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Economic Theory

Stationary Markov equilibria for overlapping generations*

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Summary. At a stationary Markov equilibrium of a Markovian economy of overlapping generations, prices at a date-event are determined by the realization of the shock, the distribution of wealth and, with production, the stock of capital. Stationary Markov equilibria may not exist; this is the case with intra-generational heterogeneity and multiple commodities or long life spans. Generalized Markov equilibria exist if prices are allowed to vary also with the realization of the shock, prices and the allocation of consumption and production at the predecessor date-event. (Stationary) Markov ϵ-equilibria always exist; as $\epsilon \to 0$, allocations and prices converge to equilibrium prices and allocations that, however, need not be stationary.

Keywords and Phrases: Stationary Markov equilibria, ϵ-equilibria.

JEL Classification Numbers: D50, D52, D60, D80, D90.

1 Introduction

In a market economy over time under uncertainty, prices of commodities and assets vary across date-events. At a competitive equilibrium, optimizing individuals must know the price process. It is thus important that the price process be simple.

A stochastic process is a Markov process if the distribution of uncertainty at a date depends on the realization at the immediately preceding date, but not before.

An economy is Markovian if the stochastic process that generates fundamentals is a Markov process.

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One way to achieve simplicity in a Markovian economy is for prices to depend on a minimal set of state-variables that follow a Markov process.

A question that arises concerns the minimal state-space that supports a dynamic competitive equilibrium.

Most simply, the state-space would consist of the state-space that generates fundamentals; nevertheless, other than in cases that are exceptional, if of interest, as in the representative individual economy in Lucas (1978), such an equilibrium may not exist. For models with overlapping generations, Spear (1985) showed that heterogeneity of individuals suffices to preclude, generically, equilibria supported by the state-space of fundamentals.

Endogenous variables a fortiori enter the state-space that supports competitive equilibria.

In a stationary Markov equilibrium the relevant state-space is determined by the payoff-relevant variables: endogenous as well as exogenous variables, sufficient for the optimization of individuals at every date-event.

In an exchange economy subject to endowment or preference shocks, it is the distribution of assets that, if added to the fundamentals, defines the state-space of stationary Markov equilibria; in a production economy, the capital stock should complete the picture. In modern macroeconomics, such equilibria are called recursive, as in Ljungqvist and Sargent (2000).

There is a variety of reasons for focusing on stationary Markov equilibria. For the case of dynamic games, Maskin and Tirole (2001) list both practical and conceptual arguments in favor of stationary equilibria. Most importantly, recursive methods can be used to approximate stationary Markov equilibria numerically. Heaton and Lucas (1996) and Krusell and Smith (1998) are examples of papers that approximate stationary Markov equilibria in models with infinitely-lived individuals; Rios-Rull (1996) and Constantinides, Donaldson and Mehra (2002) are instances of this for overlapping generations. Although an existence theorem for stationary Markov equilibria is not available, applied research, even if explicitly aware of the problem, needs to focus on such equilibria: there are no algorithms for the computation of equilibria that are not recursive.

In this paper we examine the existence of stationary Markov equilibria in Markovian economies with overlapping generations and finite life-spans.

First, we give two examples in which stationary Markov equilibria do not exist. The intuition that underlies the non-existence of stationary equilibria in a Markovian economy is straightforward: the realization of the shock and the distribution of wealth suffice for the optimization of individuals at any date-event; however, with these initial conditions for the economy starting at that date, there may be multiple equilibria; the distribution of wealth is endogenous, and it is determined by the choices of individuals at preceding date-events; these choices may lead to the current distribution of wealth only if individuals expect a particular one of the possible continuation equilibria to prevail. As a consequence, the equilibrium at a date may depend on the realization of a state at preceding events.

The non-existence of stationary Markov equilibria thus derives from the non-uniqueness of continuation equilibria. This intuition was first explained and illustrated in Hellwig (1983).

The non-existence of stationary Markov equilibria has been demonstrated in different contexts. Kubler and Schmedders (2002) gave an example that demonstrates the non-existence of stationary Markov equilibria in models with incomplete asset markets and infinitely-lived individuals. Krebs (2004) analyzes a similar model and shows that Markov equilibria never exist when there are no constraints on trades. Santos (2002) gave examples of non-existence for economies with externalities. The examples in this paper are the first that can be solved analytically.

The existence of competitive equilibria for Markovian exchange economies was demonstrated in Duffie, Geanakoplos, Mas-Colell and McLennan (1994); the setting is general and encompasses economies of infinitely lived individuals, as well as economies of overlapping generations, either with a complete or an incomplete asset market. It can be shown that the equilibrium process is a stationary Markov process, and that it has an ergodic measure. However here, following the well established terminology in dynamic games, we do not refer to these equilibria as stationary Markov equilibria — Maskin and Tirole (2001) give a formal definition of payoff-relevant states for Markov perfect equilibria. Instead, we refer to these equilibria as generalized Markov equilibria. In Markovian economies with overlapping, generations generalized Markov equilibria always exist under weak assumptions on preferences; prices and choices are allowed to vary with the realization of the shock, the prices of assets and, importantly, the consumption allocation at the predecessor date-event as well.

The existence of generalized Markov equilibria is not useful for computational methods that use recursive methods. In this paper, we examine, instead, (stationary) Markov ϵ-equilibria. An ϵ-equilibrium consists of prices and choices such that markets clear and the choices of individuals lie within ϵ of utility maximizing choices. In numerical work, because of rounding and truncation errors, only ϵ-equilibria can be computed. Furthermore, in many settings, it can be verified fairly easily whether a candidate numerical solution is an ϵ-equilibrium according to our definition.

We show that Markov ϵ-equilibria exist for all $\epsilon > 0$; moreover as $\epsilon \to 0$ all ϵ-equilibria converge to competitive equilibria.

This provides a formal justification for the above mentioned practice of focusing on Markov (ϵ)-equilibria in applied work. An algorithm that calculates Markov ϵ-equilibria for the overlapping generations model, just like Scarf's (1967) algorithm for the computation of equilibria in finite economies, produces a sequence of ϵ-equilibria that converges to an exact equilibrium, as $\epsilon \to 0$. The limit is not necessarily a stationary Markov equilibrium since the same distribution of wealth may be the limit of different sequences of distributions, and, as a consequence, the equilibrium values of endogenous variables may not be uniquely determined.

While finance often examines pure exchange economies, such as Heaton and Lucas (1996) or Constantinides et al. (2002), macroeconomics considers economies with production. With incomplete financial markets, the objective of the firm is, however, in general not well defined, and there is a large, but inconclusive literature on the subject, following Drèze (1974); it is beyond the scope of our paper to develop a comprehensive model with production. In order to circumvent this difficulty, some, such as Rios-Rull (1996), have considered firms that buy inputs for production on spot markets and do not make investment decisions: these are made by households that decide to store capital, which falls under the constant stochastic returns to scale in Diamond (1967). Importantly, the extensions of the neo-classical growth model to stochastic overlapping generations can be viewed as a special case of this formulation. We include production in this simple way into our model.

There is a large literature on the existence of Markov equilibria in discounted stochastic games. Existence of stationary Markov equilibria, in general, cannot be shown; conditions are known that guarantee the existence of subgame-perfect equilibria, in Mertens and Parthasarathy (1987), as well as the existence of Markov ϵ-equilibria, in Whitt (1980).

2 The economy

We consider an economy of overlapping generations with stochastic shocks to endowments and with individuals that live for several dates.

