Economic Theory (2007) 32: 461–475
DOI 10.1007/s00199-006-0134-2

RESEARCH ARTICLE

**S. Demichelis · H. M. Polemarchakis**

# The determinacy of equilibrium in economies of overlapping generations

Received: 8 August 2003 / Revised: 5 June 2006 / Published online: 21 July 2006


**Abstract** We study the problem of uniqueness of equilibrium paths in the overlapping generations model. We show that, despite local calculation based on counting equations and unknowns, the equilibrium path may be unique. We do this by constucting an example of an economy of overlapping generations with just one equilibrium up to time shift, beside the steady states. Time is either discrete or continuous; in either case, it extend into the infinite future and, possibly, the infinite past. There is one, non-storable commodity at each date. The economy is stationary; intertemporal preferences are logarithmic; the endowments and discount factors of individuals need not depend continuously on time. With continuous time, equilibrium paths of prices are smooth; this, even for endowments and discount factors of individuals that do not depend continuously on time. With discrete time, as the number of periods in the life-span of individuals increases, equilibrium paths converge to the continuous time solutions. If time extends infinitely into the infinite past as well as into the infinite future, in continuous time, all non-stationary equilibrium paths of prices are time-shifts of a single path; in addition, there are two stationary solutions; in discrete time, there is a one dimensional family of non-stationary solutions, up to time-shift; however, the indeterminacy vanishes as the number of periods in the life span of individuals tends to infinity. If, alternatively, time has a finite starting point, in discrete time the degree of indeterminacy increases with the life-span of individuals, and, in continuous time, it is infinite; however, these are families of exponentially decreasing oscillations which, although they may exhibit pseudo-chaotic behaviour for a while, as time tends to infinity they all get damped, and asymptotic behaviour is that of the economy that originates in the infinite past.

Stefano Lovo made interesting comments.

S. Demichelis (✉)
Dipartimento di Matematica, Universitá di Pavia, 27100 Pavia, Italy
E-mail: Sdm.golem@gmail.com

H. M. Polemarchakis
CORE, Université Catholique de Louvain, Louvain-la-Neuve, Belgium

**Keywords** Life-span · Continuous time · Overlapping generations · Determinacy

**JEL Classification Numbers** D50 · D90

## 1 Introduction

Economies of overlapping generations, introduced by Allais (1947) and Samuelson (1958), display equilibrium properties different from the equilibrium properties of economies over a finite horizon or with a finite number of individuals.[1] In particular, the determinacy of equilibrium which obtains for finite economies under standard assumptions, fails for economies of overlapping generations.

The failure of the determinacy of competitive allocations in economies of overlapping generations challenges the autonomy of a competitive market as a mechanism for the allocation of resources; while the divergence between the equilibrium properties of models of overlapping generations and models with finitely many individuals poses a modeling dilemma.

Ever since Samuelson's (1958) remark that "we can try to cut the gordian knot by our special assumption of stationariness" and Gale's (1973) exposition of the problem, it has been recognized that competitive equilibriun paths are indeterminate, and that the extent of indeterminacy depends on the number of commodities and the number of periods in the life-spans of individuals, as well as the presence of aggregate debt.

The degree of indeterminacy is the generically maximal dimension of an open set of distinct equilibrium allocations. It is not hard to find upper bounds for the dimension of the set of solutions by truncating the economy: the argument in Geanakoplos (1987) and Geanakoplos and Brown (1982, 1985) is simple and convincing: in an economy with two-period life-spans and $L$ commodities per period, the degree of indeterminacy is $2L-1$ if time extends infinitely into the past as well as into the future; it is $L-1$ if time has a finite starting point; in the latter case, the degree of indeterminacy increases to $L$ for competitive equilibria with debt.

The computation of the degree of indeterminacy follows by considering a finite truncation of the economy and counting the effective degrees of freedom, i.e. the excess of endogenous, equilibrating varibles, the prices of commodities, over the number of independent equilibrium conditions, the vanishing excess demand for commodities.

Though a conclusive demonstration is lacking, the results in the literature hint that the degree of indeterminacy increases with the number of commodities at each date or, equivalently, the length of the life-spans of individuals. This is particularly bothersome for policy analysis, where empirically relevant models involve life-spans equal to the number of years in the economic life of an individual.

However, arguments as the one mentioned above or similar ones based on local analysis near the steady state solutions, can give only local information relying as they do on the implicit function theorem (that is counting the number of variables minus the number of unknowns). In fact indeterminacy for a truncation of the economy is a first step but not a conclusive argument for indeterminacy over

[1] Geanakoplos and Polemarchakis (1991) survey the properties of competitive equilibria in economies of overlapping generations.

an infinite horizon, this because it may happen that the size of the set of solutions shrinks when the truncation time goes to infinity.

