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Monetary policy and dynamic efficiency in economies of overlapping generations

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In an economy of overlapping generations, money, distinct from debt, provides liquidity and is dominated as a store of value. Nominal rates of interest that are low, but do not vanish, eliminate equilibrium allocations far from Pareto optimal allocations.

Key words nominal rate of interest, monetary equilibrium, intertemporal optimality

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1 Introduction

In economies of overlapping generations (Gale 1973; Samuelson 1958), competitive equilibrium allocations need not be Pareto optimal. This failure of efficiency occurs under perfectly competitive markets and it is caused by a failure of valuation for consumption paths that extend over an infinite horizon or of transversality conditions. Indeed, as individuals operate only over short horizons, long-period social costs and benefits might not be reflected correctly by prices. Productive reals assets, with streams of dividends that extend over an infinite horizon, when traded in asset markets, guarantee that equilibrium prices provide consistent intertemporal valuation and restore the optimality of competitive allocations (Wilson 1981).

In this short note, we present an argument for positive nominal interest. In fact, we argue that low, but not vanishing, nominal rates of interest shield the economy from intertemporal suboptimality at the cost of some static inefficiency. Unlike other arguments for a positive nominal interest, our argument does not appeal to nominal rigidities, asymmetric information or any imperfection or incompleteness of financial markets.

We modify a canonical economy of overlapping generations by introducing cash-in-advance constraints on trades in commodities. Money, distinguished from other stores of

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values, serves as a means for the exchange of commodities and it is demanded for transaction purposes. A monetary authority pegs the nominal interest rate and accommodates the demand for balances by issuing (or withdrawing) nominal short-term bonds. An initial public liability, consisting of inherited money balances and bonds, is outstanding. Apart from the structure of overlapping generations, this construction is common in the recent published literature on monetary policy and, in particular, on the Fiscal Theory of the Price Level (e.g. Woodford 1994).

At equilibrium, the market value of initial debt is at least equal to the present value of seignorage. Seignorage corresponds to the intertemporal value of net transactions, which is, therefore, finite. A gains to trade condition (within a generation) ensures that, even at intergenerational autarky, commodities are non-negligibly traded over the entire infinite horizon; that is, provided that the nominal interest is small enough, as in Dubey and Geanakoplos (2005). As net transactions are finitely valued, so is the aggregate endowment at equilibrium.

As long as the nominal rate of interest is arbitrarily low, but bounded away from zero, the static inefficiency associated with non-vanishing nominal rates remains but is essentially negligible; more importantly, with the stream of seignorage bounded away from zero, the bank substitutes for the long-lived productive assets that guarantee intertemporal optimality.

To simplify matters, we consider here a canonical stationary economy of overlapping generations, whereas we develop a more general argument in a companion paper (Bloise and Polemarchakis 2005).

2 Fundamentals

Periods of trade are $\mathcal{T} = \{0, \ldots, t, \ldots\}$. In every period of trade, a finite set of physical commodities $\mathcal{L} = \{\ldots, l, \ldots,\}$ is traded. Each generation consists of a finite set $\mathcal{I} = \{\ldots, i, \ldots\}$ of individuals and, apart from the initial generation, it is active over a horizon of two periods.

The economy is stationary: the preferences and endowments of individuals do not vary across generations. A consumption plan of an individual i in $\mathcal{I}$ is denoted by (x^{i1}, x^{i2}) in $\mathbb{R}^{\mathcal{L}}_{+} \times \mathbb{R}^{\mathcal{L}}_{+}$, where 1 and 2 refer to the first and to the second period in the economic life of an individual. Individual i in $\mathcal{I}$ has endowment of commodities (e^{i1}, e^{i2}) in $\mathbb{R}^{\mathcal{L}}_{+} \times \mathbb{R}^{\mathcal{L}}_{+}$ and intertemporal preferences represented by a utility function $u^i : \mathbb{R}^{\mathcal{L}}_{+} \times \mathbb{R}^{\mathcal{L}}_{+} \to \mathbb{R}$ that satisfies canonical conditions of continuity, strict quasi-concavity and strict monotonicity. For the initial generation, preferences are represented by a utility function $u^i_0 : \mathbb{R}^{\mathcal{L}}_{+} \to \mathbb{R}$.

