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Economic Theory

Monetary equilibria over an infinite horizon*

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Summary. Money provides liquidity services through a cash-in-advance constraint. The exchange of commodities and assets extends over an infinite horizon under uncertainty and a sequentially complete asset market. Monetary policy sets the path of rates of interest and accommodates the demand for balances through open market operations or loans. A public authority, which, most pertinently, inherits a strictly positive public debt, raises revenue from taxes and seignorage, and it distributes possible budget surpluses to individuals through transfers. Competitive equilibria exist, under mild solvency conditions. But, for a fixed path of rates of interest, there is a non-trivial multiplicity of equilibrium paths of prices of commodities. Determinacy requires that, subject to no-arbitrage and in addition to rates of interest, the prices of state-contingent revenues be somehow determined.

Keywords and Phrases: Money, Equilibrium, Indeterminacy, Monetary policy, Fiscal policy.

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1 Introduction

We prove the existence of general competitive equilibria in a monetary economy under interest rate pegging; and we show that they display indeterminacy that we characterize.

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We modify the canonical Walrasian model by introducing money balances that facilitate transactions. The economy extends over an infinite horizon under uncertainty. Elementary securities make for a sequentially complete asset market. The transaction technology takes the simple form of the cash-in-advance constraint. Money balances are supplied by a central bank, which produces these at no cost and lends them at set short-term nominal rates of interest, meeting demand. The profits of the central bank, seignorage, accrue to a public authority. The primitives include initial nominal claims held by individuals that, in the aggregate, are the counterpart of an initial public debt of the public authority.

The public authority covers its expenditures, including initial debt, through public revenues, which consist of taxes and seignorage. Taxes are lump-sum commodity (*i.e.*, wealth) taxes collected from individuals at predetermined real levels. Importantly, the public authority distributes its eventual budget surpluses as lump-sum transfers to individuals, while no further instruments are available to correct eventual budget deficits.

Over a finite horizon, with no public debt and no taxes, Drèze and Polemarchakis [7] proved the existence of competitive equilibria, for arbitrarily set nominal rates of interest and price levels at all terminal nodes. Alternatively stated, the overall price level is arbitrary and the variability of short-term rates of inflation is unrestricted. This important indeterminacy feature reflects the intuitive property that the rate of interest at every date-event pins down expected inflation, but not inflation variability. We extend these results over an infinite horizon. The introduction of a public debt requires some qualifications.

Initial public debt must be met by public revenues from taxes and seignorage. If predetermined tax levels are positive, a suitable lower bound on the overall price level guarantees public solvency. Otherwise, the public authority must rely on seignorage, the yield of which is, roughly speaking, proportional to the overall price level. Public solvency then requires positive nominal rates of interest and positive transactions, and, hence, demand for money balances, reflecting gains to trade. The condition, due to Dubey and Geanakoplos [8, 9], requires that nominal rates of interest do not exceed gains to trade.

Our main results assert

(1) the existence of equilibria at all overall price levels above a lower bound, provided that conditions on gains to trade, if needed, are satisfied, and
(2) the indeterminacy of rates of inflation, up to no-arbitrage conditions.

Our work extends that of Woodford [18] to the case of heterogenous individuals and multiple commodities, which is in turn similar to cash-in-advance economies with a representative individual of Wilson [17] and Lucas and Stokey [13]. Differently from this literature, our more general formulation provides a framework that is suitable for the study of incomplete asset markets.

Recent literature (Woodford [18, 19] and Cochrane [6]) proposes a fiscal theory of price determination. This asserts that the price level is determined so as to balance the initial public debt and public revenues from taxes and seignorage. We here obtain indeterminacy of equilibria since, differently from that literature, we assume that the public authority can redistribute its eventual budget surpluses.

Similarly, Dubey and Geanakoplos [8,9] consider the case of a given initial stock of outside money and an additional injection of inside money, which allows for an unambiguous determination of the nominal rate of interest: seignorage revenue should absorb the outside money. The analogy with the fiscal theory is strong and, indeed, Dubey and Geanakoplos [8,9] obtain a determinate equilibrium as they do not allow for a distribution of public budget surpluses.[1]

2 A monetary economy

2.1. Time and the resolution of uncertainty are described by an event-tree, a countable set, $\mathcal{S}$, endowed with a (partial) order, $\succeq$. For every date-event, σ, an element of $\mathcal{S}$, t_σ denotes its date. The unique initial date-event is ϕ, with $t_\phi = 0$. For a given date-event, σ, $\sigma_+ = \{\tau \succ \sigma : t_\tau = t_\sigma + 1\}$ denotes the set of its immediate successors, a finite set; $\mathcal{S}_\sigma = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}$ the set of all its (weak) successors, a subtree; $\mathcal{S}^t = \{\sigma \in \mathcal{S} : 0 \leq t_\sigma \leq t\}$ the set all date-events up to date t; $\mathcal{S}_t = \{\sigma \in \mathcal{S} : t_\sigma = t\}$ the set all date-events at date t. Date-events are points in time. For accounting purposes, all values are defined as of the beginning of the time interval separating a date-event from its successors.

2.2. Markets are sequentially open for commodities, assets and balances, that are numéraire. At every date-event, there is a finite set $\mathcal{N} = \{\dots, \nu, \dots\}$ of tradable commodities, which are perfectly divisible and perishable.[2] The commodity space coincides with the space of all bounded elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. Prices of commodities p are a positive element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. These are spot nominal prices.

[1] Our work also contributes to a long debate on general equilibrium with incomplete financial markets. Following the demonstration of real indeterminacy with nominal assets in [2,5,11], some (in particular, Magill and Quinzii [14]) argued that the very notion of nominal assets is a misconception and only real assets should be considered as fruitful for economic analysis. The argument goes further: nominal assets are meaningful only if money is somehow introduced; if money were introduced, however, the real value of money would be determined, roughly speaking, by some sort of quantity theory equations, which would make real any asset initially described as nominal. In this perspective, our conclusions cast doubt of the cogency of the above argument.

[2] We shall use $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$ to denote the vector space of real-valued maps on $\mathcal{S}\times\mathcal{N}$. A typical element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$ is $x = (\dots, x_\sigma, \dots)$, where each $x_\sigma = (\dots, x_{\sigma\nu}, \dots)$ is a vector in $\mathbb{R}^{\mathcal{N}}$. An element x of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$ is positive if $x_{\sigma\nu} \geq 0$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$; it is strictly positive if $x_{\sigma\nu} > 0$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$; it is uniformly strictly positive if there is $\epsilon > 0$ such that $x_{\sigma\nu} \geq \epsilon$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$; it is bounded if there is $\epsilon > 0$ such that $\epsilon \geq x_{\sigma\nu} \geq -\epsilon$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$. For (x,z) in $\mathbb{R}^{\mathcal{S}\times\mathcal{N}} \times \mathbb{R}^{\mathcal{S}\times\mathcal{N}}$,

$$x_\sigma \cdot z_\sigma = \sum_{\nu\in\mathcal{N}} x_{\sigma\nu} z_{\sigma\nu} \text{ and } \|x_\sigma\| = \sum_{\nu\in\mathcal{N}} |x_{\sigma\nu}| .$$

For an element x of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$, its positive and negative parts are, respectively,

$$x^+ = (\dots, (\dots, \max\{x_{\sigma\nu}, 0\}, \dots), \dots) \text{ and } x^- = x^+ - x.$$

Similar definitions apply to the vector space $\mathbb{R}^{\mathcal{S}}$. Notice that a typical element of $\mathbb{R}^{\mathcal{S}}$ is $x = (\dots, x_\sigma, \dots)$, where each x_σ lies in $\mathbb{R}$. Notice that, throughout the paper, positive means greater than or equal to zero.