Fundamentals

Dates are $t = 0, \dots$.

Exogenous shocks, s_t, follow a Markov process with finite support $\mathcal{S} = \{s\}$ and transition $\Pi = \{\pi(s'|s)\}$.

Date-events or nodes are histories of shocks $\sigma_t = (s_1, \dots, s_t)$. The n'th predecessor of σ_t is σ_{t-n} and $\sigma_{t+k} = (\sigma_t, s_{t+1}, \dots, s_{t+k})$ is a generic k'th successor.

At each date-event, finitely many, perishable commodities, $l \in \mathcal{L}$, are available for consumption; a bundle of commodities is $x(\sigma_t)$.

The demographic structure is that of overlapping generations; at each date-event, finitely many individuals, $i \in \mathcal{I}$, commence their economic lives; they live for $(N+1)$ dates; an individual, (i, σ_t), exchanges and consumes commodities at the date-event σ_t, and the successor date-events, $(\sigma_t, s_{t+1}), \dots, (\sigma_t, s_{t+1}, \dots, s_{t+N})$; the age of an individual is $a = 0, \dots, N$, and the shock at the date that an individual is of age a is s_{t+a}.

At each date-event, there is a firm that produces in spot markets with a constant returns to scale technology; in addition, individuals have access to a (possibly risky) storage technology.

The restriction of the firm to production within a period eliminates the need for inter-temporal optimization under uncertainty for firms in an incomplete asset market. The storage of commodities by individuals allows for inter-temporal production.

The economy is stationary:

1. The endowment of individual (i, σ_t) at the date-event σ_{t+a} is
$$e^{i,\sigma_t}(\sigma_{t+a}) = e^{i,a}(s_{t+a});$$
2. his intertemporal, von Neumann-Morgenstern utility function is
$$U^{i,\sigma_t} = \mathrm{E}_{s_t} \sum_{a=0}^{N} u^{i,a}(x(\sigma_{t-1}, s_t, \dots, s_{t+a}), s_{t+a});$$

3. the production set of the firm at the date-event σ_t is

$$Y^{\sigma_t} = Y(s_t);$$

4. for each commodity there is, at each date-event, a vector

$$\delta_l^{\sigma_t} = \delta_l(s_t) = (\ldots, \delta_l(s_{t+1}|s_t), \ldots),$$

where $0 \leq \delta_l(s_{t+1}|s_t) \leq 1$ is the amount of commodity l at the state of the world s_{t+1} left from having stored one unit at the state, s_t; across commodities, the storage technology is described by the vector $\delta(s_t)$ or, alternatively, by the diagonal matrix $\Delta(s_t) = \text{diag}(\delta(s_t))$.

Standard restrictions on preferences, endowments and production sets are imposed to prove the existence of an equilibrium.

Assumption 1. *For every individual,*

1. *the function U^{i,σ_t} is strictly quasi-concave;*
2. *the function $u^{i,a}(., s)$ is monotonically increasing and continuous;*
3. *it is strictly increasing in commodity $l = 1$ and unbounded: $u^{i,a}(x, s) \to \infty$, as $x_1 \to \infty$;*
4. *the endowment in commodity $l = 1$ is strictly positive at every date-event in the individual's life-span: $e_1^{i,a}(s_{t+a}) > 0$, $a = 0, ..., N$.*

At every realization of the shock,

1. *the production technology, $Y(s)$, is a closed, convex cone, $Y(s) \cap \mathbb{R}_+^L = \{0\}$, and $Y(s) \cap -Y(s) = \{0\}$;*
2. *commodity, $l = 1$, is essential for production and not storable: There exists a $k > 0$ such that for all shocks s, $\delta_1(s) = 0$, and $y \in Y(s)$, $\Rightarrow$ $y_1 \leq 0$ and $\|y\| < k|y_1|$.*

In order to complete the specification, it is necessary to suppose that at the root node, $\sigma_0 = s_0$, there are individuals of all types and ages $(i, -a)$, $a = 1, \ldots, N$; their initial holdings in securities and storable commodities determine the initial conditions of the economy.

Markets

There are complete spot markets for commodities, at each date-event. Prices of commodities are $p(\sigma_t)$, a row vector; commodity $l = 1$ is numéraire, and $p_1(\sigma_t) = 1$.

In order to smooth consumption, individuals can trade in assets or store commodities.

There are fintely many assets, $j \in \mathcal{J}$.

The payoff, at the date-event σ_t, of asset j traded at the date-event σ_{t-1} is $d_j(s_t) = (d_{j,1}(s_t), d_{j,2}(s_t))$, where $d_{j,1}(s_t)$ are units of the numéraire commodity, while $d_{j,2}(s_t)$ are units of the asset j; across assets, the vector of payoffs of assets in the numéraire is

$$D_1(s_t) = (\ldots, d_{j,1}(s_t), \ldots)',$$

a column vector, the matrix of payoffs in assets is diagonal,

$$D_2(s_t) = \mathrm{diag}(d_{j,2}(s_t)),$$

and

$$D(s_t) = (D_1(s_t), D_2(s_t))'.$$

Following much of the applied literature we assume that an asset either pays one unit of itself or pays zero units at all dates event: Assets $j \in \mathcal{J}_1$ are short-lived assets, of one date maturity: $d_{j,2}(s) = 0$, at all states of the world; assets $j \in \mathcal{J}_2$ are long-lived or trees: $d_{j,2}(s) = 1$, at all states of the world. There are J_1 short-lived assets and J_2 long-lived assets.

A portfolio of assets is $\theta(\sigma_t) = (\theta_1(\sigma_t), \theta_2(\sigma_t))'$.

The aggregate supply of short-lived assets is 0, while the aggregate supply of long-lived assets is 1.

Prices of assets are $q(\sigma_t) = (q_1(\sigma_t), q_2(\sigma_t))$, a row vector.

The payoffs of short-lived assets, $\{d_{j,1} : j \in \mathcal{J}_1\}$, where $d_{j,1} = (\ldots, d_{j,1}(s_t), \ldots)'$, are linearly independent; this eliminates redundant assets.

For long-lived assets, individuals face a short-sales constraint: $\theta_2(\sigma_t) \geq 0$; this guarantees the continuity of demand.

Storage of commodities is $\phi(\sigma_t)$.

Equilibrium

Since firms generate zero profits, the budget set of an individual (i, σ_t) is

$$\mathcal{B}^{i,\sigma_t}(p,q) = \left\{ (x,\theta,\phi) : \begin{array}{l} p(\sigma_t)(x(\sigma_t) - e^{i,\sigma_t}(\sigma_t)) \leq \\ \quad -q(\sigma_t)\theta(\sigma_t) - p(\sigma_t)\phi(\sigma_t), \\ p(\sigma_{t+a})(x(\sigma_{t+a}) - e^{i,\sigma_t}(\sigma_{t+a})) \leq \\ \quad (1, q(\sigma_{t+a}))D(s_{t+a})\theta(\sigma_{t+a-1}) + \\ \quad\quad p(\sigma_{t+a})\Delta(s_{t+a})\phi(\sigma_{t+a-1}) - \\ \quad\quad\quad q(\sigma_{t+a})\theta(\sigma_{t+a}) - p(\sigma_{t+a})\phi(\sigma_{t+a}), \\ \quad\quad\quad a = 1, \ldots, N-1, \\ p(\sigma_{t+N})(x(\sigma_{t+N}) - e^{i,\sigma_t}(\sigma_{t+N})) \leq \\ \quad (1, q(\sigma_{t+N}))D(s_{t+N})\theta(\sigma_{t+N-1}) + \\ \quad\quad p(\sigma_{t+N})\Delta(s_{t+N})\phi(\sigma_{t+N-1}), \\ \phi \geq 0, \quad x \geq 0, \quad \theta_2 \geq 0 \end{array} \right\}.$$