An allocation $(x^{i1}_t, x^{i2}_t)_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ is a collection of consumption plans.[1] It is feasible whenever, for every period of trade t in $\mathcal{T}$,

$$\sum_{i\in\mathcal{I}} x^{i1}_t + \sum_{i\in\mathcal{I}} x^{i2}_t \leq \sum_{i\in\mathcal{I}} e^{i1} + \sum_{i\in\mathcal{I}} e^{i2}.$$

[1] Notice that (x^{i1}_t, x^{i2}_{t+1}) is the consumption plan of the individual i in $\mathcal{I}$ who is active over periods of trade $\{t, t+1\}$.

3 Monetary equilibrium

3.1 Money as a medium of exchange

Money, as commonly introduced in overlapping generations economies, is only a store of value generally understood as debt. In contrast to this tradition, here we introduce money that serves as a medium of exchange; it is distinct from debt, and it is held at equilibrium even though it is dominated by other assets in terms of returns. Money balances allow for transactions in commodities through liquidity (or cash-in-advance, or Clower) constraints. A monetary authority pegs nominal interest and accommodates the demand for balances by issuing bonds. Hence, importantly, and differently from more canonical representations of monetary equilibria, the supply of money balances is not modified by means of direct transfers to individuals.

3.2 Budget and liquiditity constraints

We here describe budget and liquidity constraints that restrict the market participation of individuals. In every period of trade, p_t in $\mathbb{R}_+^{\mathcal{L}}$ represents the vector of prices of physical commodities. Available assets are money balances and short-term nominal bonds, the latter yielding a constant nominal rate of interest r and not restricted in short sales. To avoid arbitrage opportunities, nominal interest is assumed to be non-negative. It simplifies notation to introduce discount factors, which are defined by $a_0 = 1$ and

$$a_{t+1} = \left(\frac{1}{1+r}\right)a_t = \left(\frac{1}{1+r}\right)^{t+1} a_0.$$

For heuristic purposes, assume that each period of trade is divided into two sub-periods (say, the beginning-of-period and the end-of-period). An individual enters a period of trade with an amount of nominal wealth, depending on previous transactions in financial instruments and commodities, and invests such a wealth in money balances and one-period bonds, at the prevailing nominal interest; the beginning-of-period holding of balances is modified by transactions in commodities, at market prices for commodities, so as to determine an end-of-period amount of balances; the beginning-of-period nominal wealth in the following period of trade consists of the yields of one-period bonds and the previous end-of-period amount of balances. Within a period of trade, money balances circulates across individuals. This description of trades, which admittedly embodies rigidities, is canonical in the published literature on cash-in-advance constraints. The sequence of constraints for a typical individual is hereafter presented.

An individual, in the first period of his economic life, faces the budget constraint

$$m_t^{i1} + \left(\frac{1}{1+r}\right)b_t^{i1} \leq 0, \tag{1}$$

where $m_t^{i1} \geq 0$ denotes holding of balances and b_t^{i1} denotes holding of one-period bonds. The budget constraint restricts trades in financial instruments only. Balances are used for transactions in commodities, according to the liquidity constraint[2]

$$p_t \cdot \left(x_t^{i1} - e^{i1}\right)^+ \leq m_t^{i1}, \tag{2}$$

which yield a quantity of balances

$$0 \leq m_t^{i1} - p_t \cdot \left(x_t^{i1} - e^{i1}\right) = n_t^{i1}. \tag{3}$$

The available wealth in the second period of economic activity is, consistently, given by

$$w_{t+1}^i = n_t^{i1} + b_t^{i1}. \tag{4}$$

An individual, in the second period of his economic life, faces the budget constraint

$$m_{t+1}^{i2} + \left(\frac{1}{1+r}\right) b_{t+1}^{i2} \leq w_{t+1}^i, \tag{5}$$

where, again, $m_{t+1}^{i2} \geq 0$ denotes holding of balances and b_{t+1}^{i2} denotes holding of one-period bonds. Transactions in commodities, which are restricted by the liquidity constraint

$$p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^+ \leq m_{t+1}^{i2}, \tag{6}$$

yield a quantity of balances

$$0 \leq m_{t+1}^{i2} - p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right) = n_{t+1}^{i2}. \tag{7}$$

A final no debt restriction imposes

$$0 \leq n_{t+1}^{i2} + b_{t+1}^{i2}. \tag{8}$$

Therefore, debts are settled at the end of the economic life of an individual.[3]