The asset market is sequentially complete. It simplifies, at no loss of generality, to assume that all securities that are traded at a date-event deliver a payoff only at the immediately succeeding date-events. A security plan v is an element of $\mathbb{R}^{\mathcal{S}}$. At a date-event, $(v_\tau : \tau \in \sigma_+)$ represents the deliveries (or payoffs) of the security plan at the immediately following date-events. State prices a are a strictly positive element of $\mathbb{R}^{\mathcal{S}}$, normalized so that $a_\phi = 1$. At a date-event σ, the market value of a security plan, with payoffs $(v_\tau : \tau \in \sigma_+)$ across its immediately succeeding date-events, is

$$\frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau v_\tau,$$

where $a_\sigma^{-1} a_\tau$ is the spot price at σ of an elementary (Arrow) security with payoff at $\tau \in \sigma_+$.

At given state prices, (one-period) nominal rates of interest r are a positive element of $\mathbb{R}^{\mathcal{S}}$. By the absence of arbitrage opportunities, they satisfy

$$\frac{1}{1 + r_\sigma} = \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau. \tag{1}$$

Nominal rates of interest are positive because, balances being storable, arbitrage opportunities would otherwise emerge. Given nominal rates of interest r, state prices a are consistent with those nominal rates of interest if they fulfill the above no arbitrage conditions (1) at all date-events.

2.3. There is a finite set $\mathcal{I} = \{\ldots, i, \ldots\}$ of individuals. An individual is described by preferences $\succeq^i$ over the consumption space, the space of all positive bounded elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$, and an endowment e^i of commodities, a positive bounded element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. The choice of the consumption space fits in a well-established tradition, beginning with Bewley [3]. Preferences and endowments of commodities are restricted by two common assumptions.

(P) Preferences. *Preferences $\succeq^i$ are continuous in the (relative) Mackey topology, convex and strictly monotone.*

(E) Endowments. *The endowment e^i is uniformly strictly positive.*

Continuity of preferences in the Mackey topology, introduced in Bewley [3], is a strong requirement. It encompasses, for example, preferences that are represented by an additively separable utility function,

$$\sum_{\sigma \in \mathcal{S}} \mu_\sigma \beta^{t_\sigma} u^i \left(x_\sigma^i\right),$$

where μ_σ is the probability of σ, $0 < \beta < 1$ is the discount factor, and $u^i : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$ is bounded, continuous, increasing and concave. In particular, the continuity of preferences in the Mackey topology implies that the individual is impatient: sufficiently distant modifications of consumption plans do not reverse the order of preference. Uniform impatience across individuals would be a stronger requirement. The much stronger assumption of a uniform rate of impatience across date-events, in some recent literature on incomplete asset markets over an infinite horizon, as in Hernández and Santos [12] and Magill and Quinzii [15], is not needed here.

Individuals are also characterized by shares $\left(\ldots, \zeta^i, \ldots\right) \geq 0$, with $\sum_i \zeta^i = 1$, and initial nominal wealths $\left(\ldots, \delta^i, \ldots\right) \geq 0$, with $\sum_i \delta^i = \delta$. The former will be used to distribute transfers across individuals. The latter represent given initial claims in terms of the numéraire.

Fundamentals are thus $\left(\ldots, \left(\succeq^i, e^i, \zeta^i, \delta^i\right), \ldots\right)$.

2.4. A public authority (or a government, or a central bank) sets nominal rates of interest, possibly contingent on date-events. Nominal rates of interest r are, thus, a positive element of $\mathbb{R}^{\mathcal{S}}$. The supply of balances m is a positive element of $\mathbb{R}^{\mathcal{S}}$. As nominal rates of interest are given, the supply of balances accommodates the demand. Although our analysis could be adapted to cope with all arbitrarily set nominal rates of interest, we impose a restriction that facilitates presentation.

(M) Nominal rates of interest. *Nominal rates of interest r are bounded.*

The public authority also sets a fiscal plan. Taxes $\left(\ldots, g^i, \ldots\right)$ are a positive bounded element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}\times\mathcal{I}}$. It is interpreted as establishing that, at a date-event, an individual is required to deliver $p_\sigma \cdot g^i_\sigma$ units of account to the public authority. In the aggregate, taxes are $g = \sum_i g^i$, a positive bounded element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. We restrict fiscal plans so as to avoid problems of solvency and, more importantly, to carry out a limit argument in the proof of existence of equilibria.

(F) Fiscal policy. *The net endowment $e^i - g^i$ is uniformly strictly positive.*

Notice that, in particular, it can be assumed that $g^i = \theta^i e^i$ for some $0 \leq \theta^i < 1$, with $\left(\ldots, \theta^i, \ldots\right) \geq 0$ being tax rates across individuals. Our commodity taxes then reduce to a wealth tax. An alternative would introduce taxes on net supplies $\left(x^i - e^i\right)^-$, corresponding to VAT or income taxes, with interesting implications for public revenue.

The public authority also issues transfers and trades in securities, subject to sequential budget constraints. Transfers h are a positive element of $\mathbb{R}^{\mathcal{S}}$. It is interpreted as positive deliveries of units of account from the public authority to individuals.

(T) Transfers. *Transfers h are distributed to individuals according to the given shares. Thus, $h^i = \zeta^i h$.*

Public liabilities w are an element of $\mathbb{R}^{\mathcal{S}}$, with a given initial value $w_\phi = \delta$. Notice that the initial public liability δ corresponds to initial nominal claims $\left(\ldots, \delta^i, \ldots\right)$ of individuals, that is, $\delta = \sum_i \delta^i$. To simplify the presentation, at no loss of realism, we assume that there is a strictly positive initial public liability.

(L) Initial public liability. *The initial public liability δ is strictly positive.*

Given nominal rates of interest and taxes, a public plan consists of transfers h and public liabilities w. A public plan (h, w) is subject, at every date-event, to a sequential public budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau w_\tau = w_\sigma + h_\sigma - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g_\sigma. \tag{2}$$

The interpretation of budget constraint (2) is the following: The public authority enters a date-event σ with a given public liability w_σ. This requires a delivery of units of account due to past investments in securities and balances. The public authority issues a transfer h_σ and supplies securities $(v_\tau : \tau \in \sigma_+)$, so as to balance its budget

$$m_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau v_\tau = w_\sigma + h_\sigma, \tag{3}$$

given that balances m_σ are supplied so as to accommodate the market demand. At every immediately succeeding date-event $\tau \in \sigma_+$, public liabilities amount to

$$w_\tau = v_\tau + m_\sigma - p_\sigma \cdot g_\sigma. \tag{4}$$

The convention that the value of taxes $p_\sigma \cdot g_\sigma$ is delivered by individuals at the end of date-event σ (or, better, at the immediately following date-event $\tau \in \sigma_+$) is only made to simplify notation. Substituting (4) into (3) and using (1), one obtains (2), with all terms evaluated at the beginning of date-event σ. In particular, v is a portfolio of securities, whereas w consolidates securities and balances. The latter is termed public liabilities since it is the amount of units of account that must be covered by issuing balances and supplying securities.

It is assumed that the public authority only trades is (one-period) safe bonds. Thus, an additional constraint requires that, at a date-event, $v_{\tau'} = v_{\tau''}$ for all (τ', τ'') in $\sigma_+ \times \sigma_+$. Equivalently,

$$w_{\tau'} = w_{\tau''} \text{ for all } (\tau', \tau'') \in \sigma_+ \times \sigma_+. \tag{5}$$

Indeed, under such a restriction, there is an element b of $\mathbb{R}^{\mathcal{S}}$ such that, at a date-event,

$$(w_\tau - m_\sigma : \tau \in \sigma_+) = b_\sigma (\ldots, 1, \ldots)$$

is interpreted as the stock of (one-period) safe bonds issued by the public authority.