In this paper we analyse the global structure of solutions and prove that shrinking effectively happens. We exhibit an example where positivity constraints on consumption and prices do not change the dimension of the solution manifold for finite truncations but become effective when time extends from minus infinity to plus infinity. The conclusion is that there is only one solution left up to time translation.

In this light, increasing the life span does not introduce new indeterminacies, actually in the limit in which the number of periods goes to infinity it becomes easier to recognize all solution as time translation of one another. So the conclusion is that indeterminacy, here, is an artefact coming from truncation and discretization and disappears when time is taken as infinite and continuous.

More in detail our results are as follows.

In discrete time, there is a one parameter family of solutions, apart from the two steady states, the parameter space is identified in a natural way with a circle whose size goes to zero when the number of periods in the life-span of individuals tends to infinity.

In continuous time, there is a unique non-stationary equilibrium path going from one steady state to the other.

Equilibrium paths in discrete time converge to the equilibrium path in continuous time.

Our analysis gives further results whose interpretation is, in our opinion, of economic relevance. In continuous time the path of prices is smooth, even when endowments and preferences are irregular—they exhibit jumps or, more generally, they are a measurable function. This is an indication that the nondifferentiability that is observed in the trajectories of prices must come from uncertainty only and is not due to irregularities in the foundamentals of the economy.

In our example preferences are logarithmic so that the trajectories of prices satisfy difference or integral equations that are linear, this allows us to apply some tools from complex analysis to compute explicitly the solutions.

We should emphasize that, geometrically, the results should be very intuitive and point to generalizations in the case of non-logarithmic preferences: solutions in a truncated economy are associated with trajectories in a (according to the life-span finite or infinite dimensional) manifold with boundary, the boundary is given by the positivity constraints. When time goes to infinity in one direction or the other all trajectories, but one, hit the boundary. In this way uniqueness up to time translation is obtained. Hitting the boundary as time goes to plus or minus infinity splits trajectories into unstable and stable manifolds. This is good evidence that our methods, together with standard properties of hyperbolic and Anosov flows, may lead to local uniqueness of paths for open sets of utility functions. This line of research will be prosecuted in a further work.

## 2 The economy

Alternative specifications consider time discrete or continuous; it is instructive to consider both and contrast the results.

The economy is stationary: the distribution of the fundamentals does not vary with calendar time.

One commodity is available at each date; there is no storage or production.

## 2.1 Discrete time

Discrete time extends into the infinite future as well as the infinite past:

$$\ldots, -(t/n), \ldots, -(1/n), 0, (1/n), \ldots, (t/n), \ldots$$

alternatively, it has a finite starting point:

$$0, (1/n), \ldots, (t/n), \ldots.$$

The indexation of time by $t/n$, where $t$ is an integer, allows for comparisons of equilibrium paths for different values of $n$, the reciprocal of the length of a period of time.

At each date, an individual $\tau = t + 1$ is born, and his life span extends until date $t + n$. At date $s = \tau - 1, \ldots \tau + n - 2$, the consumption of the individual is $x_{\tau,s}$, and his endowment is $e_{s-\tau+2}$, a non-negative amount; across the life-span of the individual,

$$\sum_{s=\tau-1}^{\tau+n-2} e_{s-\tau+2} = 1,$$

and the intertemporal utility function is

$$u = \sum_{s=\tau-1}^{\tau+n-2} k_{s-\tau+2} \ln(x_{\tau,s}),$$

where $k_{s-\tau+2}$ is the discount factor, a non-negative coefficient, with

$$\sum_{s=\tau-1}^{\tau+n-2} k_{s-\tau+2} = 1.$$

It is convenient to define

$$e_s = k_s = 0, \quad s \neq 1, \ldots, n.$$

The price of the commodity at date $t$ is $p_t$.