A competitive equilibrium, for initial holdings of assets $\overline{\theta}^{i,-a}$ and commodities $\overline{\phi}^{i-a}$, is a collection of consumption plans, production plans, portfolio holdings, storage decisions and prices of commodities and assets, $(x^{i,\sigma_t}, \theta^{i,\sigma_t}, \phi^{i,\sigma_t}, y^{\sigma_t}, p, q)$, such that

1. all markets clear at every date-events:

$$\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}(x^{i,\sigma_{t-a}}(\sigma_t)-e^{i,\sigma_{t-a}}(\sigma_t))=$$
$$\sum_{j\in\mathcal{J}_2}(d_{j,1}(s_t),\mathbf{0}^{L-1})+y(\sigma_t)+$$
$$\sum_{i\in\mathcal{I}}\left(\sum_{a=1}^{N}\Delta(s_t)\phi^{i,\sigma_{t-a}}(\sigma_{t-1})-\sum_{a=0}^{N-1}\phi^{i,\sigma_{t-a}}(\sigma_t)\right),$$
$$\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\theta_1^{i,\sigma_{t-a}}(\sigma_t)=\mathbf{0}^{J_1},\quad \sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\theta_2^{i,\sigma_{t-a}}(\sigma_t)=\mathbf{1}^{J_2};$$

2. individuals and the firm optimize:

$$(x^{i,\sigma_t},\theta^{i,\sigma_t},\phi^{i,\sigma_t})\in\arg\max\{U^{i,\sigma_t}(x):(x,\theta,\phi)\in\mathcal{B}^{i,\sigma_t}(p,q)\},$$
$$y^{\sigma_t}\in\arg\max\{p(\sigma_t)y:y\in Y^{\sigma_t}\}.$$

3 Non-existence of stationary Markov equilibria

A Markov equilibrium is characterized by a pair of functions, a policy function that maps the current state into current date endogenous variables and a transition function that describes the evolution of the state. This description does not address what variables can be used to describe the current state of the economy or how the set of admissible states should be defined.

The applied literature assumes that the shock and beginning-of-date capital and portfolio-holdings constitute a sufficient statistics for the future evolution of the economy.

Definition 1. *A stationary or recursive Markov equilibrium is described by a set of holdings of assets and storable commodities, $\mathcal{H}$, that contains an open set, and a collection of functions, $\{f_x,f_y,f_\theta,f_\phi,f_p,f_q\}$ with domain $\mathcal{H}\times\mathcal{S}$, such that for every $(\overline{\theta},\overline{\phi})\in\mathcal{H}$, there exist a competitive equilibrium with*

$$x^{i,\sigma_t}(\sigma_{t+a})=f_x^{i,a}(\theta(\sigma_{t+a-1}),\phi(\sigma_{t+a-1}),s_t),$$
$$y^{\sigma_t}(\sigma_t)=f_y(\theta(\sigma_{t-1}),\phi(\sigma_{t-1}),s_t),$$
$$\theta^{i,\sigma_t}(\sigma_{t+a})=f_\theta(\theta(\sigma_{t+a-1}),\phi(\sigma_{t+a-1}),s_t),$$
$$\phi^{i,\sigma_t}(\sigma_{t+a})=f_\phi(\theta(\sigma_{t+a-1}),\phi(\sigma_{t+a-1}),s_t),$$
$$p(\sigma_t)=f_p(\theta(\sigma_{t-1}),\phi(\sigma_{t-1}),s_t),$$
$$q(\sigma_t)=f_q(\theta(\sigma_{t-1}),\phi(\sigma_{t-1}),s_t).$$

We present two simple examples to illustrate that, even in standard econo- mies, recursive equilibria may fail to exist; and it seems impossible to derive general conditions that ensure the existence of a recursive equilibrium. There is no production or storage in these examples: $Y(s)=\emptyset$ and $\delta(s)=0$, at all states of the world. It is easy to extend the examples to include production but the point is easiest understood in pure exchange economies.

Subsequently, we introduce alternative concepts, and we prove the existence of competitive equilibria, that might not be recursive, and of Markov ϵ-equilibria.

The examples follow Hellwig's (1983) initial intuition. If, at some date event, given some distribution of initial portfolios there are multiple competitive equilibria, the Walrasian auctioneer, in selecting among these equilibria, must take into account individuals' expectations at the immediate predecessor node. These expectations are generally *not* uniquely identified by the distribution of assets. The auctioneer must therefore take into account past realizations of exogenous and endogenous variables.

In order to explain this intuition with concrete examples, one must find a specification of the model that can be solved analytically and that has multiple equilibria. In stochastic overlapping generations, this generally implies that there cannot be trade between generations and that the non-uniqueness must occur on commodity spot markets. Our construction is as follows:

There are two individuals per generation, $i = 1, 2$ two commodities, $l = 1, 2$, and two dates in the life-spans of individuals, $a = 0, 1$.

Shocks, $s_t \in \mathcal{S}$ are equiprobable, and they are distributed independently over time.

At each date-event there is a complete set of elementary securities of one date maturity; there are no long-lived assets.

Each agent has time-separable utility, and, when young, individuals are only endowed with commodity $l = 1$ and only derives utility from consumption of this commodity.

It follows that there is no intergenerational trade and that intertemporal competitive equilibria decompose into equilibria of sub-economies, σ_t, each with two individuals, and two dates. Recursive equilibria may fail to exist if in the second period spot markets of a sub-economy, there is a multiplicity of equilibria conditional on the distribution of wealth at the start of the second date.

Constructing examples of non-existence thus reduces to construction examples of multiple equilibria in standard exchange economies — in this case with two commodities and two individuals. Mas-Colell (1991) gives an overview of sufficient conditions for global uniqueness in these economies. If individual endowments are not collinear, only very strong conditions on preferences ensure global uniqueness. In general overlapping generation models with intergenerational trade, the problem seems even harder, and no conditions are known that ensure global uniqueness. We construct two explicit examples of economies with multiple equilibria, when we add a first period to these examples, the realization of the shock in the first period then determines which of the possible equilibria realizes.

The first example is chosen to ensure that one can solve analytically for all competitive equilibria. In the second example, which is from Kehoe (1992), individuals have homothetic preferences, an assumption commonly made in the applied literature to incorporate growth. There are many other examples of multiple equilibria in standard exchange economies in the literature, for example, in Shapley and Shubik (1977), and these examples can be used as well to construct examples of non-existence of recursive equilibria.

Example 1

Utilities are independent of the shock, which only affects the endowments of individuals.

The intertemporal utility function of individual $(1, \sigma_t)$ is

$$U^{1,\sigma_t} = -\frac{1}{x_1(\sigma_t)} + \mathrm{E}_{s_t}\left[x_1(\sigma_t, s_{t+1}) - \frac{1}{x_2(\sigma_t, s_{t+1})}\right],$$

and for individual $(2, \sigma_t)$,

$$U^{2,\sigma_t} = -\frac{1}{x_1(\sigma_t)} + \mathrm{E}_{s_t}\left[-\frac{1}{x_1(\sigma_t, s_{t+1})} + x_2(\sigma_t, s_{t+1})\right].$$

The endowments of individuals in the first date of their lives are stochastic, while in the second they are not.