2.5. An individual formulates a plan (x^i, m^i, w^i). The demand for balances m^i is a positive element of $\mathbb{R}^{\mathcal{S}}$. The wealth plan w^i is an element of $\mathbb{R}^{\mathcal{S}}$, with a given initial value $w^i_\phi = \delta^i$. At a date-event, such a plan is subject to a budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m^i_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau w^i_\tau + p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq w^i_\sigma + h^i_\sigma - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g^i, \tag{6}$$

a liquidity (or cash-in-advance) constraint,

$$p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- - m^i_\sigma \leq 0, \tag{7}$$

and a solvency constraint,

$$-\frac{1}{a_\sigma}\sum_{\tau\in\mathcal{S}_\sigma} a_\tau\left(h^i_\tau+\left(\frac{1}{1+r_\tau}\right)p_\tau\cdot\left(e^i_\tau-g^i_\tau\right)\right)\leq w^i_\sigma. \tag{8}$$

In the budget constraint (6), the nominal interest rate represents the opportunity cost of collecting proceeds of sales with a one-period lag. Solvency constraints serve to eliminate Ponzi schemes, as in Santos and Woodford [16]. They are equivalent to the restriction that an individual can incur any amount of nominal debt that can be repayed in finite time. The value of the endowment in commodities at a date-event is taxed at the nominal interest rate, since revenues from sales are carried over in the form of balances that do not earn interest. Equivalently, one could restrict wealth plans through some sort of transversality conditions.

By imposing sequential constraints (6)–(7), we faithfully reproduce the sequence of trades that is described by Woodford [18] in a cash-in-advance economy with a representative individual. At a date-event σ, an individual inherits some wealth w^i_σ from previous transactions and receives a transfer h^i_σ. The individual demands securities $\left(v^i_\tau:\tau\in\sigma_+\right)$ and balances n^i_σ so as to satisfy a budget constraint of the form

$$n^i_\sigma+\frac{1}{a_\sigma}\sum_{\tau\in\sigma_+}a_\tau v^i_\tau\leq w^i_\sigma+h^i_\sigma. \tag{9}$$

Notice that such a budget constraint only refers to transactions in financial instruments. The individual then trades in commodities, under a cash-in-advance constraint,

$$p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right)^+-n^i_\sigma\leq 0. \tag{10}$$

Transactions in commodities modify the holdings of balances, which now amount to

$$m^i_\sigma=n^i_\sigma-p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right)^++p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right)^-=n^i_\sigma-p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right). \tag{11}$$

At an immediately following date-event $\tau\in\sigma_+$, the inherited wealth, net of taxes, is

$$w^i_\tau=v^i_\tau+m^i_\sigma-p_\sigma\cdot g^i_\sigma. \tag{12}$$

Substituting (12) into (9) and using (1) and (11), one obtains (6). Also, because of (11), (7) is equivalent to (10). Finally, notice that, under market clearing for commodities, $\sum_i m^i=\sum_i n^i$, so that one can express the aggregate demand for balances in terms of final holdings after transactions in commodities, namely, m^i_σ.

3 Equilibrium

Given nominal rates of interest r and taxes $\left(\ldots, g^i, \ldots\right)$, an equilibrium consists of prices p, state prices a, consistent with set nominal rates of interest r, a collection of plans for individuals $\left(\ldots, \left(x^i, m^i, w^i\right), \ldots\right)$ and a public plan (h, w) such that the following conditions are satisfied:

(a) For every individual i, the plan $\left(x^i, m^i, w^i\right)$ is $\succeq^i$-maximal subject to sequential budget, cash-in-advance and solvency constraints (6)–(8).
(b) The public plan (h, w) satisfies sequential public budget constraint (2), at the supply of balances $m = \sum_i m^i$, with trades only in safe bonds (5).
(c) Markets clear for commodities, $\sum_i x^i = \sum_i e^i$, and assets, $\sum_i w^i = w$.

An equilibrium is said to be with no transfers if $h = 0$. It is said to be with unrestricted public portfolio if public liabilities do not consist of safe bonds only (that is, condition (5) is omitted).

Equilibria with no transfers are those studied by the literature on the fiscal theory of price determination. Allowing for positive transfers corresponds to the hypothesis that eventual public budget surpluses can be distributed to individuals, though no instruments are available to correct eventual public budget deficits. Apart from this natural assumption of public budget surplus disposability, our notion of an equilibrium is exactly that of Woodford [18] extended to a monetary economy with multiple commodities and heterogenous individuals.

4 Consolidation

Since the asset market is complete, the sequence of budget constraints faced by an individual reduces to a single constraint at the initial date-event.

Lemma 4.1. *At equilibrium, $\sum_{\sigma \in \mathcal{S}} a_\sigma \left\| p_\sigma \right\|$ is finite.*

At equilibrium, therefore, the intertemporal budget constraint of an individual,

$$\begin{aligned} &\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma m_\sigma^i + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i \right) \\ &\leq \delta^i + \zeta^i \sum_{\sigma \in \mathcal{S}} a_\sigma h_\sigma - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1 + r_\sigma} \right) a_\sigma p_\sigma \cdot g_\sigma^i, \end{aligned} \tag{13}$$

is well-defined.

Lemma 4.2. *At equilibrium, a consumption plan is attainable under sequential budget, cash-in-advance and solvency constraints (6)–(8) if and only if it is attainable under the unique intertemporal budget constraint (13) and sequential cash-in-advance constraints (7). Optimality of a consumption plan requires that,* *at every date-event,*

$$a_\sigma w^i_\sigma = \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{r_\tau}{1+r_\tau}\right)a_\tau m^i_\tau + \sum_{\tau\in\mathcal{S}_\sigma} a_\tau p_\tau\cdot\left(x^i_\tau - e^i_\tau\right)$$
$$+\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right)a_\tau p_\tau\cdot g^i_\tau - \zeta^i\sum_{\tau\in\mathcal{S}_\sigma} a_\tau h_\tau$$

and

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot\left(x^i_\sigma - e^i_\sigma\right)^- = \left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma m^i_\sigma.$$

The transversality condition takes the form

$$\lim_{t\to\infty}\sum_{\sigma\in\mathcal{S}_t} a_\sigma w^i_\sigma = 0.$$

As the cash-in-advance constraint is binding whenever the nominal rate of interest is strictly positive, the intertemporal budget constraint of an individual reduces to

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)\pi_\sigma\cdot\left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}}\pi_\sigma\cdot\left(x^i_\sigma - e^i_\sigma\right)$$
$$\leq \delta^i + \zeta^i\sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma - \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)\pi_\sigma\cdot g^i_\sigma,$$

where

$$\pi = \left(\ldots,\left(\ldots,\pi_{\sigma\nu},\ldots\right),\ldots\right) = \left(\ldots,\left(\ldots,a_\sigma p_{\sigma\nu},\ldots\right),\ldots\right)$$

are present value prices of commodities. Given present value prices of commodities, the optimal consumption plan of an individual is affected by state prices only through modifications of the outside (nominal) claims of such an individual,

$$\delta^i + \zeta^i\sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma.$$

This is a consequence of a complete asset market. Similarly, any proportional alteration of present value prices of commodities induces an adjustment in consumption plans because of a redistribution of the real value of outside nominal claims across individuals.

At equilibrium, aggregation across individuals yields

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot\sum_i\left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot g_\sigma = \delta + \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma,$$

which is the intertemporal public budget ‘constraint’. Such a constraint emerges only as an equilibrium restriction, as fiscal plans are considered to be given exogenously. In particular, it is a consequence of intertemporal Walras’ Law.