The wealth of individual $\tau$ is

$$w_\tau = \sum_{s=\tau-1}^{\tau+n-2} p_s e_{s-\tau+2}.$$

Since the individual maximizes his intertemporal utility subject to the intertemporal budget constraint

$$\sum_{s=\tau-1}^{\tau+n-2} p_s x_{\tau,s-\tau+2} \leq w_\tau,$$

his consumption demand is

$$x_{\tau,s} = \frac{k_{s-\tau+2}}{p_s} w_\tau.$$

Individuals active at date $t$ are $\tau = t+2-n, \ldots, t+1$; since, at equilibrium, aggregate demand must coincide with the aggregate endowment,

$$\overline{x}_t = \sum_{\tau=t+2-n}^{t+1} e_{\tau,t} = 1,$$

it is necessary and sufficient that

$$\sum_{\tau=t+2-n}^{t+1} \frac{k_{t-\tau+2}}{p_t} w_\tau = 1.$$

Substituting for $w_\tau$ yields the equilibrium equation for prices

$$p_t = \sum_{\tau=t+2-n}^{t+1} \sum_{s=\tau-1}^{\tau+n-2} k_{t-\tau+2} p_s e_{s-\tau+2}.$$

Changing the order of summation,

$$p_t = \sum_{s=t-n+1}^{t+n-1} p_s \sum_{\tau=t+2-n}^{\min\{t+1,s+1\}} k_{t-\tau+2} e_{s-\tau+2},$$

and, by a sequence of changes of variables, of $s+t$ for $s$ and of $r$ for $t+\tau-2$,

$$p_t = \sum_{s=1-n}^{n-1} p_{s+t} \sum_{r=-n}^{r=\min\{-1,s-1\}} k_r e_{s+r}.$$

Equilibrium paths of prices satisfy

$$p_t = \sum_{s=1-n}^{n-1} c_s p_{t+s},$$

where the coefficients

$$c_s = \sum_{r=-n}^{r=\min\{-1,s-1\}} k_r e_{s+r} \geq 0$$

satisfy

$$\sum_{s=1-n}^{n-1} c_s = 1.$$

To avoid degeneracies, none of the $c_s$ vanishes; for this it is enough to require that all endowments and all discount factors are strictly positive. The equation is a linear difference equation, whose positive solutions characterize equilibrium paths.

**Proposition 1** *In the generic case, if time extends into the infinite past, equilibrium prices satisfy*

$$p_t = a + bq^t.$$

*If time has a finite starting point, equilibrium prices satisfy*

$$p_t = a + bq^t + \sum_{k=1}^{m} q_k^t (a_k \cos(\omega_k t) + b_k \sin(\omega_k t)) + \sum_{i=1}^{s} d_i q_i^t (-1)^t,$$

*where* $0 < q$, *while* $0 < q_k, q_i < \min\{1, q\}$ *and* $2m + s = n - 2$, *so that the* $a_k, b_k, d_i$ *are* $2n - 2$ *chosen so that the price is positive.*

The dimension of the set of equilibrium paths is equal to $n - 1$, when time has a finite starting point; however, positivity constraints reduce the dimension to 1, as opposed to $2n - 3$ when time extends to the infinite past. Furthermore, in a sense that can be made exact in continuous time, the single dimension of indeterminacy amounts to translations in time.

## 2.2 Continuous time

Time extends continuously into the infinite future as well as the infinite past:

$$-\infty < t < +\infty.$$

Alternatively, it has a finite starting point:

$$0 \leq t < +\infty.$$

At each date, an individual, $\tau = t$ is born, and his life span extends to $\tau + 1$. At date $\tau \leq s \leq \tau + 1$, the consumption of the individual is $x_\tau(s)$, and his endowment is $e(s - \tau)$, where $e$ is an integrable,[2] positive function on the interval [0, 1]; over the life-span of the individual,

$$\int_{\tau}^{\tau+1} e(s - \tau)\mathrm{d}s = 1,$$

and the intertemporal utility function of the individual is

$$u = \int_{\tau}^{\tau+1} k(s - \tau) \ln x_\tau(s)\mathrm{d}s,$$

where $k(s - \tau)$ is the discount factor $k$ is an integrable, positive function on the interval [0, 1], which is absolutely continuous except for finitely many jumps and such that

$$\int_{\tau}^{\tau+1} k(s - \tau)\mathrm{d}s = 1.$$

[2] $L^1$.

