

Information at equilibrium^{*}

E. Minelli¹ and H. Polemarchakis²

¹ CORE, Université Catholique de Louvain, 1348 Lovain-la-Neuve, BELGIUM
(e-mail: minelli@core.ucl.ac.be)

² Department of Economics, Brown University, Providence, RI 02912, USA
(e-mail: herakles_polemarchakis@brown.edu)

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Summary. In a game with rational expectations, individuals simultaneously refine their information with the information revealed by the strategies of other individuals. At a Nash equilibrium of a game with rational expectations, the information of individuals is essentially symmetric: the same profile of strategies is also an equilibrium of a game with symmetric information; and strategies are common knowledge. If each player has a veto act, which yields a minimum payoff that no other profile of strategies attains, then the veto profile is the only Nash equilibrium, and it is an equilibrium with rational expectations and essentially symmetric information; which accounts for the impossibility of speculation.

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1 Introduction

Private information may differ across individuals: it may be asymmetric.

If expectations are rational, individuals refine their information with the information revealed by the acts of others. If an event is common knowledge, individuals know it has occurred, they know that others know it, they know that others know that they know it,

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Thus, at a rational expectations equilibrium as well as for an event which is common knowledge, gains to knowledge from the exchange of information are exhausted.

Rational expectations were formalized by Radner (1979) in the context of walrasian equilibria. Common knowledge was formalized by Aumann (1976) without reference to the optimizing or strategic behavior of individuals.

Both rational expectations and common knowledge are powerful conceptual tools that lead to surprising and often similar conclusions. The impossibility of speculative exchange has been claimed by Milgrom and Stokey (1982) as a consequence of the common knowledge of individuals of their willingness to trade, and by Tirole (1982) as a property of equilibria with rational expectations.

Rational expectation is a property of the information of individuals at equilibrium. It applies across states of the world or of private information. Common knowledge is a property of events relative to the information of individuals. It applies at each state of the world or of private information.

The comparison of rational expectations and common knowledge should then be posed as follows: what events are common knowledge at a rational expectations equilibrium?

At a Nash equilibrium of a game with uncertainty and private information, according to the formalization of Harsanyi (1967 - 1968), individuals do not extract information from the acts of other individuals in the same round of play; this takes literally the simultaneity of moves. But it is naive.

At a Nash equilibrium of a game with rational expectations, individuals extract information from the simultaneous acts of other individuals.

Individuals know the strategies of others, and they observe their realized elementary acts; the no - regret condition that characterizes Nash equilibrium requires robustness to the new knowledge that individuals obtain at equilibrium. This is the motivation for the definition of a game with rational expectations.

A Nash equilibrium for a game with rational expectations abstracts from the strategic aspects of the revelation of information. This is in the spirit of Nash equilibrium, but may fall short of interpretation of rational expectations as a reduced form of a process of information revelation and learning; Dubey, Geanakoplos and Shubik (1987) and Forges and Minelli (1997a,b) explore the connection between dynamic or repeated games with asymmetric information and games with asymmetric information and rational expectations; Kalai (2000) discusses the issues involved.

At a Nash equilibrium of a game with rational expectations the information of individuals is essentially symmetric: any differences in information do not affect equilibrium acts; and the acts of individuals are common knowledge.

If the structure of payoffs in a games is such that at a Nash equilibrium, information is essentially symmetric, equilibria are *à fortiori* equilibria of the game with rational expectations, where the acts of individuals are common knowledge: this is the case for veto games, a powerful insight introduced and developed in Geanakoplos (1995).

Common knowledge of acts immediately implies consensus among individuals trying to guess the value of a random variable. Speculative behaviour, on the

other hand, cannot be ruled out at a rational expectations equilibrium of a game. Indeed, an example shows that common knowledge of the equilibrium acts need not imply common knowledge of speculation. Speculation can thus occur at the Nash equilibrium of a veto game.

In a strong veto game, every individual plays its veto act at a Nash equilibrium, speculation cannot occur and, as a consequence, it cannot occur at a competitive equilibrium.

2 Games with rational expectations

A game with private information is a collection

$$G_{\mathcal{P}} = \{\mathcal{I}, \mathcal{S}, (\mathcal{A}^i, u^i, \mathcal{P}^i) : i \in \mathcal{I}\}.$$

Individuals are $i \in \mathcal{I}$, a finite set. States of the world are $s \in \mathcal{S}$, a finite set. A profile of private information is $\mathcal{P} = \{\dots, \mathcal{P}^i, \dots\}$, where $\mathcal{P}^i = \{P^i(s) : s \in \mathcal{S}\}$, is a partition of the set of states of the world that represents the private information of the individual. For an individual, an elementary act is $a^i \in \mathcal{A}^i$, and an act or a strategy is f^i , an element of the set of feasible strategies ¹

$$\mathcal{F}^i = \{f^i : \mathcal{S} \rightarrow \mathcal{A}^i, \text{ measurable with respect to } \mathcal{P}^i\}.$$

Across individuals, a profile of elementary acts is $a \in \mathcal{A}$, where $a = (\dots, a^i, \dots)$, and $\mathcal{A} = \times_{i \in \mathcal{I}} \mathcal{A}^i$, and a profile of strategies is $f \in \mathcal{F} = \times_{i \in \mathcal{I}} \mathcal{F}^i$.