For individual $(1, \sigma_t)$,

$$e^{1,\sigma_t}(\sigma_t) = (e_1^1(s_t), 0), \quad e^{1,\sigma_t}(\sigma_{t+1}) = (\kappa^1, 0),$$

and for individual $(2, \sigma_t)$,

$$e^{2,\sigma_t}(\sigma_t) = (e_1^2(s_t), 0), \quad e^{2,\sigma_t}(\sigma_{t+1}) = (0, \kappa^2),$$

with

$$\kappa^2 > \frac{e_1^2(s_t)}{e_1^1(s_t)} > \kappa^1.$$

Note that this example does not satisfy Assumption 1.4. However, it can be easily seen that this does not cause the problems below.

The economy decomposes into sub-economies, σ_t. Economies associated with different realizations of the shock, s_t, are different, since the shock affects the endowments of individuals in the first date.

Randomness translates into heterogeneity across sub-economies, which, however, are not subject to uncertainty.

A perfect capital market is operative in each sub-economy; the only asset needed is a risk-free bond.

The competitive equilibrium at each sub-economy is unique — however the prices and allocations in the second period depend on the realization of the endowment shock in the first period.

The construction is as follows:

1. There is no trade in commodity $(2, \sigma_t)$.
2. The value, discounted at σ_t, of the aggregate demand of the old for the numéraire commodity at σ_{t+1} is

$$q(\sigma_t)x^{1,\sigma_t}(1, \sigma_{t+1}) + q(\sigma_t)x^{2,\sigma_t}(1, \sigma_{t+1}) =$$
$$e_1^1(s_t) + q(\sigma_t)\kappa^1 - \sqrt{q(\sigma_t)} - q(\sigma_t)\sqrt{p(\sigma_{t+1})} + q(\sigma_t)\sqrt{p(\sigma_{t+1})},$$

where, by a slight abuse of notation, $p(\sigma_{t+1})$ is the price of the non-numéraire commodity, for which, the value of the aggregate demand is

$$q(\sigma_t)p(\sigma_{t+1})x^{1,\sigma_t}(2,\sigma_{t+1}) + q(\sigma_t)p(\sigma_{t+1})x^{2,\sigma_t}(2,\sigma_{t+1}) =$$
$$q(\sigma_t)\sqrt{p(\sigma_{t+1})} + e_1^2(s_t) + q(\sigma_t)p(\sigma_{t+1})\kappa^2 -$$
$$\sqrt{q(\sigma_t)p(\sigma_{t+1})} - q(\sigma_t)\sqrt{p(\sigma_{t+1})}.$$

3. Since the value of the supply (among the old) of the numéraire commodity is $q(\sigma_t)\kappa^1$, and of the non-numéraire commodity $q(\sigma_t)p(\sigma_{t+1})\kappa^2$, the unique competitive equilibrium prices are

$$q(\sigma_t) = [e_1^1(s_t)]^2, \quad p(\sigma_{t+1}) = \left[\frac{e_1^2(s_t)}{e_1^1(s_t)}\right]^2.$$

While individuals' portfolios are independent of the exogenous shock, the price of the non-numéraire commodity is determined by the realization of the shock at the predecessor of a date-event. Recursive equilibria fail to exist.

Example 2

Utilities are independent of the shock, which only affects the endowments of individuals.

The intertemporal utility function of individual $(1,\sigma_t)$ is

$$U^{1,\sigma_t} = -\frac{1024}{x_1^4(\sigma_t)} + \mathrm{E}_{s_{t+1}|s_t}\left[-\frac{1024}{x_1^4(\sigma_{t+1})} - \frac{1}{x_2^4(\sigma_{t+1})}\right],$$

and for individual $(2,\sigma_t)$,

$$U^{2,\sigma_t} = -\frac{1}{x_1^4(\sigma_t)} + \mathrm{E}_{s_{t+1}|s_t}\left[-\frac{1024}{x_2^4(\sigma_{t+1})} - \frac{1}{x_1^4(\sigma_{t+1})}\right].$$

The endowments of individuals in the first date of their lives are stochastic, while in the second they are not.

For individual $(1,\sigma_t)$,

$$e^{1,\sigma_t}(\sigma_t) = \begin{cases} (10.4, 0), & \text{if } s_t = s_1, \\ (8.6313, 0), & \text{if } s_t = s_2, \end{cases} \qquad e^{1,\sigma_t}(\sigma_{t+1}) = (12, 1),$$

and for individual $(2,\sigma_t)$,

$$e^{2,\sigma_t}(\sigma_t) = \begin{cases} (2.6, 0), & \text{if } s_t = s_1, \\ (4.3687, 0), & \text{if } s_t = s_2, \end{cases} \qquad e^{2,\sigma_t}(\sigma_{t+1}) = (1, 12).$$

As above, there is no intergenerational trade, and equilibria decompose into equilibria for sub-economies.

A perfect capital market is operative in each sub-economy; the only asset needed is a risk-free bond.

The competitive equilibrium at each sub-economy is unique.

The construction is as follows:

1. There is no trade in commodity $(2, \sigma_t)$.
2. Since the aggregate supply of the numéraire commodity is the same at the two dates and the cardinal utility index for the commodity at the two dates is the same, the price of the risk-free bond is $q(\sigma_t) = 1$.
3. The aggregate excess demand for the non-numéraire commodity at date 2 is

$$\frac{e^{1,\sigma_t}(s_t) + 12 + p(\sigma_{t+1})}{p(\sigma_{t+1})^{1/5}(8 + p(\sigma_{t+1})^{4/5})} + \frac{4(e^{2,\sigma_t}(s_t) + 1 + 12p(\sigma_{t+1}))}{p(\sigma_{t+1})^{1/5}(2 + 4p(\sigma_{t+1})^{4/5})} - 13 = 0.$$

4. For $s = 1, 2$ competitive equilibria are unique.
5. Whenever $s_t = 1$, prices of both elementary securities are equal to $1/2$, and the spot price of the non-numéraire commodity in the following date is $p(\sigma_{t+1}) = 1$; the consumption of individuals in the second dates of their lives is (10.4,2.6), for individual $i = 1$, and (2.6,10.4), for individual $i = 2$.
   Whenever $s_t = 2$, the prices of both elementary securities are equal to $1/2$, and the spot price of the non-numéraire commodity in the following date is be $p(\sigma_{t+1}) = 7.8574$; the consumption of individuals in the second date of their lives is (8.6313,1.4288) for individuals $i = 1$, and (4.3687,11.5812), for individuals $i = 2$.

As in Example 1, the failure of existence of a recursive equilibrium relies on the fact that, in the sub-economy, at date 2, there are multiple spot-market equilibria. This multiplicity in the second period is robust, which is not the case in Example 1 — a perturbation in second period endowments leads to a unique second period equilibrium in that example.

4 Generalized Markov equilibria

In this section we define an equilibrium set as the set of all endogenous variables that realize in a competitive equilibrium. We prove that, under our assumptions on preferences, technologies, assets and endowments generalized Markov equilibria always exist and that the equilibrium set is closed and bounded. The method of proof follows Duffie et al. (1994). We first state a simplified version of a proposition there that allows us to establish existence by verifying that equilibria exist for truncated economies and that equilibrium variables remain uniformly bounded for all truncations.