It is convenient to define

$$e(s) = k(s) = 0, \quad s \notin [0, 1].$$

The price of the commodity at date $t$ is $p(t)$, where $p$ is a locally integrable, positive function, such that $p(t) \leq Ke^{N|t|}$, for some large, real numbers $K$ and $N$; indeed, it will follow that, at equilibrium, the price function is smooth.[3]

The wealth of individual $\tau$ is

$$w_\tau = \int_\tau^{\tau+1} e(s-\tau)p(s)\mathrm{d}s.$$

Since he maximizes his intertemporal utility subject to the intertemporal budget constraint

$$\int_\tau^{\tau+1} x_\tau(s-\tau)p(s)\mathrm{d}s \leq w_\tau,$$

his demand is

$$x_\tau(s) = \frac{k(s-t)}{p(s)} w_\tau.$$

At date $t$, individuals $t-1 \leq \tau \leq t$ are active; aggregate demand is

$$x(t) = \int_{t-1}^{t} x_\tau(t)\mathrm{d}\tau.$$

Since, at equilibrium, aggregate demand must coincide with the aggregate endowment,

$$\overline{x}(t) = \int_{t-1}^{t} e(t-\tau)\mathrm{d}\tau = 1,$$

it is necessary and sufficient that

$$\int_{t-1}^{t} \frac{k(t-\tau)}{p(t)} w_\tau \mathrm{d}\tau = 1$$

or

$$p(t) = \int_{t-1}^{t} k(t-\tau) \left( \int_\tau^{\tau+1} e(s-\tau)p(s)\mathrm{d}s \right) \mathrm{d}\tau.$$

[3] $C^\infty$.

The function $g$, defined by

$$g(t-s) = \int_{-\infty}^{+\infty} k(t-\tau)e(s-\tau)\mathrm{d}\tau,$$

is continuous function, with generalized derivative, $g'$, which is integrable; moreover $g(t-s) = 0$ if $|t-s| > 1$. The equilibrium equation is

$$p(t) = \int_{t-1}^{t+1} g(t-s)p(s)\mathrm{d}s,$$

which writes in compact form as the convolution equation

$$p = g * p.$$

If p is locally integrable and satisfies the equilibrium equation, then it must be a smooth function; and the argument extends to non-logarithmic preferences.

The solutions of the equilibrium equation can be found by using a combination of Laplace and Fourier transforms and their form depends on the zeroes of the function

$$F(\lambda) = \left( \int_{-\infty}^{+\infty} e^{\lambda t} g(t)\mathrm{d}t \right) - 1.$$

As a real function, it is a convex function of $\lambda$, and, in the generic case, it has two distinct real zeroes: 0 and another, $\lambda_0$; in non-generic cases the two coincide at 0.

**Proposition 2** *In the generic case, if time extends into the infinite past, equilibrium prices satisfy*

$$p(t) = a + be^{\lambda_0 t}.$$

*If time has a finite starting point, equilibrium prices satisfy*

$$p(t) = \sum_{\lambda_k \leq \min\{0,\lambda\}} e^{\lambda_k t}(a_k \cos \mu_k t + b_k \sin \mu_k t).$$

*where the $a_k$ and $b_k$ are real numbers such that the sum converges and is positive, and $\lambda_k + i\mu_k$ are the complex zeroes of the function*

$$F(\lambda + i\mu) = \int_{-\infty}^{+\infty} e^{(\lambda + i\mu)t} g(t)\mathrm{d}t - 1.$$

To fix ideas, $\lambda_0 > 0$, and $q = e^{\lambda_0} > 1$.

There are three types of solutions: the non-stationary ones, for non-zero $a$ and $b$, starts from a value close to a constant for $t$ near $-\infty$ and increases exponentially with rate q when $t$ goes to $+\infty$; and two stationary ones, when one of the values of $a$ or b is equal to zero, that give the two steady states.

Since what matters are relative prices, one studies the inflation rate

$$r_{a,b}(t) = \frac{1}{p(t)}\frac{\mathrm{d}p(t)}{\mathrm{d}t} = \frac{\lambda_0 e^{\lambda_0 t}}{\frac{a}{b} + e^{\lambda_0 t}};$$

along non-stationary solutions, the rate of inflation converge to 0, the steady state of constant prices as $t \to -\infty$, and to $\lambda_0$, the autarkic stationary state as $t \to +\infty$.

**Proposition 3** *The non-stationary solutions are all equivalent up to time shifting.*

If one is interested in solutions for positive time only, it follows from the form of the equations that one has convergence of $r_{a,b}(t)$ to the largest of 0 and $\lambda_0$, with exponentially decreasing fluctuations.

In order to compare discrete time with $n$ periods and continuous time, one assumes that the coefficients $e_j$ and $k_j$ are approximations of the corresponding functions in continuous time in the sense that

$$e_j = \int_{(j-1)/n}^{j/n} e(t)\mathrm{d}t, \quad \text{and} \quad k_j = \int_{(j-1)/n}^{j/n} k(t)\mathrm{d}t,$$

and writes the solution of the discretized equation as

$$p_n(t) = 1 + bq_n^{[nt]},$$

where $[nt]$ is the integer part of a real number; this is a piece-wise constant function corresponding to prices of the n-period model. In the same way one defines the rate of inflation $r_{a,b,n}(t)$.