As far as consumption opportunities are concerned, the sequence of restrictions (1)–(8) is equivalent to a single intertemporal budget constraint of the form

$$\left(\frac{r}{1+r}\right) a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right)^- + \left(\frac{r}{1+r}\right) a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^- + a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right) + a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right) \leq 0. \tag{9}$$

This single intertemporal budget constraint (9) could be equivalently written as

$$\left(\frac{1}{1+r}\right) w_{t+1}^i + \left(\frac{r}{1+r}\right) p_t \cdot \left(x_t^{i1} - e^{i1}\right)^- + p_t \cdot \left(x_t^{i1} - e^{i1}\right) \leq 0, \tag{10}$$

$$\left(\frac{r}{1+r}\right) p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^- + p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right) \leq w_{t+1}^i. \tag{11}$$

[2] For a vector x in $\mathbb{R}^{\mathcal{L}}$, x^+ and x^- denote, respectively, its positive and its negative part, so that $x = x^+ - x^-$.

[3] Clearly, if individuals were active over longer horizons than two periods only, the distinction between money and the other available stores of value would be more relevant. This more general framework would not modify our assertions concerning efficiency.

The formulation (10)–(11) makes it evident that w_{t+1}^i represents the amount of wealth that is transferred from the first into the second period of economic life of the individual. The first two terms of budget constraint (9) (as well as the second term of constraint (10) and the first term of constraint (11)) correspond to the effects of operative cash-in-advance constraints.

Notice that, assuming that constraints (10)–(11) are satisfied with the equality, that cash-in-advance constraints hold with the equality (if $r > 0$) and that markets for commodities clear, sequential Walras' Law imposes

$$\left(\frac{1}{1+r}\right)\sum_{i\in\mathcal{I}} w_{t+1}^i + \left(\frac{r}{1+r}\right)\sum_{i\in\mathcal{I}} m_t^{i1} + \left(\frac{r}{1+r}\right)\sum_{i\in\mathcal{I}} m_t^{i2} = \sum_{i\in\mathcal{I}} w_t^i, \tag{12}$$

where

$$\sum_{i\in\mathcal{I}} m_t^{i1} + \sum_{i\in\mathcal{I}} m_t^{i2} = \sum_{i\in\mathcal{I}} n_t^{i1} + \sum_{i\in\mathcal{I}} n_t^{i2}.$$

The latter relation establishes that the beginning-of-period amount of balances coincides with the end-of-period amount of balances under market clearing for commodities. Recovering holdings of one-period bonds through the original constraints (1)–(8), condition (12) becomes

$$\sum_{i\in\mathcal{I}} m_t^{i1} + \sum_{i\in\mathcal{I}} m_t^{i2} + \left(\frac{1}{1+r}\right)\sum_{i\in\mathcal{I}} b_t^{i1} + \left(\frac{1}{1+r}\right)\sum_{i\in\mathcal{I}} b_t^{i2} = \sum_{i\in\mathcal{I}} w_t^i. \tag{13}$$

Restriction (13) requires that aggregate wealth be exhausted by holdings of balances and holdings of one-period bonds.

3.3 Monetary policy

A monetary authority pegs a constant nominal rate of interest, $r \geq 0$, by accommodating the demand for one-period nominal bonds. In addition, it supplies money balances subject to a sequential budget constraint. Importantly, it does not make direct transfers of balances to individuals.

In every period of trade, a monetary authority inherits public liabilities $w_t \geq 0$ from previous transactions. A sequential budget constraint imposes

$$w_t \leq m_t + \left(\frac{1}{1+r}\right) b_t, \tag{14}$$

where $m_t \geq 0$ and b_t represent, respectively, the supply of money balances and the supply of (in fact, the demand for, if negative) bonds at the prevailing market price. The sequential budget constraint establishes that outstanding liabilities (consisting of money balances and bonds) are to be covered by issuing money balances and bonds, at the pegged nominal rate of interest. Public liabilities evolve according to

$$w_{t+1} = m_t + b_t. \tag{15}$$

In addition, the monetary authority acts so as to prevent negative public liabilities, therefore imposing

$$0 \leq w_{t+1}. \tag{16}$$

Finally, an initial public liability $w_0 > 0$ is inherited from the unrepresented past and is to be taken as given.

Constraints (14)–(16) reduce to the single constraint

$$\left(\frac{1}{1+r}\right) w_{t+1} + \left(\frac{r}{1+r}\right) m_t = w_t, \tag{17}$$

which corresponds to (12) under market clearing. In addition, public liabilities $(w_t)_{t \in \mathcal{T}}$ cannot be negative, beginning with an initial predeterminate value $w_0 > 0$.