The structure of the equilibrium set shall imply the existence of Markov ϵ-equilibria.

4.1 Equilibrium sets for expectation correspondences

The state-space consist of all exogenous and endogenous variables that describe the state of the economy at some date-event:

$$\Omega = \mathcal{S} \times \mathcal{Z},$$

where $\mathcal{S}$ is the finite set of exogenous shocks and $\mathcal{Z}$, a subset of Euclidean space, is a comprehensive set of possible values for the endogenous variables at any date-event.

An expectations correspondence,

$$G : \Omega \Rightarrow \boldsymbol{\mathcal{Z}}, \quad \boldsymbol{\mathcal{Z}} = \bigotimes_{s \in \mathcal{S}} \mathcal{Z},$$

embodies all short run equilibrium conditions: (first order) conditions for individual optimization and market clearing conditions; to every pair of a realization of the shock and values of endogenous variables, it assigns values of the endogenous variables at all possible realization of the shock at the following date.

An equilibrium set for an expectations correspondence, G, is a compact subset $\boldsymbol{\mathcal{Z}}^* \subset \boldsymbol{\mathcal{Z}}$, such that

$$[(z_1, ..., z_S) \in \boldsymbol{\mathcal{Z}}^*] \quad \Rightarrow \quad [G(s, z_s) \cap \boldsymbol{\mathcal{Z}}^* \neq \emptyset, \text{ for all } s \in \mathcal{S}].$$

A T-horizon equilibrium for the correspondence G consists of a subset $\tilde{\Omega} \subset \Omega$ and $(s_t, z(\sigma_t)) : t = 1, \ldots, T,\ z(\sigma_t) \in G_{s_t}(s_{t-1}, z(\sigma_{t-1}))$, that satisfies $(s_t, z(\sigma_t)) \in \tilde{\Omega}$; the set $\tilde{\Omega}$ supports the equilibrium.

Proposition 1. *If G is an expectations correspondence, such that*

1. *there exists a compact subset $\mathcal{K} \subset \Omega$ that supports a T-horizon equilibrium for $T = 1, \ldots$, and*
2. *the graph of G is closed,*

then, an equilibrium set for G exists.

Proof. Define sets $\mathcal{T}_0 = \mathcal{K}$, and, recursively,

$$\mathcal{T}_n = \{(\bar{s}, \bar{z}_{\bar{s}}) \in \mathcal{K} : \text{ there exists } (z_1, ..., z_S) \in G(\bar{s}, \bar{z}_{\bar{s}}), \text{ with } (s, z_s) \in \mathcal{T}_{n-1}\}.$$

Since, by assumption, the expectations correspondence has a closed graph and since $\mathcal{K}$ is compact each $\mathcal{T}_n$ is compact.

We show by induction that for each $n > 1$, $\mathcal{T}_n \subset \mathcal{T}_{n-1}$. By definition $\mathcal{T}_1 \subset \mathcal{T}_0$. If $\mathcal{T}_n \subset \mathcal{T}_{n-1}$ and for some $(\bar{s}, \bar{z}) \in \mathcal{K}$ there exists $(z_1, ..., z_S) \in G(\bar{s}, \bar{z})$ with

$$(s, z_s) \in \mathcal{T}_n \subset \mathcal{T}_{n-1},$$

then, obviously, $(\bar{s}, \bar{z}) \in \mathcal{T}_n$ and $\mathcal{T}_{n+1} \subset \mathcal{T}_n$.

Since, by assumption, all $\mathcal{T}_n$ are non-empty, the set

$$\Omega^* = \bigcap_{n=0}^{\infty} \mathcal{T}_n$$

is non-empty. Define $\boldsymbol{\mathcal{Z}}^*$ as follows:

$$\boldsymbol{\mathcal{Z}}^* = \{z = (z_1, ..., z_S) : (s, z_s) \in \Omega^* \text{ for all } s \in \mathcal{S}\}$$

It follows from the construction that this is an equilibrium set for G □

Existence of competitive equilibria

For the stochastic economy of overlapping generations, endogenous state variables are portfolios and stored commodities at the preceding date and current excess demand for commodities (in excess of individual endowments and net storage), portfolios, storage decisions, production decisions and prices for commodities and assets:

$$\mathcal{Z} = \left\{ (\theta_-, \phi_-, \xi, y, \theta, \phi, p, q) : \begin{array}{l} \sum_{i\in\mathcal{I}} \sum_{a=0}^{N-1} \theta_1^{i,a} = \mathbf{0}^{J_1}, \\ \sum_{i\in\mathcal{I}} \sum_{a=0}^{N-1} \theta_2^{i,a} = \mathbf{1}^{J_2} \end{array} \right\}.$$

The expectations correspondence is defined by

$$G(s,z) = \left\{ z : \begin{array}{l} \theta_-^{i,a}(s) = \bar{\theta}^{i,a-1}, \quad \phi_-^{i,a}(s) = \bar{\phi}^{i,a-1}, \\ p(s)\xi^{i,0}(s) = -\theta^{i,0}(s)q(s), \\ p(s)\xi^{i,a}(s) = (1, q(s))D(s)\theta_-^{i,N}(s) - \theta^{i,a}(s)q(s), \\ p(s)\xi^{i,N}(s) = (1, q(s))D(s)\theta_-^{i,N}(s), \\ -\bar{q}_j\bar{\lambda}^{i,a} + \beta \mathrm{E}_{s|\bar{s}} d_{j,1}(s)\lambda^{i,a}(s) = 0, \qquad j \in \mathcal{J}_1, \\ -\bar{q}_j\bar{\lambda}^{i,a} + \beta \mathrm{E}_{s|\bar{s}} (q_j(s) + d_{j,1}(s))\lambda^{i,a}(s) \le 0, \quad j \in \mathcal{J}_2, \\ -\bar{q}_j\bar{\lambda}^{i,a} + \beta \mathrm{E}_{s|\bar{s}} (q_j(s) + d_{j1}(s))\lambda^{i,a}(s) = 0, \\ \quad j \in \mathcal{J}_2, \quad \bar{\theta}_j^{i,a-1} > 0, \\ -\bar{p}_l\bar{\lambda}^{i,a} + \beta \mathrm{E}_{s|\bar{s}} \delta_l(s) p_l(s)\lambda^{i,a}(s) \le 0, \\ -\bar{p}_l\bar{\lambda}^{i,a} + \beta \mathrm{E}_{s|\bar{s}} \delta_l(s) p_l(s)\lambda^{i,a}(s) = 0, \quad \bar{\phi}_l^{i,a-1} > 0, \\ D_x u^{i,a}(x^{i,a}(s), s) - \lambda^{i,a}(s)p(s) \ge 0, \\ D_{x_l} u^{i,a}(x^{i,a}(s), s) - \lambda^{i,a}(s)p_l(s) = 0, \quad x_l^{i,a}(s) > 0, \\ p(s)\cdot\tilde{y} > p(s)\cdot y(s) \Rightarrow \tilde{y} \notin Y(s), \\ \sum_{i\in\mathcal{I}} \sum_{a=0}^{N} \xi^{i,a}(s) = y(s) + \sum_{j\in\mathcal{J}_2} (d_{j,1}(s), \mathbf{0}^{L-1}). \end{array} \right\}$$

where

$$x^{i,a}(s) = e^{i,a}(s) + \xi^{i,a}(s) - \phi^{i,a}(s), \quad a = 0,$$
$$x^{i,a}(s) = e^{i,a}(s) + \xi^{i,a}(s) - \phi^{i,a}(s) + \Delta(s)\bar{\phi}^{i,a-1}, \quad a = 1, ..., N-1,$$
$$x^{i,a}(s) = e^{i,a}(s) + \xi^{i,a-1}(s) + \Delta(s)\bar{\phi}^{i,a}, \quad a = N, \quad \text{and}$$
$$\bar{p}_l\bar{\lambda}^{i,a} = D_{x_l} u^{i,a}(e^{i,a-1}(\bar{s}) + \bar{\xi}^{i,a-1}, \bar{s}) \text{ for } e_l^{i,a-1}(\bar{s}) + \bar{\xi}_l^{i,a-1} > 0.$$

In order to prove that this expectations correspondence has an equilibrium set, we apply Proposition 1.