**Proposition 4** *As $n \to \infty$, $r_{a,b,n}(t)$ converges uniformly to $r_{a,b}(t)$.*

This proposition also justifies the growth condition on prices assumed for continuous time.

## 3 Proofs

The proofs use similar ideas in continuous and discrete time; the latter is technically simpler. In both cases, one first looks for real valued solutions and then imposes the positivity constraint. To solve the equation in continuous time, the essential tool is the Laplace tranform, which transforms convolutions to products, easier to analize. For solutions that are bounded, or bounded by a polynomial, the standard tools of Fourier analysis yield that, generically, the only solutions are the constants. For solutions that are exponential, the argument requires a refinement, as follows: first one splits the solution, $p(t)$ into two parts, $p_-(t)$ and $p_+(t)$, that have support in a half line and satisfy the convolution equation up to an error term, $f(t)$, with compact support. To these, one applies the Laplace transform and uses estimates to prove that $p(t)$ is a sum of exponentials, exact solutions of the convolution equation and a bounded error term, r(t), that has to be a solution as well. To the bounded term, the standard argument in Fourier analysis applies to show that it is necessarily constant.

*Proof of Proposition 1* Complex solutions of the equilibrium equation depend on the roots of[4]

$$P(z) = \sum_{r=1-n}^{n-1} c_r z^r = 1,$$

with $z = \rho e^{i\omega}$ a complex number. For simplicity we assume they are all simple zeroes — the case of multiple zeroes can be easily worked out by the reader. If the set of roots is $\{z_1, \ldots, z_{2n-2}\}$, complex solutions are

$$p_t = \sum_{k=1}^{2n-2} a_k z_k^t,$$

for some $a_1, \ldots, a_k, \ldots, a_{n-2}$.

In order to study the positivity constraint on prices, one considers the function $P(\rho e^{i\omega})$.

**Lemma 1** *As a function on the positive real line, $P(\rho)$ is a convex function that tends to infinity as $\rho$ tends to infinity or zero, and is such that*

1. $P(1)1$, *and there is another root,* $P(q) = 1$, *where,* $q \neq 1$ *in the generic case, while the two roots coincide in the non-generic case.*
2. *if $\rho$ is strictly between 1 and $q$, then $P(\rho) < 1$.*
3. *Other than 1 and $q$, there are no other real or complex roots of $P(\rho e^{i\omega}) = 1$ with $1 \leq \rho \leq q$; or $1 \geq \rho \geq q$; in particular, there are no solutions with $\|z\| = 1$.*
4. *There are $2(n-2)$ other roots, all complex or negative, of which $n-2$ with $\rho < 1$ and $n-2$ with $\rho > q$.*

*Proof* (1) The function $P(\rho)$ is convex on the positive real line, since it is the sum of convex functions, integer powers of $z$, with positive coefficients. Since $\sum_{r=1-n}^{n-1} c_r = 1$, $P(1) = 1$. If the first derivative $P'(1) < 0(>0)$, there is another root, $P(q) = 1$, with $q > 1(<1)$; in the non-generic case of $P' = 0$, there is a double root at 1.

(2) It follows immediately, since $P$ is convex.

(3) Since $P(z) < 1$, for $z$ real, with $1 < z < q$, and all the coefficients in $P(z)$ are non-negative, for $1 < \|z\| < q$, $\|P(z)\| < P(\|z\|) < 1$, while, if $\|z\| = 1$ or $\|z\| = q$, and $z$ is not real and positive, still $\|P(z)\| < 1$, with strict inequality, since, in the sum that defines $P(z)$, not all terms can be positive.

(4) The family of polynomials $P(z)$ as the coefficients $c_r$ vary is parametrized by the open simplex $\Delta^{2n-2} = \{c_r > 0 : \sum_{r=1-n}^{n-1} c_r = 1\}$, and, thus it is connected. Given any two polynomials in it, $P_0(z)$ and $P_1(z)$, the path joining them

[4] Goldberg (1958).

is $[P_s(z) : 0 \leq s \leq 1]$. From (3), no root of $P_s(z)$, other than 1 or $q$ can cross the annulus $1 \leq \|z\| \leq q$ (or $q \leq \|z\| \leq 1$); it follows that the number of solutions, with multiplicity, inside the disc $\|z\| \leq \min(1, q)$ is constant along the path, as is the number outside it. If $P_0(z) = P_1(z^{-1})$, both polynomials are in the family, and every solution of $P_0(z) = 1$ with $\|z\| < 1$ yields a solution of $P_1(z^{-1}) = 1$ with $\|z\| > 1$. Since the two polynomials can be joined by a path along which the number of solutions inside or outside the disc is constant, it follows that they are both equal to $n - 2$. □

