\mathrm{d}t_1} + 1\right) + \frac{\mathrm{d}\tau}{\mathrm{d}t_1} = -\frac{\alpha_1^1(\alpha_1^2 - \alpha_1^1)}{2(\alpha_1^1 + \alpha_1^2)^2}$$

By the envelope theorem and the fact that the marginal utility of consumption is 1, the change in utility from consumption to agent 1 is just $\mathrm{d}x_3^1/\mathrm{d}t_1$. The effect on his total utility also includes the externalities from the changes in 2's consumption, namely

$$\frac{\mathrm{d}u_1^1}{\mathrm{d}t_1} = \frac{\mathrm{d}x_3^1}{\mathrm{d}t_1} + \frac{\mathrm{d}x_1^2}{\mathrm{d}t_1}\lambda_{2,1}^1 + \frac{\mathrm{d}x_2^2}{\mathrm{d}t_1}\lambda_{2,2}^1 + \frac{\mathrm{d}x_3^2}{\mathrm{d}t_1}\lambda_{2,3}^1 = \frac{\alpha_1^1(\alpha_1^2 - \alpha_1^1)}{2(\alpha_1^1 + \alpha_1^2)^2} + \frac{-1}{\alpha_1^1 + \alpha_1^2}\lambda_{2,1}^1 + 0 - \frac{\alpha_1^1(\alpha_1^2 - \alpha_1^1)}{2(\alpha_1^1 + \alpha_1^2)^2}\lambda_{2,3}^1$$

By symmetry the other welfare effects can be immediately deduced to be:

$$D_t\vec{u} = \begin{pmatrix} D_t u^1 \\ D_t u^2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u^1}{\partial t_1} & \dfrac{\partial u^1}{\partial t_2} \\ \dfrac{\partial u^2}{\partial t_1} & \dfrac{\partial u^2}{\partial t_2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\alpha_1^1(\alpha_1^2 - \alpha_1^1)}{2(\alpha_1^1 + \alpha_1^2)^2}(1 - \lambda_{2,3}^1) - \dfrac{1}{\alpha_1^1 + \alpha_1^2}\lambda_{2,1}^1 & \dfrac{\alpha_2^2(\alpha_2^2 - \alpha_2^1)}{2(\alpha_2^1 + \alpha_2^2)^2}(1 - \lambda_{2,3}^1) + \dfrac{1}{\alpha_2^1 + \alpha_2^2}\lambda_{2,2}^1 \\ \dfrac{\alpha_1^1(\alpha_1^1 - \alpha_1^2)}{2(\alpha_1^1 + \alpha_1^2)^2}(1 - \lambda_{1,3}^2) + \dfrac{1}{\alpha_1^1 + \alpha_1^2}\lambda_{1,1}^2 & \dfrac{\alpha_2^2(\alpha_2^1 - \alpha_2^2)}{2(\alpha_2^1 + \alpha_2^2)^2}(1 - \lambda_{1,3}^2) - \dfrac{1}{\alpha_2^1 + \alpha_2^2}\lambda_{1,2}^2 \end{pmatrix}.$$

In the absence of external effects, for $\vec{\lambda} = 0$, the matrix $D_t\vec{u}$ is singular, in particular $D_t u^1 + D_t u^2 = 0$, and Pareto improving taxes do not exist; the Walrasian equilibrium is Pareto optimal.

Pareto improving taxes $\mathrm{d}t$, that solve $(D_t\vec{u})\,\mathrm{d}t \gg 0$, exist if the matrix $D_t\vec{u}$ has full row rank.

Since the coefficients $1/(\alpha_\ell^i + \alpha_\ell^{-i})$ are all non-zero (in fact, all positive), it is clear that by perturbing the variables $\lambda_{-i,l}^i$ one can perturb the matrix in any way desired. Thus the matrix $D_t\vec{u}$ is invertible for almost all choices of the externality variables $\lambda_{-i,l}^i$. Since a regular economy $\vec{e}$ has a finite number of equilibria, almost all choices $\lambda_{-i,l}^i$ will simultaneously make all the equilibrium welfare effect matrices $D_t\vec{u}$ invertible.

7. Notation

- $'$ is the transpose.
- A vector $a = (\ldots, a_k, \ldots)$, is non-negative: $a \geq 0$, if $a_k \geq 0$, for every k; it is positive: $a > 0$, if $a_k \geq 0$, for every k, with strict inequality, $a_k > 0$, for some; it is strictly positive: $a \gg 0$, if $a_k > 0$, for every k; analogously, the vector

a is non-positive: $a \leq 0$, negative: $a < 0$, and strictly negative: $a \ll 0$. For vectors a and b, $a \geq b$ if $(a-b) \geq 0$, $a > b$ if $(a-b) > 0$, and $a \gg b$ if $(a-b) \gg 0$; analogously, $(a-b) \leq 0$, $(a-b) < 0$, and $(a-b) \ll 0$.

- For a, a real number, $a_+ = \max\{a, 0\}$, and $a_- = -\min\{a, 0\}$; for $a = (\ldots, a_k, \ldots)$, vector, $a_+ = (\ldots, a_{k+}, \ldots)$, and $a_- = (\ldots, a_{k-}, \ldots)$.
- $\mathbb{R}^k$ is Euclidean space of dimension k —for simplicity, one writes $\mathbb{R}$ for $\mathbb{R}^1$; the non-negative orthant is $\mathbb{R}^k_+$, and its interior, the strictly positive orthant, is $\mathbb{R}^k_{++}$.
- $\mathcal{S}^k$ is the sphere of dimension k; its intersection with the non-negative orthant is $\mathcal{S}^k_+$, and with the strictly positive orthant $\mathcal{S}^k_{++}$.
- [] is the span of a collection of vectors or the column span of a matrix.
- If g is a function of $(\ldots, y_k, \ldots)$, then "g_{y_k}" is the function defined by $g_{y_k}(\ldots, y_{k-1}, y_{k+1}, \ldots) = g(\ldots, y_{k-1}, y_k, y_{k+1}, \ldots)$.
- $D_y g$ is the gradient of a function, g, with respect to y —for simplicity, one writes Dg; if $y = (\ldots, y_k, \ldots)'$, then

$$D_y g = (\ldots, \frac{\partial g}{\partial y_k}, \ldots);$$

  if $g = (\ldots, g_l, \ldots)'$, then

$$D_y g = (\ldots, D_{y_k} g, \ldots) = \begin{pmatrix} & \vdots & \\ \cdots & \dfrac{\partial g_l}{\partial y_k} & \cdots \\ & \vdots & \end{pmatrix}.$$

  is the Jacobean matrix.
- $D^2_y g$ is the Hessian matrix of second derivatives of a function, g, with respect to y —for simplicity, one writes $D^2 g$; if $y = (\ldots, y_k, \ldots)'$, then

$$D^2_y g = \begin{pmatrix} & \vdots & \\ \cdots & \dfrac{\partial^2 g}{\partial y_{k_1} y_{k_2}} & \cdots \\ & \vdots & \end{pmatrix}.$$

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