T-horizon equilibria for the expectations correspondence, G, are competitive equilibria for a truncated economy.

The T-truncated economy prohibits trades in commodities after date T and trade in assets after date $T-1$; the consumption of individuals coincides, after T, with their endowment, including the payoffs of long-lived assets. With no loss of generality, the truncated economy consists of individuals born before date T, while the separability of preferences makes reference to consumption after date T unnecessary.

At a competitive equilibrium, individuals optimize, and markets for commodities and assets clear.

For every T, competitive equilibria exist; we outline the argument, which is standard — Geanakoplos and Polemarchakis (1986) and Radner (1972).

Economic activity terminates at date T; the utility functions and endowments of individuals born after $T-N$ are adjusted accordingly.

At each date-event, prices of commodities are non-negative: $p_T(\sigma_t) \geq 0$, and prices of assets lie in the closure of the set that does not not allow for arbitrage: $q_T(\sigma_t) = \sum \mu(\sigma_{t+1}|\sigma_t)(p_{T,1}(\sigma_{t+1}), q_T(\sigma_{t+1}))D(\sigma_{t+1})$, for some $\mu(\sigma_t) \gg 0$. The characterization of non-arbitrage is nested, as is required by long-lived assets; prices of assets vanish at terminal date-events, while the distinction between short-lived and long-lived assets vanishes at date $T-1$. Normalized prices of commodities and assets satisfy $\|(p_T(\sigma_t), q_T(\sigma_t))\| = 1$; importantly, here normalization does not single out commodity $l = 1$ as numéraire.

We restrict attention to normalized prices with $p_{T,1}(\sigma_t) \geq \epsilon > 0$; and we impose an upper bound $k > 0$, on net trades in commodities and assets by individuals and commodities by firms.

Given bounds k and ϵ, the budget correspondences of every individual is continuous.

It then follows from a standard fixed point argument that there exist for every individual, $x^{i,\sigma_t}_{T,\epsilon,k}$, $\theta^{i,\sigma_t}_{T,\epsilon,k}$, and $\phi^{i,\sigma_t}_{T,\epsilon,k}$, and, for every firm, $y^{i,\sigma_t}_{T,\epsilon,k}$, as well as prices of commodities and assets $p_{T,\epsilon,k}$ and $q_{T,\epsilon,k}$, such that individuals and firms optimize, subject to the additional bound on net trades, while, at every date-event,

$$(p_{T,\epsilon,k}(\sigma_t), q_{T,\epsilon,k}(\sigma_t)) \in \arg\max\{p(\sigma_t)\xi_{T,\epsilon,k}(\sigma_t) + q(\sigma_t)(\zeta_{T,\epsilon,k}(\sigma_t)) : p_1(\sigma_t) \geq \epsilon\},$$

where

$$\begin{aligned}\xi_{T,\epsilon,k}(\sigma_t) &= \textstyle\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}(x^{i,\sigma_{t-a}}_{T,\epsilon,k}(\sigma_t) - e^{i,\sigma_{t-a}}(\sigma_t)) - \\ &\quad \textstyle\sum_{j\in\mathcal{J}_2}(d_{j,1}(s_t), \mathbf{0}^{L-1}) + y_{T,\epsilon,k}(\sigma_t) - \\ &\qquad \textstyle\sum_{i\in\mathcal{I}}\left(\sum_{a=1}^{N}\Delta(s_t)\phi^{i,\sigma_{t-a}}_{T,\epsilon,k}(\sigma_{t-1}) - \sum_{a=0}^{N-1}\phi^{i,\sigma_{t-a}}_{T,\epsilon,k}(\sigma_t)\right),\end{aligned}$$

and

$$\begin{aligned}\zeta_{T,\epsilon,k}(\sigma_t) &= (\zeta_{T,\epsilon,k,1}(\sigma_t), \zeta_{T,\epsilon,k,2}(\sigma_t)) = \\ &\quad \textstyle(\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\theta^{i,\sigma_{t-a}}_{T,\epsilon,k,1}(\sigma_t), \sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\theta^{i,\sigma_{t-a}}_{T,\epsilon,k,2}(\sigma_t) - \mathbf{1}^{J_2})\end{aligned}$$

are, respectively, the aggregate excess demands for commodities and assets.

For fixed k, demands and prices for commodities and assets lie in a compact set, and, as a consequence, as $\epsilon \to 0$, they converge, up to a subsequence.

Since by the budget constraints,

$$p_{T,\epsilon,k}(\sigma_t)\xi_{T,\epsilon,k}(\sigma,t) + q_{T,\epsilon,k}(\sigma_t)\zeta_{T,\epsilon,k}(\sigma_t) \leq 0,$$

at the limit,

$$\xi_{T,k}(\sigma_t) = 0 \quad \text{and} \quad \phi_{T,k}(\sigma_t) = 0.$$

Continuity of demand requires that the price of the numéraire remain positive: at the limit, $p_{T,k}(\sigma_t) > 0$, at all date-events.

If, at some terminal terminal event, $p_{T,\epsilon,k,1}(\sigma_T) \to 0$, demand for the numéraire must explode, since utility is strictly increasing in that good and the price of some other commodity and, as a consequence, the revenue of individuals who commence their lives then, stay positive — no assets are traded at terminal events. If, at some non-terminal $p_{T,\epsilon,k,1}(\sigma_t) \to 0$, demand would only stay bounded if the prices of all other commodities in that date event tended to zero; but then, the price of some asset would remain non-zero; if this asset were long-lived, some individuals with positive holdings could trade the asset for commodity 1, while, if the asset were short-lived, individuals at the first period of their lives would trade it for commodity 1.

By market clearing and the independence of the payoffs of short-lived assets, the bounds on trades do not bind, for k sufficiently large, and $(p_T, q_T) = (p_{k,T}, q_{k,T})$ are competitive equilibrium prices for the truncated economy.

It now suffices to establish that there exist uniform bounds such that for every T all equilibrium asset holdings and asset prices lie within these bounds — uniform bounds on consumption and production at equilibrium are immediate.

Since individuals have finite lives and there are short-sale constraints for the long-lived assets bounds on portfolios are immediate. By the assumption on utility, if the price of some commodity or some asset would be sufficiently large, individuals who in the equilibrium consume the commodity or are able to sell some the asset can do so and strictly increase their utility by buying some of commodity 1.

The expectations correspondence, G has a closed graph.

From Proposition 1 and the fact that first order conditions are necessary and sufficient for optimization, it follows that that there exists a generalized Markov equilibrium where portfolios, prices and consumptions today just depend on the wealth distribution, the exogenous shocks as well as the shock, asset prices and consumptions at the predecessor node.