5 Gains to trade

At equilibrium, the overall public revenue must (weakly) exceed the initial public liability, that is,

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot\sum_i\left(x_\sigma^i-e_\sigma^i\right)^-+\sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot g_\sigma\geq\delta>0.$$

The initial public liability $\delta > 0$ is a given nominal magnitude. The sources of public revenue are seignorage and taxes, respectively, the first and the second term in the left-hand side of the above inequality. If taxes are strictly positive, any high enough overall price level suffices to balance an intertemporal public budget, possibly by the exhaustion of a budget surplus through transfers. Otherwise, the initial public liability is to be honored by means of seignorage. This requires strictly positive nominal rates of interest and, more relevantly, that individuals trade at set nominal rates of interest, so as to hold strictly positive quantities of balances in aggregate. Trade is guaranteed by an assumption on the gains to trade.

Following Dubey and Geanakoplos [8,9], we make explicit the existence of gains to trade as follows. An allocation $\left(\ldots,x^i,\ldots\right)$ is feasible if $\sum_i x^i \leq \sum_i e^i$. Allocation $\left(\ldots,z^i,\ldots\right)$ weakly Pareto dominates allocation $\left(\ldots,x^i,\ldots\right)$ if, for every individual i, $z^i \succ^i x^i$. We impose that, if a feasible allocation involved no trade at some date-event, then a Pareto improvement would be obtained through a reallocation of consumptions if this readjustment involved real costs of the same magnitude as liquidity costs.

(R) Public revenue. *Either (i) aggregate taxes $g = \sum g^i$ are strictly positive or (ii) nominal rates of interest r are strictly positive and every feasible allocation $\left(\ldots,x^i,\ldots\right)$, that coincides with the initial allocation $\left(\ldots,e^i,\ldots\right)$ at some date-event,[3] is weakly Pareto dominated by an allocation $\left(\ldots,z^i,\ldots\right)$ satisfying, at every date-event,*

$$\sum_i z_\sigma^i+\left(\frac{r_\sigma}{1+r_\sigma}\right)\sum_i\left(z_\sigma^i-x_\sigma^i\right)^-\leq\sum_i x_\sigma^i.$$

The hypothesis of gains to trade involves fundamentals and nominal rates of interest. Weak Pareto ordering only simplifies presentation, as the hypothesis could be equivalently stated in terms of the true Pareto ordering. Also, the requirement that trade is beneficial at all date-events is stronger than necessary, as it would suffice to require that only feasible allocations that involve no trade starting from some date-event would be weakly Pareto dominated. Though this is not necessary, it is clear that it could be assumed that the alternative allocation $\left(\ldots,z^i,\ldots\right)$ does not modify the allocation $\left(\ldots,x^i,\ldots\right)$ out of the single date-event that exhibits no trade. Finally, but importantly, postulating time additively separable preferences, one could easily provide robust examples of economies that exhibit gains to trade at all nominal rates of interest that do not exceed some upper bound.

[3] To be clearer, there is date-event σ such that $\left(\ldots,x_\sigma^i,\ldots\right)=\left(\ldots,e_\sigma^i,\ldots\right)$.

It is to be noticed that the above assumption guarantees that the intertemporal revenue from taxes and seignorage is strictly positive at every date-event. Such a condition is necessary for the existence of an equilibrium with no transfers. Indeed, starting from a date-event, the public revenue covers outstanding public liabilities, as required by

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) a_\tau p_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i\right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot g_\tau = a_\sigma w_\sigma .$$

As the public authority only trades in safe bonds and there are no transfers, the outstanding public liabilities at a date-event are determined at the immediately preceding date-event. If date-events τ' and τ'' are two immediate successors of date-event σ and the public revenue vanishes starting from τ', but not from τ'', there would not be any equilibrium. Under unrestricted public portfolio, existence of an equilibrium with no transfers only requires gains to trade at the initial date-event.

6 Existence and indeterminacy

An equilibrium exists under assumptions on fundamentals that are not more restrictive than those needed for the existence of Walrasian equilibrium, provided that public revenue is guaranteed (assumption (R)).

Proposition 6.1 (Existence). *Given nominal rates of interest r, there exists an equilibrium with no transfers.*

Equilibria with vanishing transfers are those considered under the fiscal theory of the price level. With no transfers, the overall price level is typically determinate, as the intertemporal public budget constraint requires that

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot \sum_i \left(x_\sigma^i - e_\sigma^i\right)^- + \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma = \delta,$$

and increases in the overall price level generate a public surplus that cannot be exhausted through transfers. In addition, at a date-event, the intertemporal public budget also imposes

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) a_\tau p_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i\right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot g_\tau = a_\sigma w_\sigma .$$

Since the public authority only trades in safe bonds, public liabilities w_σ are determined at the preceding date-event. Dividing by a_σ, the constraint at σ is of the same form as the constraint at the initial date-event, so that the overall price level at σ is typically determinate as well. State price a_σ is then determined by present value prices at the initial date-event. This delivers a complete determination of state prices.

With no transfers, existence requires that public liabilities remain strictly positive at every date-event with a strictly positive value of public revenue. Positive public liabilities at all date-events are also required for determinateness of state prices. In addition, the fact that the public authority trades in safe bonds only (better, in a given bundle of securities only) is crucial for the conclusion. In fact, even with no transfers, a full indeterminacy of state prices, up to consistency with nominal rates of interest, would obtain if the composition of the public portfolio of securities were not restricted exogenously (that is, if no condition equivalent to (5) were imposed). Indeed, in this case, sequential public budget constraint (2) would be implied by sequential budget constraints (6) of individuals at equilibrium, so involving no additional restrictions. As the real allocation remains unchanged, indeterminacy of state prices has no real effects. This straightforward conclusion is stated in the following proposition.

Proposition 6.2 (Indeterminacy with unrestricted public portfolio). *Given nominal rates of interest r, when public portfolio is unrestricted, every equilibrium allocation remains an equilibrium for all state prices a set arbitrarily, up to consistency with nominal rates of interest r.*

Allowing for the distribution of public budget surpluses, price determination fails even when public liabilities consist of safe bonds only.

Proposition 6.3 (Indeterminacy). *Given nominal rates of interest r, there is $c^* > 0$ such that, for every $c \geq c^*$ and for all state prices a set arbitrarily, up to consistency with nominal rates of interest r, there exists an equilibrium with $\sum_{\sigma \in \mathcal{S}} a_\sigma \|p_\sigma\| = c$.*

That is, the overall price level, $\sum_{\sigma \in \mathcal{S}} a_\sigma \|p_\sigma\|$, is indeterminate up to a lower bound. A proportional increase in prices, in general, bears real effects because it redistributes wealth across individuals. No real effects obtain when, for instance, initial nominal wealths are proportional to transfers (that is, $\delta^i = \zeta^i \delta$).

For a given overall price level, if transfers are strictly positive, state prices exhibit degrees of purely nominal multiplicity, also when portfolio policy is pegged (that is, condition (5) is imposed). Indeed, without altering the real allocation and present value prices of commodities, at a date-event, intertemporal public budget constrain only imposes

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau p_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1 + r_\tau} \right) a_\tau p_\tau \cdot g_\tau$$
$$= a_\sigma w_\sigma + \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau .$$

Thus, a slight variation in state prices, up to consistency with nominal rates of interest, can be compensated by a slight variation in the intertemporal profile of transfers, so as to satisfy the requirement of a balanced intertemporal public budget at that date-event. However, as only positive transfers are allowed, large variations in state prices could not be consistent with the given overall price level.