The utility or payoff to the individual at $f \in \mathcal{F}$ is $u^i(f)$, and his utility function is $u^i : \mathcal{F} \rightarrow \mathcal{R}$.

For an individual, the complementary set of individuals is $\{-i\} = \{\mathcal{I} \setminus \{i\}\}$. At strategies $f^{-i} \in \mathcal{F}^{-i}$ by the complementary set of individuals, where $f^{-i} = (\dots, f^{i-1}, f^{i+1}, \dots)$, $\mathcal{F}^{-i} = \times_{i' \in \{-i\}} \mathcal{F}^{i'}$, and $f = (f^i, f^{-i})$, the optimization problem of the individual is to

$$\begin{aligned} &\max u^i(f^i, f^{-i}), \\ &\text{s.t. } f^i \in \mathcal{F}^i. \end{aligned}$$

The solution to the optimization problem is $\varphi^i(f^{-i}) \subseteq \mathcal{F}^i$, which may be empty, when a solution does not exist, or not a singleton, when the solution is not unique. The choice or reaction correspondence is $\varphi^i : \mathcal{F}^{-i} \rightarrow \mathcal{F}^i$.

A Nash equilibrium is a profile of strategies, f^{I*} , such that $f^{i*} \in \varphi^i(f^{-i*})$, for every individual.

Individuals optimize ex - ante, prior to the resolution of uncertainty, and the utility function evaluates profiles of strategies. Under conditions that are well understood, Debreu (1959), the utility function of an individual is additively separable across states of the worlds: $u^i(f) = \sum_{s \in \mathcal{S}} v^i(f(s), s)$. Under stronger conditions, Savage (1954), it has a state - independent expected utility representation:

¹ A function $f : \mathcal{S} \rightarrow \mathcal{A}$ is measurable with respect to a partition \mathcal{P} if $P(s) = P(s') \Rightarrow f(s) = f(s')$.

$u^i(f) = E^i v^i(f(s))$, where the the probability measure under which expectations are computed is as much a characteristic that may vary across individuals as the cardinal utility index.

An ex - post formulation of games with uncertainty and private information is possible. For separable utility functions, the optimization problem of an individual at a state of the world is

$$\begin{aligned} \max \sum_{s' \in \mathcal{P}^i(s)} v^i((f^i(s), f^{-i}(s')), s'), \\ \text{s.t } f^i(s) \in \mathcal{A}^i. \end{aligned}$$

For non - separable utility functions, an ex - post formulation is possible but contrived. With this ex - post formulation of games with private information, the solutions to the individual optimization problems coincide for s and s' , with $\mathcal{P}^i(s) = \mathcal{P}^i(s')$. They yield unambiguously a solution to the ex - ante optimization problem, which, in particular, is measurable with respect to the information available to the individual ².

Information is symmetric if , for some partition, \mathcal{P}^0 , of the set of states of the world, $\mathcal{P}^i = \mathcal{P}^0$, for every individual: the information of individuals coincides. A game with uncertainty and symmetric information is

$$\mathcal{G}_{\mathcal{P}^0} = \{\mathcal{I}, \mathcal{S}, \mathcal{P}^0, (\mathcal{A}^i, u^i) : i \in \mathcal{I}\}.$$

If the utility functions of all individuals are separable across states of the world and information is symmetric, the game decomposes into a collection of games, indexed by the elements of the common partition.

At a Nash equilibrium of a game with private information, the information of individuals is essentially symmetric, with respect to a partition, \mathcal{P}^0 , if the profile of acts is a Nash equilibrium of the game with symmetric information \mathcal{P}^0 .

In a game with rational expectations, individuals refine their information with the information revealed by the elementary acts of other individuals at each state of the world.

For an individual, the feasible act correspondence is defined by ³

$$\Phi^i(f^{-i}) = \{f^i : \mathcal{S} \rightarrow \mathcal{A}^i, \text{ measurable with respect to } \mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^j}\}.$$

² An alternative ex-post formulation is for an individual to solve, at a state of the world, the optimization problem

$$\begin{aligned} \max \sum_{s' \in \mathcal{P}^i(s)} v^i((f^i(s), f^{-i}(s')), s'), \\ \text{s.t } f^i(s) \in \mathcal{A}^i; \end{aligned}$$

solutions need not coincide even if $\mathcal{P}(s) = \mathcal{P}(s')$, and they need not yield a solution to the ex - ante optimization problem subject to the measurability constraint.