3.4 Equilibrium

Given a nominal rate of interest, $r \geq 0$, an equilibrium consists of an allocation, $(x_t^{i1}, x_t^{i2})_{(i,t) \in \mathcal{I} \times \mathcal{T}}$, prices, $(p_t)_{t \in \mathcal{T}}$, a supply of money balances, $(m_t)_{t \in \mathcal{T}}$, and a supply of public liabilities, $(w_t)_{t \in \mathcal{T}}$, that satisfy the following conditions.

(a) Markets clear; that is, for every t in $\mathcal{T}$,

$$\sum_{i \in \mathcal{I}} x_t^{i1} + \sum_{i \in \mathcal{I}} x_t^{i2} = \sum_{i \in \mathcal{I}} e^{i1} + \sum_{i \in \mathcal{I}} e^{i2}.$$

(b) Consumption plans are individually optimal subject to budget constraint; that is, for every (i, t) in $\mathcal{I} \times \mathcal{T}$, the consumption plan (x_t^{i1}, x_{t+1}^{i2}) maximizes utility u^i subject to

$$\left(\frac{r}{1+r}\right) a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right)^- + \left(\frac{r}{1+r}\right) a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^- + a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right) + a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right) \leq 0;$$

for every i in $\mathcal{I}$, the consumption plan x_0^{i2} maximizes utility u_0^i subject to

$$\left(\frac{r}{1+r}\right) p_0 \cdot \left(x_0^{i2} - e^{i2}\right)^- + p_0 \cdot \left(x_0^{i2} - e^{i2}\right) \leq w_0^i,$$

where w_0^i is the initial holding of public liabilities.

(c) Asset markets clear; that is, for every t in $\mathcal{T}$,

$$p_t \cdot \sum_{i \in \mathcal{I}} \left(x_t^{i1} - e^{i1}\right)^- + p_t \cdot \sum_{i \in \mathcal{I}} \left(x_t^{i2} - e^{i2}\right)^- \leq m_t,$$

with the equality if $r > 0$, and

$$0 \leq \left(\frac{r}{1+r}\right) p_t \cdot \sum_{i \in \mathcal{I}} \left(x_t^{i2} - e^{i2}\right)^- + p_t \cdot \sum_{i \in \mathcal{I}} \left(x_t^{i2} - e^{i2}\right) = w_t,$$

given the initial public liability satisfies $w_0 = \sum_{i \in \mathcal{I}} w_t^i > 0$.

Notice that, when nominal interest vanishes, $r = 0$, equilibrium coincides with the most canonical notion of an equilibrium with (positive) outside money.[4] Indeed, condition (c) implies $w_t = w_0 > 0$ for every t in $\mathcal{T}$. It is well-known that efficiency fails in such equilibria.

4 Dynamic efficiency

4.1 Finitely-valued net trades

At equilibrium, the sequential public budget constraint, as public liabilities cannot be negative, implies that

$$\left(\frac{r}{1+r}\right)\sum_{t\in\mathcal{T}} a_t m_t \le w_0.$$

This delivers an obvious property of equilibrium prices and trades.

Proposition 1 (finitely-valued aggregate net trades) *At equilibrium, if the nominal rate of interest is positive, $r > 0$,*

$$\sum_{t\in\mathcal{T}} a_t p_t \cdot \sum_{i\in\mathcal{I}} \left(x_t^{i1} - e^{i1}\right)^+ + \sum_{t\in\mathcal{T}} a_t p_t \cdot \sum_{i\in\mathcal{I}} \left(x_t^{i2} - e^{i2}\right)^+ \le \left(\frac{1+r}{r}\right) w_0.$$

This is the basic elementary observation that we shall elaborate on. Our purpose is to show that, under fairly general assumptions, finitely valued net trades imply a finitely valued aggregate endowment at equilibrium. In turn, in line with a well-established body of results in the published literature (Wilson 1981), the latter property allows us to draw strong conclusions of efficiency for a vanishing nominal interest.