As the overall price level increases, public budget exhibit larger and larger surpluses, to be exhausted through transfers to individuals. As such budget surpluses could be made arbitrarily large, there are in fact no restrictions on state prices at equilibrium. Though the asset market is sequentially complete, it can no longer be assumed that the allocation does not vary with arbitrarily set state prices. Some equilibrium allocations might not be preserved without large increases in the overall price level at those state prices.

7 Efficiency

Neither of the basic Welfare Theorems holds in a monetary economy under strictly positive nominal rates of interest: (a) equilibrium allocations, in general, fail to be Pareto efficient; (b) Pareto efficient allocations cannot, in general, be sustained as equilibrium allocations (though they could, trivially, under suitable redistributions of endowments of commodities). The Pareto inefficiency follows from the wedge driven by the cash-in-advance constraint between buying and selling prices of commodities. More importantly, one can construct robust examples of economies exhibiting Pareto-ranked equilibria at given nominal rates of interest.

To clarify our last claim, we provide a simple example without aiming at being exhaustive. There are two individuals and two physical commodities. Let the nominal rate of interest, $r > 0$, be given. Assuming a common rate of time preference across individuals, we can treat a stationary infinite horizon economy as a simple one-period economy. Individual 1's preferences are represented by $x_1^1 + (1+r)^{-1} x_2^1$ and endowments are $(0,1)$. Individual 2's preferences are represented by $(1+r)^{-1} x_1^2 + x_2^2$ and endowments are $(1,0)$. A symmetric allocation is represented by $0 \leq \theta \leq 1$, with consumptions $x_\theta^1 = (\theta, 1-\theta)$ and $x_\theta^2 = (1-\theta, \theta)$. The strictly positive amount of public debt is equally distributed across the two individuals. It is simple to verify that, for every $0 < \theta \leq 1$, $\left(x_\theta^1, x_\theta^2\right)$ is an equilibrium with prices π_θ proportional to $(1,1)$. There is thus a continuum of real equilibria ranking from the no-trade to the symmetric Pareto-efficient allocation. Notice that all such equilibria involve no transfers and can be indexed by the overall price level, up to a lower bound.

The concept of constrained efficiency suitable for monetary economies is not evident. However, given nominal rates of interest, one could obtain an analogue of the two Welfare Theorems using a notion of supportability of an allocation in place of the standard notion of Pareto efficiency (Bloise, Drèze and Polemarchakis [4, Sect. 7]). When nominal rates of interest vanish, supportability coincides with Pareto efficiency. Otherwise, the interpretation is unclear. Still, it allows for a complete characterization of the full set of equilibria (with transfers), at given nominal rates of interest, in terms of fundamentals only. In addition, it helps in understanding what distinguishes cash-in-advance economies from economies with real intermediation costs, as those that are described by Foley [10].

8 Remarks

8.1. Various contributions over the last decade (among others, Drèze and Polemarchakis [7] and Dubey and Geanakoplos [8,9]) have pointed out that finite time is suitable to meaningfully address issues of monetary analysis. Our current work is intended to confirm this view, as our arguments are independent for substance of the horizon being finite or infinite. Remarkably, the finite-horizon model provides a tractable disaggregate framework for a short-term analysis of, for instance, financial markets and nominal price rigidities.

8.2. Throughout our analysis, we have maintained the assumption of a sequentially complete asset market. This has allowed for a focus only on balances needed for transaction purposes. A sequentially incomplete asset market would enrich our analysis in a number of ways and, in particular, it would make the variability of inflation rates of real allocative relevance.

8.3. Our analysis points at a limited relevance of the fiscal theory of the price level (Woodford [18,19] and Cochrane [6]). Differently from the framework of that theory, we only assume that eventual public budget surpluses are distributed to individuals through transfers. This seems innocuous and, yet, dramatically changes the conclusions on the determinacy of prices.

Proofs

Proof of Lemma 4.1. Solvency constraints imply that

$$\sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(h_\tau^i + \left(\frac{1}{1+r_\tau} \right) p_\tau \cdot \left(e_\tau^i - g_\tau^i \right) \right)$$

takes finite value at every non-initial date-event and, hence, at every date-event. By assumptions (T) and (F),

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau} \right) a_\tau p_\tau \cdot \left(e_\tau^i - g_\tau^i \right)$$

is finite. Hence, by assumptions (M) and (F), the claim easily follows.

Proof of Lemma 4.2. The argument is standard. Suppose that a plan $\left(x^i, m^i, w^i\right)$ satisfies sequential budget, liquidity and solvency constraints, given initial nominal claims. Multiplication of the sequential budget constraints by a_σ and summation over $\mathcal{S}^t$ yield

$$\sum_{\sigma \in \mathcal{S}_{t+1}} a_\sigma w_\sigma^i + \sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma} \right) a_\sigma m_\sigma^i + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot x_\sigma^i \leq$$

$$\delta^i + \sum_{\sigma \in \mathcal{S}^t} a_\sigma h_\sigma^i + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot e_\sigma^i - \sum_{\sigma \in \mathcal{S}^t} \left(\frac{1}{1+r_\sigma} \right) a_\sigma p_\sigma \cdot g_\sigma^i .$$

The solvency constraint at every date-event, then, implies

$$\sum_{\sigma\in\mathcal{S}^t}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma m^i_\sigma+\sum_{\sigma\in\mathcal{S}^t}a_\sigma p_\sigma\cdot x^i_\sigma-\sum_{\sigma\in\mathcal{S}^t}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot e^i_\sigma\le$$
$$\delta^i+\sum_{\sigma\in\mathcal{S}}a_\sigma h^i_\sigma+\sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot e^i_\sigma-\sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot g^i_\sigma.$$

Since the left-hand side is bounded, the first term is non-decreasing and the other two terms converge, taking the limit as $t\to\infty$ implies

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma m^i_\sigma+\sum_{\sigma\in\mathcal{S}}a_\sigma p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right)\le$$
$$\delta^i+\sum_{\sigma\in\mathcal{S}}a_\sigma h^i_\sigma-\sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot g^i_\sigma.$$

Therefore, $\left(x^i,m^i\right)$ satisfy the intertemporal budget constraint and sequential liquidity constraints.

Conversely, suppose that a plan $\left(x^i,m^i\right)$ satisfies the intertemporal budget constraint and sequential liquidity constraints and define w^i, at all non-initial date-events, by

$$a_\sigma w^i_\sigma=\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{r_\tau}{1+r_\tau}\right)a_\tau m^i_\tau+\sum_{\tau\in\mathcal{S}_\sigma}a_\tau p_\tau\cdot\left(x^i_\tau-e^i_\tau\right)$$
$$+\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right)a_\tau p_\tau\cdot g^i_\tau-\sum_{\tau\in\mathcal{S}_\sigma}a_\tau h^i_\tau.$$

Solvency constraints are satisfied, since liquidity constraints imply that

$$-a_\sigma^{-1}\sum_{\tau\in\mathcal{S}}\left(\frac{1}{1+r_\tau}\right)a_\tau p_\tau\cdot e^i_\tau\le$$
$$-a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right)a_\tau p_\tau\cdot\left(x^i_\tau-e^i_\tau\right)^-+a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma}a_\tau p_\tau\cdot\left(x^i_\tau-e^i_\tau\right)^+\le$$
$$w^i_\sigma+a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma}a_\tau h^i_\tau-a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right)a_\tau p^i_\tau\cdot g^i_\tau.$$

To see that sequential budget constraints are satisfied as well, observe that, at every non-initial date-event, the definition of w^i implies that

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)m^i_\sigma+a_\sigma^{-1}\sum_{\tau\in\sigma_+}a_\tau w^i_\tau+p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right)=w^i_\sigma+h^i_\sigma-\left(\frac{1}{1+r_\sigma}\right)p_\sigma\cdot g^i_\sigma.$$

At the initial date-event, the intertemporal budget constraint and the definition of w^i imply that

$$\left(\frac{r_\phi}{1+r_\phi}\right) m_\phi^i + \sum_{\sigma \in \phi^+} a_\sigma w_\sigma^i + p_\phi \cdot \left(x_\phi^i - e_\phi^i\right) \leq \delta^i + h_\phi^i - \left(\frac{1}{1+r_\phi}\right) p_\phi \cdot g_\phi^i.$$

At an optimal plan, the intertemporal budget constraint must hold with equality since preferences are strictly monotone. Moreover, it is clear that the liquidity constraint is non-binding only if the nominal rate of interest is zero.