The formula for the equilibrium price, $p_t$, implies that, as $t$ tends to infinity, the argument of $p_t$ is determined by the argument of the term $a_k z_k^t$, where $z_k$ is the root with largest absolute value. It follows that this root must be real and equal to the largest of 1 and $q$; and that all other $z_k$ with $a_k \neq 0$ must lie inside the unit circle.

This gives the formula for the equilibrium paths when time has a finite starting point: $t \geq 0$. If positivity of prices is required for $t \leq 0$, and, in particular as $t \to -\infty$, this eliminates also the $z_k$ with $\|z_k\| < 1$, and the only solutions are $p_t = a + bq^t$.

*Proof of Proposition 2* In continuous time the equilibrium equation is

$$p(t) = \int_{t-1}^{t+1} g(t-s)p(s)\mathrm{d}s$$

or, in compact form,

$$p = g * p,$$

where $g$ is a non-negative, absolutely continuous function with support on the closed interval $[-1, +1]$, such that $\int_{-1}^{+1} g(t)\mathrm{d}t = 1$.

One seeks the solution in the space of locally integrable functions that satisfy the growth condition $p(t) < Ke^{N|t|}$ almost everywhere, for some $K$ and $N$.

**Lemma 2** *The function $p(t)$ is smooth.*

*Proof* The derivative, $g'$, in the sense of distributions, of $g$ is an integrable function. If $p'$ is the distributional derivative of $p$, then $p' = g' * p$, and since the convolution of two integrable functions is an integrable function, $p'$ is integrable and $p$ is absolutely continuous; in this manner, all derivatives of $p$ are continuous. □

Next one characterizes complex solutions of the equilibrium equation.

**Lemma 3** *If $p$ solves the equilibrium equation and $g$ is generic in the sense to be specified below, then $p$ is of the form*

$$p(t) = \sum_{-N \leq \lambda_k \leq N} c_k e^{(\lambda_k + i\mu_k)t}.$$

*Proof* If $\phi(t)$ is a smooth function such that $\phi(t) \geq 0$, $\phi(t) = 0$, for $t < 0$, and $\phi(t) = 1$, for $t > 1$, and if $p_+(t) = \phi(t)p(t)$, and $p_-(t) = (1 - \phi(t))p(t)$, then it follows from the equilibrium equation that $p_+$ satisfies the equation

$$p_+(t) - p_+(t) * g(t) = f_+(t),$$

where $f_+(t)$ is a smooth function with support in $[-1, 2]$.

If

$$\overline{p}_+(z) = \int_{-\infty}^{+\infty} e^{-zt} p_+(t)\mathrm{d}t$$

is the Laplace transform[5] of $p(t)$, where $z = \lambda + i\mu$ is a complex number, then $\overline{p}_+(z)$ exists and is an analytical function on the semi-plane $\lambda \geq N$; similarly, one defines $\overline{f}_+(z)$ and $\overline{g}(z)$, which are defined and analytic on the whole plane, because $f$ and $g$ have compact support and so the integral converges for every $z$. □

**Lemma 4** *For all $l$ and $K$, there exist $C_{l,K}$, such that*

$$|\overline{f}_+(\lambda + i\mu)| < \frac{C_{l,K}}{|\mu|^l}, \quad |\mu| \to \infty$$

*is an estimate of the decay of $f$, while*

$$|\overline{g}(\lambda + i\mu)| \leq \frac{o_K(\mu)}{|\mu|}, \quad |\mu| \to \infty.$$

*is an estimate the decay of $g$.*

Here, $C_{l,K}$ is a constant that does not depend on $\lambda$, as long as $\lambda \in [-K, K]$; $o_K(\mu)$ is a function with the same properties that tends to zero as $|\mu|$ tends to infinity.