4.2 Gains to trade

Heterogeneity within each generation guarantees gains to trade for small nominal rates of interest. To this purpose, assume that, for every nominal rate of interest, $r > 0$, small enough, the autarkic allocation within a generation is strongly Pareto dominated by an alternative allocation $(y^{i1}, y^{i2})_{i\in\mathcal{I}}$ (that is, $u^i(y^{i1}, y^{i2}) > u^i(e^{i1}, e^{i2})$ for every individual i in $\mathcal{I}$) satisfying

$$\left(\frac{r}{1+r}\right)\sum_{i\in\mathcal{I}}(y^{i1} - e^{i1})^- + \sum_{i\in\mathcal{I}} y^{i1} \le \sum_{i\in\mathcal{I}} e^{i1}, \tag{18}$$

$$\left(\frac{r}{1+r}\right)\sum_{i\in\mathcal{I}}(y^{i2} - e^{i2})^- + \sum_{i\in\mathcal{I}} y^{i2} \le \sum_{i\in\mathcal{I}} e^{i2}. \tag{19}$$

[4] The reference to outside money, which is traditional in the unpublished literature, might not be consistent with our terminology.

This is a mild requirement, which we shall refer to as the Hypothesis of Gains to Trade.

Lemma 1 (non-negligible trade) *Given a nominal rate of interest*, $r > 0$, *small enough, such that the Hypothesis of Gains to Trade is satisfied, there is* $\epsilon > 0$ *such that, at equilibrium, in every period of trade* t *in* $\mathcal{T}$,

$$\left\| \sum_{i \in \mathcal{I}} \left(x_t^{i1} - e^{i1} \right)^+ \right\|_\infty + \left\| \sum_{i \in \mathcal{I}} \left(x_{t+1}^{i2} - e^{i2} \right)^+ \right\|_\infty \geq \epsilon.$$

As a consequence, under the Hypothesis of Gains to Trade, non-negligible amounts of commodities are traded along any equilibrium provided that nominal interest is sufficiently small.

4.3 Finitely-valued endowment

At equilibrium, with a positive nominal rate of interest, $r > 0$, small enough, net trades are finitely valued (Proposition 1). In addition, non-negligible amounts of commodities are traded along periods of trade (Lemma 1). These two observations imply a finitely valued aggregate endowment at equilibrium.

Lemma 2 (finitely-valued aggregate endowment) *Given a nominal rate of interest*, $r > 0$, *small enough, such that the Hypothesis of Gains to Trade is satisfied, there is* $\eta > 0$ *such that, at equilibrium,*

$$\sum_{t \in \mathcal{T}} a_t p_t \cdot \sum_{i \in \mathcal{I}} e^{i1} + \sum_{t \in \mathcal{T}} a_t p_t \cdot \sum_{i \in \mathcal{I}} e^{i2} \leq \eta.$$

The above lemma shows that, whenever nominal interest is positive and small enough, the aggregate endowment is finitely valued at equilibrium.

4.4 Dynamic efficiency

A positive small nominal interest implies a finitely valued aggregate endowment at equilibrium, which, in turn, rules out dynamic inefficiency, although a sort of static inefficiency is introduced as a result of monetary transactions. This statement is made clear by the following proposition.

Proposition 2 (almost dynamic efficiency) *Given a nominal rate of interest*, $r > 0$, *small enough, such that the Hypothesis of Gains to Trade is satisfied, equilibrium allocation* $(x_t^{i1}, x_t^{i2})_{(i,t) \in \mathcal{I} \times \mathcal{T}}$ *is not Pareto dominated by an alternative allocation* $(y_t^{i1}, y_t^{i2})_{(i,t) \in \mathcal{I} \times \mathcal{T}}$ *satisfying, for every* t *in* $\mathcal{T}$,

$$\sum_{i \in \mathcal{I}} y_t^{i1} + \sum_{i \in \mathcal{I}} y_t^{i2} \leq \left(\frac{1}{1+r} \right) \sum_{i \in \mathcal{I}} e^{i1} + \left(\frac{1}{1+r} \right) \sum_{i \in \mathcal{I}} e^{i2}.$$

At equilibrium,

$$\left(\frac{r}{1+r}\right)\sum_{t\in\mathcal{T}} a_t p_t \cdot \sum_{i\in\mathcal{I}} \left(x_t^{i1} - e^{i1}\right)^- + \left(\frac{r}{1+r}\right)\sum_{t\in\mathcal{T}} a_t p_t \cdot \sum_{i\in\mathcal{I}} \left(x_t^{i2} - e^{i2}\right)^- = w_0$$

plays the role of an asset pricing condition. It basically asserts that the value of outstanding initial public liabilities is equal to the intertemporal value of money services. Under the Gains to Trade Hypothesis, money is productive, in the sense that money services represent a non-negligible share of aggregate resources. Therefore, as in the case of a productive enough real asset, dynamic inefficiency is ruled out.