Concerning transversality, a plan satisfies solvency constraints only if

$$\liminf \sum_{\sigma \in \mathcal{S}_t} a_\sigma w_\sigma^i \geq 0.$$

It, then, suffices to show that a plan is maximal only if

$$\limsup \sum_{\sigma \in \mathcal{S}_t} a_\sigma w_\sigma^i \leq 0.$$

If not, then, for infinitely many dates, n, and some $\epsilon > 0$,

$$\epsilon + \sum_{\sigma \in \mathcal{S}^n} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma^i + \sum_{\sigma \in \mathcal{S}^n} a_\sigma p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right) \leq$$
$$\delta^i + \sum_{\sigma \in \mathcal{S}^n} a_\sigma h_\sigma^i - \sum_{\sigma \in \mathcal{S}^n} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma^i.$$

From the limit, since all series must converge, it follows that

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma^i + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right) <$$
$$\delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h_\sigma^i - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma^i,$$

which violates optimality.

Proof of Propositions 6.1–6.3. The proof is organized as follows. First (I), we introduce a notion of abstract equilibrium, which allows for the determination of present value prices of commodities independently of state prices. Second (II), we show that an abstract equilibrium exists in every truncated economy. Third (III), we prove that the limit of truncated equilibria is an abstract equilibrium of the infinite-horizon economy. Fourth (IV), we show that every abstract equilibrium corresponds to an equilibrium with no transfers for some state prices and, if the overall price level is high enough, to an equilibrium with transfers for arbitrarily set state prices.

(I) Abstract equilibrium. Let X^i be the consumption space of individual i, the positive cone of $\ell_\infty\left(\mathcal{S} \times \mathcal{N}\right)$, and Π the space of normalized present value prices of commodities, the subset of the positive cone of $\ell_1(\mathcal{S}\times\mathcal{N})$ satisfying the normalization $\|\pi\|_1 = 1$. For (π, x) in $\ell_1(\mathcal{S}\times\mathcal{N})\times\ell_\infty(\mathcal{S}\times\mathcal{N})$, $\pi\cdot x = \sum_{\sigma\in\mathcal{S}}\pi_\sigma\cdot x_\sigma$ denotes the duality operation.[4]

An abstract equilibrium consists of present value prices of commodities, π, an allocation, $(\ldots, x^i, \ldots)$, an index for (the reciprocal of) the overall price level, $\mu \geq 0$, and an aggregate transfer, $\beta \geq -\mu\delta$, such that:

(a) market clearing is achieved,

$$\sum_i x^i - \sum_i e^i = 0;$$

(b) for every individual,

$$z^i \succ^i x^i \text{ implies } \pi\cdot z^i + \left(\frac{r}{1+r}\right)\pi\cdot\left(z^i - e^i\right)^- > \pi\cdot x^i + \left(\frac{r}{1+r}\right)\pi\cdot\left(x^i - e^i\right)^-$$

and

$$\pi\cdot\left(x^i - e^i\right) + \left(\frac{r}{1+r}\right)\pi\cdot\left(x^i - e^i\right)^- = \mu\delta^i + \zeta^i\beta - \left(\frac{1}{1+r}\right)\pi\cdot g^i.$$

To offset the redundancy stemming from the choice of the unit of account, we add the normalization $\pi\in\Pi$. Notice that, in an abstract equilibrium, $\mu = 0$ is allowed.

(II) Truncations. Suppose that all vector spaces are of finite dimension, which corresponds to a truncated economy. We show that, given any $\mu \geq 0$ small enough, an abstract equilibrium exists for some $\beta \geq -\mu\delta$; alternatively, setting $\beta = 0$, an abstract equilibrium exists for some $\mu \geq 0$.

Choose any $\mu \geq 0$ small enough so that, for every individual i,

$$0 < \epsilon \leq \mu\left(\delta^i - \zeta^i\delta\right) + \left(\frac{1}{1+r}\right)\pi\cdot\left(e^i - g^i\right), \tag{14}$$

for all $\pi\in\Pi$. (This can be done as the net endowment is interior and nominal rates of interest are bounded.) Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \beta\right) \in \cdots\times X^i\times\cdots\times\Pi\times B = F,$$

where X^i is the consumption space of individual i, Π is the space of normalized present value prices and $B = \{\beta\in\mathbb{R} : \beta \geq -\mu\delta\}$. A correspondence $\hat{f}\mapsto\bar{f}$ is defined by:

[4] Concerning notation, for a countable set $\mathcal{A}$, $\ell(\mathcal{A})$, $\ell_\infty(\mathcal{A})$ and $\ell_1(\mathcal{A})$ denote, respectively, the vector space of all real-valued maps, bounded real-valued maps and summable real-valued maps on $\mathcal{A}$, where summable means that $\|x\|_1 = \sum_{\alpha\in\mathcal{A}}|x_\alpha|$ is finite. For (x, z) in $\ell(\mathcal{S})\times\ell(\mathcal{S}\times\mathcal{N})$, $xz = zx$ is the element of $\ell(\mathcal{S}\times\mathcal{N})$ obtained by point-wise product, $(\ldots, x_\sigma(\ldots, z_{\sigma\nu}, \ldots), \ldots) = (\ldots, (\ldots, z_{\sigma\nu}, \ldots)x_\sigma, \ldots)$. Moreover, we use

$$\left(\frac{1}{1+r}\right) = \left(\ldots, \left(\frac{1}{1+r_\sigma}\right), \ldots\right) \quad\text{and}\quad \left(\frac{r}{1+r}\right) = \left(\ldots, \left(\frac{r_\sigma}{1+r_\sigma}\right), \ldots\right)$$

for notational convenience. Aliprantis and Border [1] provide a useful treatment of infinite-dimensional analysis.

(a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \mu\delta^i + \zeta^i \hat{\beta} - \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g^i;$$

(b) $\bar{\beta}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g = \mu\delta + \beta;$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy (Drèze and Polemarchakis [7]). Therefore, in a truncated economy, an abstract equilibrium exists for all arbitrarily chosen $\mu \geq 0$ small enough.

Alternatively, set $\beta = 0$. Notice that, for every $\mu \geq 0$,

$$0 < \epsilon \leq \mu\delta^i + \left(\frac{1}{1+r}\right) \pi \cdot \left(e^i - g^i\right). \tag{15}$$

Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \beta\right) \in \cdots \times X^i \times \cdots \times \mathit{\Pi} \times M = F,$$

where X^i is the consumption space of individual i, $\mathit{\Pi}$ is the space of normalized present value prices and $M = \{\mu \in \mathbb{R} : \mu \geq 0\}$. A correspondence $\hat{f} \mapsto \bar{f}$ is defined by:

(a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \mu\delta^i - \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g^i;$$

(b) $\bar{\mu}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g = \mu\delta;$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy (Drèze and Polemarchakis [7]). Therefore, in a truncated economy, an abstract equilibrium exists with no aggregate transfer, $\beta = 0$.