*Proof* This is standard, given that $f_+(t)$ is smooth and $g$ is absolutely continuous.[6]

In the case of $g$, integration by parts yields

$$|\overline{g}(\lambda + i\mu)| = \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g(t)\mathrm{d}t \right| = \frac{1}{|\lambda + i\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)\mathrm{d}t \right|.$$

The fact that $g'$is an integrable function, with support $[-1, +1]$ yields the bound

$$\frac{1}{|\lambda + i\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)\mathrm{d}t \right| \leq \frac{e^{\lambda}}{|\lambda + i\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)\mathrm{d}t \right|$$
$$\leq \frac{e^k}{|\mu|} \left| \int_{-\infty}^{+\infty} e^{-(\lambda+i\mu)t} g'(t)\mathrm{d}t \right|;$$

[5] Smith (1966).
[6] Rudin (1977).

the integral is the Fourier transform of $g'(t)$ and the Lebesgue theorem[7] yields that the Fourier transform of an integrable function is continuous and tends to 0 at infinity.

A similar argument, iterating the integration by parts $l$ times yields the result for $f$. □

**Corollary 1** *For any* $K > 0$, *the function* $1 - \overline{g}(\lambda + i\mu)$ *has finitely many zeroes in the strip* $|\lambda| \leq K$, *for any* $K$; *it has at most two real zeroes,* $\lambda = 0$ *and* $\lambda = \lambda_0$, *possibly coinciding in the non-generic case; there are no other complex zeroes in the closed strip* $\{\lambda + i\mu : 0 \leq \lambda \leq \lambda_0$ *or* $\lambda_0 \leq \lambda \leq 0\}$.

*Proof* The assertion about finitely many zeroes follows immediately from the decay of $\overline{g}$. For the rest, one notes that $\overline{g}(\lambda)$ is convex on the real line, since, by differentiation with respect to $\lambda$ twice under the integral sign,

$$\frac{d^2}{d\lambda^2}\overline{g}(\lambda) = \int_{-\infty}^{+\infty} t^2 e^{-\lambda t} g(t)dt > 0,$$

and tends to $\infty$ as $|\lambda| \to \infty$. Moreover, one has $\overline{g}(0) = 1$, and, thus, the result about the real zeroes of $\overline{g}$ follows. That there are no complex zeroes in the closed strip follows from the strict inequality

$$|\overline{g}(\lambda + i\mu)| = \left| \int_{-\infty}^{+\infty} e^{-(\lambda + i\mu)t} g(t)dt \right| < \int_{-\infty}^{+\infty} |e^{-\lambda t}dt| = \overline{g}(\lambda),$$

which holds when $\mu \neq 0$, since $e^{-\lambda t} g(t)$ is always positive, while the convexity of $\overline{g}(\lambda)$ implies that

$$\overline{g}(\lambda) \leq 1, \quad 0 \leq \lambda \leq \lambda_0.$$

From now on, one assumes that these zeroes are simple; this is true in the generic case, and it simplifies some of the arguments — the extension to the general case is left to the reader.

A standard property of the Laplace transform says that, if $p_+$ satisfies the auxiliary convolution equation, then

$$\overline{p}_+(z) - \overline{g}(z) * \overline{p}_+(z) = \overline{f}_+(z).$$

so that

$$\overline{p}_+(z) = \frac{\overline{f}_+(z)}{1 - \overline{g}(z)}.$$

This means that $\overline{p}_+(z)$ extends as a meromorphic function to the whole complex plane, whose poles are contained in the zeroes of $1 - \overline{g}(z)$; moreover the decay estimates and corollary 1 imply that $\bar{p}_+(\lambda + i\mu))$ has only finitely many poles in the strip $|\lambda| \leq K$, for any $K$, and that $|\overline{p}_+(\lambda + i\mu)| \leq (C/|\mu|^k)$, as $|\mu|$ tends

[7] Rudin(1977).

to infinity. By the assumption on growth, $\overline{p}_+(\lambda + i\mu)$ has no poles in the plane $\lambda \geq N$.

According to the Laplace inversion formula,[8] one can recover $p_+(t)$ as the integral

$$p_+(t) = \int_{-\infty}^{+\infty} e^{(\lambda+i\mu)t}\overline{p}_+(\lambda + i\mu)\mathrm{d}\mu,$$

for any $\lambda \geq N$.

If one denotes the last integral $I(\lambda, t)$, the decay estimates together with Cauchy formula imply that $I(\lambda, t) = I(\lambda', t)$ if there are no poles of $\overline{p}_+(t)$ in the strip between $\lambda$ and $\lambda'$, $\lambda' \leq \lambda$. On the other side, if $\{z_1; \ldots; z_k\}$ is the finite set of poles of $\overline{p}_+(t)$ in this strip, one has

$$I(\lambda, t) = I(\lambda', t) + \sum_k c_k e^{i\mu_k} e^{z_k t},$$

where $c_k e^{i\mu_k}$ are the residues of $\overline{p}_+(z)$ at $z_k$. Moreover, up to a constant, $|I(\lambda', t)| \leq e^{\lambda' t}$ for $t \to +\infty$, by the decay estimates.