5 Comments

A monetary policy that pegs a positive nominal interest sufficiently small and accommodates the demand for balances through trades in short-term nominal bonds delivers almost Pareto efficiency at equilibrium. The simple logic for this strong conclusion is easy to grasp. An initial outstanding public liability sets a bound on seignorage and, therefore, on the intertemporal value of net trades, as money serves as a means for transactions. If nominal interest is sufficiently small, commodities are traded in non-negligible amounts and, hence, the intertemporal value of aggregate endowment is finite, which restores the potential lack of transversality condition.

The mechanism sustaining efficiency applies as in the case of any real productive assets that are traded in the asset market. In real (developed) economies, however, seignorage represents a negligible share of national income, which would seem to suggest a minor relevance of nominal interest policies with respect to other productive assets (say, land). In this perspective, we annotate that the core of the argument relies on a non-negligible productivity of the real asset, with respect to the aggregate endowment, rather than on (and, as a matter of mere fact, independently of) its availability. Furthermore, we are not aware of any other instance in the published literature in which such a sufficient productivity is not introduced by means of an explicit assumption of the stream of dividends delivered by the traded asset.

A long tradition in the published literature, initiated by Samuelson's (1958) celebrated contribution, asserts that money, introduced as a store of value only, restores optimality in overlapping generations economies. This claim typically refers to stationary, or nearly stationary, equilibria. In fact, not all monetary equilibria are efficient, as well-understood hyperinflationary paths show. In contrast, introducing money as a transaction means, jointly with the specified monetary policy, delivers almost efficiency of all monetary equilibria. We explain below that the nature of the result is substantially different from Samuelson's.

Our argument for dynamic efficiency is only superficially related to the classical argument due to Samuleson. To see this, we consider the case of a single commodity ($\#\mathcal{L} = 1$) and a single individual ($\#\mathcal{I} = 1$) in each generation. In addition, we assume a specific distribution of the endowment: $e = e^1$ and $e^2 = 0$. This is the most favorable stage for the classical argument of Samuelson.

The budget constraint gives

$$\left(\frac{1}{1+r}\right) p_t \cdot (y_t - e) + \left(\frac{1}{1+r}\right) p_{t+1} \cdot z_{t+1} =$$
$$\left(\frac{r}{1+r}\right) p_t \cdot (y_t - e)^- + p_t \cdot (y_t - e) + \left(\frac{1}{1+r}\right) p_{t+1} \cdot z_{t+1} = 0,$$

where, to simplify, we use the notation $(y_t, z_{t+1}) = (x_t^1, x_{t+1}^2)$. (Notice that the peculiar distribution of the endowment implies that $(y_t - e)^- = -(y_t - e)$.) At an interior consumption plan, first-order condition imposes

$$\frac{u_1'(e - z_t, z_{t+1})}{p_t} = \frac{u_2'(e - z_t, z_{t+1})}{p_{t+1}}.$$

In addition, from sequential Walras' Law,

$$p_t \cdot z_t = w_t = w_0.$$

Hence, the only relevant equilibrium restriction is

$$u_1'(e - z_t, z_{t+1}) z_t = u_2'(e - z_t, z_{t+1}) z_{t+1}.$$

Monetary transactions do not alter the set of sustainable equilibrium trajectories in terms of allocations. In particular, dynamically inefficient equilibria remain. This happens as the Hypothesis of Gains to Trade fails.

6 Proofs

PROOF OF LEMMA 1: If not, letting $\epsilon > 0$ vanish, along an equilibrium,

$$\lim_{\epsilon \to 0} \left\| \sum_{i \in \mathcal{I}} \left(x_{t(\epsilon)}^{i1} - e^{i1}\right)^+ \right\|_\infty + \left\| \sum_{i \in \mathcal{I}} \left(x_{t(\epsilon)+1}^{i2} - e^{i2}\right)^+ \right\|_\infty = 0.$$