5 Markov ϵ-equilibria

In applied computational work it is well understood that exact equilibria can generally not be computed in finite time. This motivates the following definition.

An ϵ-equilibrium for initial holdings of assets $\overline{\theta}^{i,-a}$ and commodities $\overline{\phi}^{i-a}$, for $\epsilon > 0$, is a collection of consumption plans, production plans, portfolio holdings, storage decisions and prices of commodities and assets, $(x^{i,\sigma_t}, \theta^{i,\sigma_t}, \phi^{i,\sigma_t}, y^{\sigma_t}, p, q)$, such that

1. all markets clear at every date-events:

$$\begin{aligned}&\textstyle\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}(x^{i,\sigma_{t-a}}(\sigma_t)-e^{i,\sigma_{t-a}}(\sigma_t)) =\\ &\qquad\textstyle\sum_{j\in\mathcal{J}_2}(d_{j,1}(s_t),\mathbf{0}^{L-1})+y(\sigma_t)+\\ &\qquad\qquad\textstyle\sum_{i\in\mathcal{I}}\left(\sum_{a=1}^{N}\Delta(s_t)\phi^{i,\sigma_{t-a}}(\sigma_{t-1})-\sum_{a=0}^{N-1}\phi^{i,\sigma_{t-a}}(\sigma_t)\right),\\ &\textstyle\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\theta_1^{i,\sigma_{t-a}}(\sigma_t)=\mathbf{0}^{J_1},\quad \sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\theta_2^{i,\sigma_{t-a}}(\sigma_t)=\mathbf{1}^{J_2};\end{aligned}$$

2. firm optimize:

$$y^{\sigma_t}\in\arg\max\{p(\sigma_t)y : y\in Y^{\sigma_t}\};$$

3. individuals ϵ-optimize:

$$\|U^{i,\sigma_t}(x^{i,\sigma_t})-\max\{U^{i,\sigma_t}(x):(x,\theta,\phi)\in\mathcal{B}^{i,\sigma_t}(p,q)\}\|<\epsilon.$$

And analogously,

Definition 2. *A Markov ϵ-equilibrium is described by a set of holdings of assets and stored commodities, $\mathcal{H}$, that contains an open set, and a collection of functions, $\{f_x, f_y, f_\theta, f_\phi, f_p, f_q\}$, with domain $\mathcal{H}\times\mathcal{S}$, such that for every $(\overline{\theta},\overline{\phi})\in\mathcal{H}$, there exist an ϵ-equilibrium with*

$$\begin{aligned}&x^{i,\sigma_t}(\sigma_{t+a})=f_x^{i,a}(\theta(\sigma_{t+a-1}),\phi(\sigma_{t+a-1}),s_t)),\\ &y^{\sigma_t}(\sigma_t)=f_y(\theta(\sigma_{t-1}),\phi(\sigma_{t-1}),s_t)),\\ &\theta^{i,\sigma_t}(\sigma_{t+a})=f_\theta(\theta(\sigma_{t+a-1}),\phi(\sigma_{t+a-1}),s_t),s_t),\\ &\phi^{i,\sigma_t}(\sigma_{t+a})=f_\phi(\theta(\sigma_{t+a-1}),\phi(\sigma_{t+a-1}),s_t),\\ &p(\sigma_t)=f_p(\theta(\sigma_{t-1}),\phi(\sigma_{t-1}),s_t),\\ &q(\sigma_t)=f_q(\theta(\sigma_{t-1}),\phi(\sigma_{t-1}),s_t).\end{aligned}$$

In numerical work the quality of an approximate solution is often evaluated by imposing market clearing and reporting the relative errors in individuals' first order conditions — Judd (1998) refers to these errors as "Euler equation residuals." In some cases, if one discretizes the state-space, for example, the maximum relative error can be determined over the entire domain of the recursive equilibrium functions, $\mathcal{H}$. Since, with overlapping generations, individuals are finitely-lived, one can derive an explicit upper bound on their optimization error from the maximum Euler equation residual, as in Santos (2000). Computational work that reports maximal errors in Euler equations, indeed computes ϵ-equilibria.

5.1 Existence of Markov ϵ-equilibria

While stationary Markov equilibria may fail to exist, Markov ϵ-equilibria exist, for all $\epsilon > 0$.

Theorem 1. *For the stochastic economy of overlapping generations, a Markov ϵ-equilibrium exists, for any $\epsilon > 0$.*

Proof. Given a (compact) equilibrium set, $\boldsymbol{\mathcal{Z}}^*$, for an expectations correspondence, G, there exists, for any $\delta > 0$, a finite collection of points, $\mathcal{F}^\delta$, such that

$$\sup_{\zeta \in \boldsymbol{\mathcal{Z}}^*} \inf_{\xi \in \mathcal{F}^\delta} \|\xi - \zeta\| < \delta,$$

and such that there exist a set of beginning-of-period portfolio holdings and stored commodities $\mathcal{T}$ and a function $w : \mathcal{S} \times \mathcal{T} \to \mathcal{Z}^N$, with

$$\mathcal{F}^\delta = \text{graph}(w);$$

the former follows from compactness of $\mathcal{Z}^*$ and the latter from the finiteness of $\mathcal{F}$.

For each $(\bar{z}_1, \dots, \bar{z}_S) \in \mathcal{F}^\delta$, each $\bar{s} \in \mathcal{S}$, choose

$$\xi = (z_1, \dots, z_S) \in \mathcal{F}^\delta \text{ such that } \inf_{\zeta \in G(\bar{s}, \hat{z})} |\zeta - \xi| < \delta$$

for $\hat{z}$ with $|\hat{z} - \bar{z}_{\bar{s}}| < \delta$

Write $z_s = (\theta_-, \phi_-, \xi, y, \theta, \phi, p, q)$ and

$$\bar{z}_{\bar{s}} = (\bar{\theta}_-, \bar{\phi}_-, \bar{\xi}, \bar{y}, \bar{\theta}, \bar{\phi}, \bar{p}, \bar{q}),$$

and define z^δ by replacing the new portfolios in $\bar{z}_{\bar{s}}$, $\bar{\theta}$ by θ_-,, replacing the new storage decisions in $\bar{z}_{\bar{s}}$, $\bar{\phi}$ by ϕ_- and adjusting aggregate excess demand (and hence consumption), $\bar{\xi}$, to ensure that the budget constraints hold. Collect all z^δ in $\mathcal{F}^{*\delta}$. Since demand and utility functions are continuous and prices and choices are bounded, for each $\epsilon > 0$, there exists a $\delta > 0$, such that $\mathcal{F}^{*\delta}$ is the graph of functions describing a Markov ϵ-equilibrium. □

5.2 Convergence of ϵ-equilibria to competitive equilibria

The question arises what happens to Markov ϵ equilibria as $\epsilon \to 0$. If some ϵ equilibria were to converge to allocations and prices that are not equilibria, the use of ϵ- equilibria to approximate exact stationary would be questionable.

It turns out that, as $\epsilon \to 0$, all Markov ϵ- equilibria converge (in allocations) to exact equilibria.

The intuition behind the result can be easily illustrated via Example 2, above, where exact Markov equilibria fail to exist.