It follows that

$$p_+(t) = \left( \sum_{0 \leq \lambda_k \leq N} c_k e^{(\lambda_k + i\mu_k)t} \right) + r_+(t),$$

where $r_+(t)$ is a bounded function for $t$ going to plus infinity by the discussion above and is bounded for $t$ going to minus infinity because $p_+(t)$ is zero in this case and the sum is bounded.

One proves an analogous inequality for $p_-(t)$ and putting the two together one gets

$$p(t) = \sum_{-N \leq \lambda_k \leq N} c_k e^{(\lambda_k + i\mu_k)t} + r(t),$$

where $r(t)$ is a bounded function on the real line.

Now both $p(t)$ and the expression in the sum are solutions of the convolution equation, and it follows that the bouded function $r(t)$ is as well; an easy argument, using the Fourier transform, shows that the only bounded solutions of the convolution equation, in the generic situation of simple zeros of $1 - \overline{g}(\lambda + i\mu)$, are the constants. □

*Proof of proposition 3* If $s$ is such that $e^{\lambda_0 s} = a/b$,

$$r_{a,b}(t+s) = \frac{\lambda_0 (b/a) e^{\lambda_0 (t+s)}}{1 + \frac{b}{a} e^{\lambda_0 (t+s)}} = \frac{\lambda_0 e^{-\lambda_0 s} e^{\lambda_0 (t+s)}}{1 + e^{-\lambda_0 s} e^{\lambda_0 (t+s)}} = \frac{e^{\lambda_0 t}}{1 + e^{\lambda_0 t}} = r_{1,1}(t).$$

□

[8] Smith (1966).

*Proof of Proposition 4* One considers only the generic case $q \neq 1$.

Given $n$, the price is $p_n(t) = a + bq_n^{[nt]}$.

If $e_j$, $k_j$ and $c_j$ are related to $e(t)$, $k(t)$ and $g(t)$, as before, then $q_n^n \to q$, as $n \to \infty$, where $q$ is the solution in continuous time:

In fact by the proof of proposition 1, $q_n$ is the real root of the equation

$$P_n(z) = \sum_{r=1-n}^{n-1} c_r z^r = 1,$$

different from 1, while $\lambda_0$ is the root of

$$F(\lambda) = \int_{-1}^{1} e^{-\lambda t} g(t) \mathrm{d}t = 1,$$

with the same properties. Setting $z = e^{-\lambda/n}$, one sees that $P_n(e^{-\lambda/n})$ is the Riemann sum of the integral that defines $f(\lambda)$. It follows easily that the sequence of functions $(P_n(e^{\lambda/n}) : n = 2, \ldots)$ of $\lambda$ converges uniformly with its derivatives on compact subsets to $F(\lambda)$; this, in turn, implies that, if $P_n(e^{\lambda_n/n}) = 1$, then $\lambda_n \to \lambda_0$. Since $q_n = e^{\lambda_n/n}$ and $q = e^{\lambda_0}$, it follows that $q_n^n \to q$. □

## References

Allais, M.A.: Economie et Intérêt. Imprimérie Nationale (1947)

Gale, D.: Pure exchange equilibrium of dynamic economic models. J Econ Theory **5**, 12–36 (1973)

Geanakoplos, J.D.: Overlapping generations models of general equilibrium. In: Eatwell, J., Milgate, M., Newman, P. (eds.) The new Palgrave: a dictionary of economics. Macmillan, New York 767–779 (1987)

Geanakoplos, J.D., Brown, D.: Understanding overlapping generations economies as lack of market clearing at infinity. mimeo (1982)

Geanakoplos, J.D., Brown, D.: Comparative statics and local indeterminacy in overlapping generations economies: an application of the multiplicative ergodic theorem. mimeo (1985)

Geanakoplos, J.D., Polemarchakis, H.M.: Overlapping generations. In: Hildenbrand, W., Sonnenschein, H. (eds.) Handbook of mathematical economics, vol. IV. North Holland (1991)

Goldberg, S.: Introduction to difference equations. J Wiley, London (1958)

Rudin, W.: Functional analysis. McGraw Hill, New York (1977)

Samuelson, P.A.: An exact consumption-loan model of interest with or without the social contrivance of money. Jo Polit Econ **66**, 467–482 (1958)

Smith, M.G.: Laplace transform theory. van Nostrand (1966)