At no loss of generality, by individual rationality and strict monotonicity and strict convexity of preferences, it can be assumed that the sequence

$$\left\{ \left(x_{t(\epsilon)}^{i1}, x_{t(\epsilon)+1}^{i2}\right)_{i \in \mathcal{I}} \right\}_{\epsilon > 0}$$

converges to $(e^{i1}, e^{i2})_{i \in \mathcal{I}}$, which, by the Hypothesis of Gains to Trade, is weakly Pareto dominated by an alternative allocation $(y^{i1}, y^{i2})_{i \in \mathcal{I}}$ satisfying inequalities (18)–(19). Hence, by optimality of consumption plans at equilibrium and by continuity of preferences, for every

$\epsilon > 0$ small enough,

$$0 \geq \left(\frac{r}{1+r}\right) a_{t(\epsilon)} p_{t(\epsilon)} \cdot \sum_{i \in \mathcal{I}} (y^{i1} - e^{i1})^- + \left(\frac{r}{1+r}\right) a_{t(\epsilon)+1} p_{t(\epsilon)+1} \cdot \sum_{i \in \mathcal{I}} (y^{i2} - e^{i2})^- + a_{t(\epsilon)} p_{t(\epsilon)} \cdot \sum_{i \in \mathcal{I}} (y^{i1} - e^{i1}) + a_{t(\epsilon)+1} p_{t(\epsilon)+1} \cdot \sum_{i \in \mathcal{I}} (y^{i1} - e^{i1}) > 0,$$

where the first inequality follows from conditions (18)–(19). This is a contradiction, so proving the claim. □

PROOF OF LEMMA 2: Consider the alternative allocation defined, for some $0 < \lambda < 1$, by

$$z_t^{i1} = \lambda x_t^{i1} + \frac{1}{\#\mathcal{I}} \sum_{i \in \mathcal{I}} \left(x_t^{i1} - e^{i1}\right)^+,$$

$$z_{t+1}^{i2} = \lambda x_{t+1}^{i2} + \frac{1}{\#\mathcal{I}} \sum_{i \in \mathcal{I}} \left(x_{t+1}^{i2} - e^{i2}\right)^+.$$

(In addition, $z_0^{i2} = x_0^{i2}$.) Provided that $1 > \lambda > 0$ is large enough, invoking Lemma 1 and the stationarity of the economy, this alternative allocation Pareto dominates the equilibrium allocation. Hence,

$$\left(\frac{r}{1+r}\right) a_t p_t \cdot \left(z_t^{i1} - e^{i1}\right)^- + \left(\frac{r}{1+r}\right) a_{t+1} p_{t+1} \cdot \left(z_{t+1}^{i2} - e^{i2}\right)^- + a_t p_t \cdot \left(z_t^{i1} - e^{i1}\right) + a_{t+1} p_{t+1} \cdot \left(z_{t+1}^{i2} - e^{i2}\right) \geq 0.$$

Observing that $(z_t^{is} - e^{is})^- \leq (\lambda x_t^{is} - e^{is})^- \leq \lambda (x_t^{is} - e^{is})^- + (1 - \lambda) e^{is}$, it follows that

$$\lambda \left(\frac{r}{1+r}\right) a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right)^- + \lambda \left(\frac{r}{1+r}\right) a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^- + \lambda a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right)^+ + \lambda a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^+ - \lambda a_t p_t \cdot \left(x_t^{i1} - e^{i1}\right)^- - \lambda a_{t+1} p_{t+1} \cdot \left(x_{t+1}^{i2} - e^{i2}\right)^- + \frac{1}{\#\mathcal{I}} p_t \cdot \sum_{i \in \mathcal{I}} \left(x_t^{i1} - e^{i1}\right)^+ + \frac{1}{\#\mathcal{I}} p_{t+1} \cdot \sum_{i \in \mathcal{I}} \left(x_{t+1}^{i2} - e^{i2}\right)^+ \geq (1 - \lambda) \left(\frac{1}{1+r}\right) p_t \cdot e^{i1} + (1 - \lambda) \left(\frac{1}{1+r}\right) p_{t+1} \cdot e^{i2}.$$

This inequality, summing over individuals and periods of trade and observing that all terms on the right hand-side converge, proves the claim as $0 < \lambda < 1$. □

PROOF OF PROPOSITION 2: Observing that

$$\left(\frac{r}{1+r}\right) \sum_{i \in \mathcal{I}} \left(y_t^{is} - e^{is}\right)^- \leq \left(\frac{r}{1+r}\right) \sum_{i \in \mathcal{I}} e^{is},$$

the argument is exactly that for the proof of the First Welfare Theorem. □

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