We can construct a sequence of Markov ϵ-equilibria that converges to the unique competitive equilibrium as follows:

Let $\theta^i(s)$ denote individual i's bond-purchase when young in shock s and define $\theta^1(s_1) = \delta$, $\theta^2(s_1) = -\delta$, $\theta^1(s_2) = \theta^2(s_2) = 0$.

Define consumption in shock s_1 when young by

$$x^{1,\sigma_t}(\sigma_t) = (10.4 - \delta, 0), \quad x^{2,\sigma_t}(\sigma_t) = (2.6 + \delta, 0),$$

and consumption in shock s_2, when young by

$$x^{1,\sigma_t}(\sigma_t) = (6.6313, 0), \quad x^{2,\sigma_t}(\sigma_t) = (4.3687, 0).$$

Define consumption when old contingent on the portfolio holdings. If individuals' portfolios are zero,

$$x^{1,\sigma_t}(\sigma_{t+1}) = (8.6313, 1.4288), \quad x^{2,\sigma_t}(\sigma_{t+1}) = (4.3687, 11.5812);$$

if individuals' portfolios are $(\delta, -\delta)$,

$$x^{1,\sigma_t}(\sigma_{t+1}) = (10.4 + \delta, 2.6), \quad x^{2,\sigma_t}(\sigma_{t+1}) = (2.6 - \delta, 10.4).$$

Prices are as the prices of the exact equilibrium in Example 2.

It can be easily verified that, for each $\epsilon > 0$, there exists a $\delta > 0$, such that the constructed allocations and prices are a Markov ϵ-equilibrium.

Furthermore, the approximation of exact competitive equilibria via a sequence of Markov ϵ-equilibria is always possible; and there can be no sequences of ϵ-equilibria that converge to something other than a competitive equilibrium.

Theorem 2. *For any $\delta > 0$, there exists an $\epsilon > 0$, such that, for every ϵ- equilibrium with allocation x^ϵ, there exists an exact equilibrium with allocation $\bar{x}$, such that*

$$\sup_\sigma \|\bar{x}(\sigma) - x^\epsilon(\sigma)\| < \delta.$$

Proof. For any Markov ϵ-equilibrium, there exists an allocation $\tilde{x}^\epsilon$, such that all individuals maximize their utility subject to the budget constraint at the given (ϵ-equilibrium) prices, and such that

$$\begin{aligned}&\| \textstyle\sum_{i\in\mathcal{I}}\sum_{a=0}^{N}\left(\tilde{x}^{\epsilon\ i,\sigma_{t-a}}(\sigma_t) - e^{i,\sigma_{t-a}}(\sigma_t)\right) - \sum_{j\in\mathcal{J}_2}(d_{j,1}(s_t), \mathbf{0}) - y(\sigma_t) + \\ &\quad \textstyle\sum_{i\in\mathcal{I}}\left(\sum_{a=1}^{N}\Delta(s_t)\phi^{i,\sigma_{t-a}}(\sigma_{t-1}) - \sum_{a=0}^{N-1}\phi^{i,\sigma_{t-a}}(\sigma_t)\right)\| < \delta_1(\epsilon),\\ &\quad\quad \sigma_t = (\sigma_{t-1}, s_t),\end{aligned}$$

as well as

$$\sup_\sigma \|x^\epsilon(\sigma) - \tilde{x}^\epsilon(\sigma)\| < \delta_2(\epsilon),$$

where $\delta_i(\epsilon) \to 0$, as $\epsilon \to 0$, for $i = 1, 2$.

As $\delta_1 \to 0$, the constructed equilibria converge to exact equilibria. □

The limit equilibria might not be Markov: the examples above prove this.

5.3 Generalized Markov and ϵ-equilibria equilibria

The discussion in Section 4 implies that generalized Markov equilibria as in Duffie et al. (1994) also exist in our setting. Duffie et al. use the generalized Markov equilibrium to construct a stationary Markov equilibrium with sunspots. Choices and prices depend on the exogenous shock, on the beginning-of-period portfolio holdings and on the realization of the sunspot variable. The latter are therefore used to convexify over possible continuation equilibria.

However, for computational purposes the existence of generalized Markov equilibria or of a Markov equilibrium with sunspots is not useful. Standard recursive methods naturally use the distribution of wealth across individuals as state variables. Variables that are not pay-off relevant (such as sunspots) cannot be included in a recursive setup.

We therefore focus on Markov ϵ-equilibria. From a theoretical perspective, there is little justification for considering these equilibria (as opposed to considering sunspot Markov equilibria, for example). However, as we argued above, applied work computes Markov ϵ-equilibria.

Markov ϵ-equilibria do not necessarily approximate generalized Markov equilibria. Any competitive equilibrium with a bounded equilibrium set can be approximated by a (sequence of) Markov ϵ-equilibria. This raises the questions whether the multiplicity of Markov ϵ-equilibria affects the validity of applied computational work for comparative statics and whether some ϵ equilibria (for example, those that approximate generalized Markov equilibria) are more interesting than others. It is beyond the scope of this paper to investigate this issue further.

Notation

- "$'$" is the transpose; unless otherwise indicated, vectors are column vectors.
- A vector $a = (\dots, a_k, \dots)$, is non-negative: $a \geq 0$, if $a_k \geq 0$, for every k; it is positive: $a > 0$, if $a_k \geq 0$, for every k, with strict inequality, $a_k > 0$, for some; it is strictly positive: $a \gg 0$, if $a_k > 0$, for every k; analogously, the vector a is non-positive: $a \leq 0$, negative: $a < 0$, and strictly negative: $a \ll 0$. For vectors a and b, $a \geq b$ if $(a - b) \geq 0$, $a > b$ if $(a - b) > 0$, and $a \gg b$ if $(a - b) \gg 0$; analogously, $(a - b) \leq 0$, $(a - b) < 0$, and $(a - b) \ll 0$.
- For "k, a real number, $a_+ = \max\{a, 0\}$, and $a_- = -\min\{a, 0\}$; for $a = (\dots, a_k, \dots)$, vector, $a_+ = (\dots, a_{k+}, \dots)$, and $a_- = (\dots, a_{k-}, \dots)$.
- "$\mathbf{1}_k^K$" is k-th column, unit vector of dimension K.
- "$\mathbf{1}^K$" is the column vector of 1's of dimension K; similarly for "$\mathbf{0}^K$," though simply 0 simplifies notation when confusion would not arise.
- "I^K" is the identity matrix of dimension K.
- "$\mathbb{R}^k$ is Euclidean space of dimension k — for simplicity, one writes $\mathbb{R}$ for $\mathbb{R}^1$; the non-negative orthant is $\mathbb{R}^k_+$, and its interior, the strictly positive orthant, is $\mathbb{R}^k_{++}$.
- If g, is a function of $(\dots, y_k, \dots)$, then "g_{y_k}" is the function defined by $g_{y_k}(\dots, y_{k-1}, y_{k+1}, \dots) = g(\dots, y_{k-1}, y_k, y_{k+1}, \dots)$.
- "$D_y g$" is the gradient of a function, g, with respect to y — for simplicity, one writes Dg; if $y = (\dots, y_k, \dots)'$, then

$$D_y g = \left(\dots, \frac{\partial g}{\partial y_k}, \dots\right);$$

if $g = (\dots, g_l, \dots)'$, then

$$D_y g = (\dots, D_{y_k} g, \dots) = \begin{pmatrix} & \vdots & \\ \dots & \frac{\partial g_l}{\partial y_k} & \dots \\ & \vdots & \end{pmatrix}.$$

is the Jacobean matrix.

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