(III) Limit. We now make truncation explicit. For an element x of $\ell(\mathcal{S} \times \mathcal{N})$, let $x\chi_t$ denote its truncation at t. That is, $(x\chi_t)_\sigma = x_\sigma$, if $0 \leq t_\sigma \leq t$, and $(x\chi_t)_\sigma = 0$, otherwise. A t-truncated economy is constructed as follows: preferences on the consumption space, X^i, the positive cone of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, are recovered using $x^i \succeq^{it} z^i$ if and only if $x^i\chi_t + (e^i - e^i\chi_t) \succeq^i z^i\chi_t + (e^i - e^i\chi_t)$; truncated present value prices of commodities are elements of $\Pi^t = \{\pi \in \Pi : \pi\chi_t = \pi\}$.

Consider a sequence of abstract equilibria of t-truncated economies: for every t, the allocation is $(\ldots, x^{it}, \ldots)$, present value prices of commodities are π^t, the aggregate transfer is β^t and the index for the overall price level is μ^t. Along such a sequence of truncated equilibrium, one might assume that either $\mu^t = \mu \geq 0$ is constant or $\beta^t = \beta = 0$ is constant. To simplify, write $\alpha^{it} = \mu^t\delta^i + \zeta^i\beta^t \geq 0$.

Letting

$$\varphi^t = (\ldots, \varphi^t_\sigma, \ldots) = \left(\ldots, \left(\frac{1}{1 + r_\sigma}\right)\pi^t_\sigma, \ldots\right),$$

π^t and φ^t can be viewed as elements of $ba(\mathcal{S} \times \mathcal{N})$, the norm dual of $\ell_\infty(\mathcal{S} \times \mathcal{N})$ consisting of all finitely additive set functions on $\mathcal{S} \times \mathcal{N}$ and endowed with the norm $\|\cdot\|_{ba}$ (the norm of total variation). Let $\sigma(ba, \ell_\infty)$ denote the weak* topology of $ba(\mathcal{S} \times \mathcal{N})$. Since

$$\|\varphi^t\|_1 = \|\varphi^t\|_{ba} \leq \|\pi^t\|_{ba} = \|\pi^t\|_1 = 1$$

and since, by Alaoglu Theorem, the unit sphere in $ba(\mathcal{S} \times \mathcal{N})$ is $\sigma(ba, \ell_\infty)$ compact, without loss of generality, $\{\pi^t\}$ and $\{\varphi^t\}$ converge to π and φ, respectively, in the $\sigma(ba, \ell_\infty)$ topology. Moreover, both π and φ, as well as $\pi - \varphi$, are positive elements of $ba(\mathcal{S} \times \mathcal{N})$ and $0 < \|\varphi\|_{ba} \leq \|\pi\|_{ba} = 1$.

By Tychonov Theorem, without loss of generality, every $\{x^{it}\}$ converges to x^i in the product topology. Since the product and the Mackey topology coincide on bounded subsets of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, it follows that every $\{x_{it}\}$ converges to x^i in the Mackey topology.

As $\{\mu^t\}$ and $\{\beta^t\}$ can be assumed to be bounded, without loss of generality, they converge to μ and β, respectively. Defining $\alpha^i = \mu\delta^i + \zeta^i\beta$, it follows that every α^{it} converges to α^i.

We now show that the limit of abstract equilibria in the truncated economies is an abstract equilibrium of the economy over an infinite horizon. The proof, which is presented in a sequence of steps (1)-(5), uses standard arguments.

1. Decomposition. Since π (φ) is a positive linear functional, it follows from the Yosida-Hewitt Theorem that there is a unique decomposition $\pi = \pi_f + \pi_b$ ($\varphi = \varphi_f + \varphi_b$), where π_f (φ_f) is a positive functional in $\ell_1(\mathcal{S} \times \mathcal{N})$, the Mackey-topology dual of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, and π_b (φ_b) is a positive finitely additive measure (a pure charge) vanishing on all vectors having only a finite number of non-zero components.

2. $z^i \succeq^i x^i$ *implies*

$$\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

For a strictly positive real number, λ, $z^i + \lambda e^i \succ^{it} x^{it}$ for all t large enough, which implies that

$$\varphi^t \cdot g^i + \pi^t \cdot \left(z^i - (1-\lambda) e^i\right)^+ \geq \alpha^{it} + \varphi^t \cdot \left(z^i - (1-\lambda) e^i\right)^- .$$

Taking the limit, one obtains

$$\varphi \cdot g^i + \pi \cdot \left(z^i - (1-\lambda) e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - (1-\lambda) e^i\right)^- .$$

As lattice operations are continuous in the norm topology and π and φ are norm-continuous linear functionals, letting λ go to zero, the claim is proven.

3. $z^i \succ^i x^i$ *implies*

$$\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ > \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

Continuity of preferences implies that $\lambda z^i \succ^i x^i$ for some $0 < \lambda < 1$. Since

$$\varphi \cdot g^i + \lambda\pi \cdot \left(z^i - e^i\right) + (\pi - \varphi) \cdot \left(\lambda z^i - e^i\right)^- \geq \alpha^i + (1-\lambda)\, \pi \cdot e^i$$

and

$$\lambda \left(z^i - e^i\right)^- + (1-\lambda)\, e^i \geq \left(\lambda z^i - e^i\right)^- ,$$

one obtains

$$\begin{aligned}\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ &\geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\frac{1-\lambda}{\lambda}\right)\left(\varphi \cdot \left(e^i - g^i\right) + \alpha^i\right) \\ &\geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\frac{1-\lambda}{\lambda}\right)\epsilon,\end{aligned}$$

where the last inequality follows from solvency conditions (14)–(15).

4. $\pi_b = \varphi_b = 0$ *and*

$$\varphi \cdot g^i + \pi \cdot \left(x^i - e^i\right)^+ = \alpha^i + \varphi \cdot \left(x^i - e^i\right)^- .$$

For a vector u in $\ell\left(\mathcal{S} \times \mathcal{N}\right)$,

$$\varphi^t \cdot u = \pi^t \cdot \left(\left(\frac{1}{1+r}\right) u\right)$$

holds at every t and, hence, in the limit,

$$\varphi \cdot u = \pi \cdot \left(\left(\frac{1}{1+r}\right) u\right).$$

Using truncations, one can show that

$$\varphi_b \cdot u = \pi_b \cdot \left(\left(\frac{1}{1+r} \right) u \right).$$

It follows that $\pi_b = 0$ if and only if $\varphi_b = 0$.

Suppose that $\pi_b > 0$, so that, by interior net endowments, $\varphi_b \cdot \left(e^i - g^i \right) = \xi > 0$. Since $x^i \chi_t + \lambda e^i \chi_t \succ^i x^i$ for all t large enough and all strictly positive real numbers, λ,

$$\begin{aligned} \varphi \cdot g^i + \pi \cdot \left(x^i - e^i \right)^+ \chi_t + \lambda \pi \cdot e^i \chi_t \geq \\ \varphi \cdot g^i + \pi \cdot \left(x^i \chi_t + \lambda e^i \chi_t - e^i \right)^+ \geq \\ \alpha^i + \varphi \cdot \left(x^i \chi_t + \lambda e^i \chi_t - e^i \right)^- \geq \\ \alpha^i + \varphi \cdot \left(x^i - e^i \right)^- \chi_t - \lambda \varphi \cdot e^i \chi_t + \varphi \cdot \left(e^i - e^i \chi_t \right). \end{aligned}$$

In the limit, one obtains

$$\begin{aligned} \varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i \right)^+ + \lambda \left(\pi_f - \varphi_f \right) \cdot e^i \geq \\ \alpha^i + \varphi_f \cdot \left(x^i - e^i \right)^- + \varphi_b \cdot \left(e^i - g^i \right) \geq \\ \alpha^i + \varphi_f \cdot \left(x^i - e^i \right)^- + \xi. \end{aligned}$$

Thus,

$$\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i \right)^+ \geq \alpha^i + \varphi_f \cdot \left(x^i - e^i \right)^- + \xi.$$

To prove equality, notice that, for all $0 \leq s \leq t$,

$$\varphi^t \cdot g^i + \pi^t \cdot \left(x^{it} - e^i \right)^+ \chi_s \leq \alpha^i + \varphi^t \cdot \left(x^{it} - e^i \right)^- \chi_s + \varphi^t \cdot \left(e^i - e^i \chi_s \right).$$

Therefore, in the limit,

$$\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i \right)^+ \leq \alpha^i + \varphi_f \cdot \left(x^i - e^i \right)^- + \xi.$$

Summing over individuals,

$$\varphi_f \cdot g + \left(\pi_f - \varphi_f \right) \cdot \sum_i \left(x^i - e^i \right)^- > \sum_i \alpha^i.$$

Observe that, for all $0 \leq s \leq t$,

$$\varphi^t \cdot g \chi_s + \left(\pi^t - \varphi^t \right) \cdot \left(\sum_i \left(x^{it} - e^i \right)^- \right) \chi_s \leq \sum_i \alpha^{it}.$$

In the limit,

$$\varphi_f \cdot g + \left(\pi_f - \varphi_f \right) \cdot \sum_i \left(x^i - e^i \right)^- \leq \sum_i \alpha^i,$$

a contradiction to the previous reverse strict inequality.

5. Limit is an abstract equilibrium. By point-wise limits, one obtains

$$\varphi = \left(\frac{1}{1+r}\right)\pi,$$

thus proving the claim.

(IV) Equilibrium. In every abstract equilibrium, it is clear that present value prices π are strictly positive. We show that, at every date-event,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)\pi_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \left(\frac{1}{1+r_\sigma}\right)\pi_\sigma \cdot g_\sigma > 0. \tag{16}$$

Indeed, if $g_\sigma = 0$, then $r_\sigma > 0$ by our assumption (R). If public revenue vanishes at some date-event, it follows that the abstract equilibrium allocation $\left(\ldots, x^i, \ldots\right)$ coincides with the initial allocation $\left(\ldots, e^i, \ldots\right)$ at that date-event. By the condition on trade at equilibrium (R), there exists an allocation $\left(\ldots, z^i, \ldots\right)$ that weakly Pareto dominates allocation $\left(\ldots, x^i, \ldots\right)$ and satisfies

$$\sum_i z^i + \left(\frac{r}{1+r}\right)\sum_i \left(z^i - x^i\right)^- \leq \sum_i x^i.$$

Since $\left(z^i - e^i\right)^- \leq \left(z^i - x^i\right)^- + \left(x^i - e^i\right)^-$, it follows that

$$\sum_i z^i + \left(\frac{r}{1+r}\right)\sum_i \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right) \leq \sum_i x^i.$$

By the optimality of consumption plans, for every individual,

$$\pi \cdot z^i + \left(\frac{r}{1+r}\right)\pi \cdot \left(z^i - e^i\right)^- > \pi \cdot x^i + \left(\frac{r}{1+r}\right)\pi \cdot \left(x^i - e^i\right)^- .$$

Thus,

$$\pi \cdot \left(z^i - x^i + \left(\frac{r}{1+r}\right)\left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right)\right) > 0,$$

which, summing over individuals, implies

$$\pi \cdot \left(\sum_i z^i - \sum_i x^i + \left(\frac{r}{1+r}\right)\sum_i \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right)\right) > 0,$$

a contradiction.

Letting

$$u_\sigma = \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{r_\tau}{1+r_\tau}\right)\pi_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- + \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right)\pi_\tau \cdot g_\tau,$$

the above argument shows that, in every abstract equilibrium, u is a strictly positive element of $\ell(\mathcal{S})$ with $u_\phi = \mu\delta + \beta > 0$. In particular, this proves that, if there is no aggregate transfer, $\beta = 0$, then $\mu > 0$ in every abstract equilibrium.

To obtain an equilibrium from an abstract equilibrium with $\mu > 0$, one needs only to show the existence of state prices a, consistent with nominal rates of interest r, public liabilities w and transfers h that satisfy, at every date-event σ,

$$\frac{1}{\mu}\frac{u_\sigma}{a_\sigma} = w_\sigma + \frac{1}{a_\sigma}\sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau. \tag{17}$$

and

$$w_{\tau'} = w_{\tau''} \text{ for all } (\tau', \tau'') \in \sigma_+ \times \sigma_+. \tag{18}$$

Prices are then obtained by

$$(\ldots, (\ldots, p_{\sigma\nu}, \ldots), \ldots) = \frac{1}{\mu}\left(\ldots, \left(\ldots, \frac{\pi_{\sigma\nu}}{a_\sigma}, \ldots\right), \ldots\right).$$

Condition (17) ensures that the asset market clears, while condition (18) guarantees that public liabilities consist of safe bonds only.

Given any abstract equilibrium with $\beta = 0$, an equilibrium with no transfers, $h = 0$, corresponds to state prices a that, subject to no arbitrage (restrictions (1)), solve, at every date-event σ,

$$\frac{a_\tau}{\sum_{\tau \in \sigma_+} a_\tau} = \frac{u_\tau}{\sum_{\tau \in \sigma_+} u_\tau} \text{ for all } \tau \in \sigma_+.$$

This proves Proposition 6.1.

To prove Proposition 6.3, observe that there is an abstract equilibrium for every $\mu > 0$ small enough, with associated aggregate transfers $\beta \geq -\mu\delta$. Suppose that, as μ vanishes, there is an abstract equilibrium with

$$\left(\frac{r_\phi}{1 + r_\phi}\right)\pi_\phi \cdot \left(x^i_\phi - e^i_\phi\right)^- + \left(\frac{1}{1 + r_\phi}\right)\pi_\phi g_\phi \leq \mu\delta.$$

One can show that the limit is also an abstract equilibrium, which contradicts the strict positivity of public revenue established by condition (16). Hence, for every $\mu > 0$ small enough, there is an abstract equilibrium with

$$\left(\frac{r_\phi}{1 + r_\phi}\right)\pi_\phi \cdot \left(x^i_\phi - e^i_\phi\right)^- + \left(\frac{1}{1 + r_\phi}\right)\pi_\phi g_\phi > \mu\delta, \tag{19}$$

that is, such that the public revenue at the initial date-event exceeds the initial public liability. Consider any abstract equilibrium such that (19) is satisfied and set state prices a arbitrarily, subject to no arbitrage (restrictions (1)). Transfers h are obtained, at the initial date-event ϕ, by

$$\left(\frac{r_\phi}{1 + r_\phi}\right)\pi_\phi \cdot \sum_i \left(x^i_\phi - e^i_\phi\right)^- + \left(\frac{1}{1 + r_\phi}\right)\pi_\phi \cdot g_\phi - \mu\delta = \mu h_\phi \geq 0;$$

at every non-initial date-event σ, by

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)\pi_\sigma \cdot \sum_i \left(x_\sigma^i - e_\sigma^i\right)^- + \left(\frac{1}{1+r_\sigma}\right)\pi_\sigma \cdot g_\sigma = \mu a_\sigma h_\sigma \geq 0.$$

This construction trivially fulfills conditions (17)–(18) with public liabilities w vanishing at all non-initial date-event, that is, with $w_\sigma = 0$ for every non-initial date-event σ. The argument proves Proposition 6.3.

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