³ A partition, \mathcal{P} , is at least as coarse as another, \mathcal{P}' if and only if $P'(s) = P'(s') \Rightarrow P(s) = P(s')$; one writes $\mathcal{P}' \subseteq \mathcal{P}$. If $\{\mathcal{P}^k : k \in \mathcal{K}\}$ is a collection of Partitions, the join is defined as the partition $\overline{\mathcal{P}} = \vee_{k \in \mathcal{K}} \mathcal{P}^k$ such that $\overline{\mathcal{P}}(s) = \overline{\mathcal{P}}(s')$ if and only if $P^k(s) = P^k(s')$ for all $k \in \mathcal{K}$: it is the coarsest common refinement; the meet is the partition $\underline{\mathcal{P}} = \wedge_{k \in \mathcal{K}} \mathcal{P}^k$, the finest common coarsening. The partition, \mathcal{P}_f , induced by a function f is defined by $P^f(s) = P^f(s')$ if and only if $f(s) = f(s')$: it is the coarsest partition with respect to which the function is measurable.

A strategy is feasible for the individual if, whenever $f^i(s) \neq f^i(s')$, either $\mathcal{P}^i(s) \neq \mathcal{P}^i(s')$: the private information of the individual distinguishes states s and s' or, for some individual, $j \in \{-i\}$, $f^j(s) \neq f^j(s')$: the elementary acts of some other individual distinguishes states s and s' .

The optimization problem of the individual is⁴

$$\begin{aligned} \max u^i(f^i, f^{-i}), \\ \text{s.t. } f^i \in \Phi^i(f^{-i}). \end{aligned}$$

The solution to the optimization problem is $\varphi^i(f^{-i}) \subseteq \Phi^i(f^{-i})$, and the choice correspondence is $\varphi^i : \mathcal{F}^{-i} \rightarrow \mathcal{F}^i$.

A profile of strategies, f , is feasible if $f^i \in \Phi^i(f^{-i})$, for all individuals.

A Nash equilibrium for the game with rational expectations is a feasible profile of strategies, $f^{\mathcal{I}^*}$, such that $f^{i^*} \in \varphi^i(f^{-i^*})$, for every individual.

Proposition 1. *At a Nash equilibrium for a game with rational expectations, the information of individuals is essentially symmetric with respect to the information partition*

$$\mathcal{P}^* = \bigvee_{i \in \mathcal{I}} \mathcal{P}^{f^{i^*}}.$$

Proof. If $f^* = (\dots, f^{i^*}, \dots)$ is a profile of strategies which is a Nash equilibrium with rational expectations, then it is a Nash equilibrium for the game with symmetric information $\mathcal{G}_{\mathcal{P}^*}$, where $\mathcal{P}^* = \bigvee_{i \in \mathcal{I}} \mathcal{P}^{f^{i^*}}$.

One argues in steps, for each individual:

1. By the definition, the function, f^{i^*} is measurable with respect to the partition $\mathcal{P}^{f^{i^*}}$, and, hence, also with respect to the finer partition \mathcal{P}^* .
2. By the definition of a Nash equilibrium with rational expectations, the function f^{i^*} is measurable and optimal with respect to the partition $\tilde{\mathcal{P}}^i = \mathcal{P}^i \bigvee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$.
3. Since the partition $\mathcal{P}^{f^{i^*}}$ is the coarsest partition with respect to which the function f^{i^*} is measurable, the partition $\tilde{\mathcal{P}}^i$ is at least as fine as the partition $\mathcal{P}^{f^{i^*}}$. It follows that the partition $\tilde{\mathcal{P}}^i \bigvee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$ is at least as fine as the partition $\mathcal{P}^{f^{i^*}} \bigvee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$. Since $\tilde{\mathcal{P}}^i \bigvee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}} = \tilde{\mathcal{P}}^i$, while $\mathcal{P}^{f^{i^*}} \bigvee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}} = \mathcal{P}^*$, the partition $\tilde{\mathcal{P}}^i$ is at least as fine as the partition \mathcal{P}^* .
4. Since the function f^{i^*} is measurable with respect to the partition \mathcal{P}^* , it is measurable and optimal with respect to the partition $\tilde{\mathcal{P}}^i$, and the partition $\tilde{\mathcal{P}}^i$ is at least as fine as the partition \mathcal{P}^* , the function f^{i^*} is measurable and optimal with respect to the partition \mathcal{P}^* . \square

At a Nash equilibrium for a game with rational expectations,

$$\mathcal{P}^i \bigvee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}} \subseteq \bigvee_{j \in \mathcal{I}} \mathcal{P}^{f^{j^*}} \subseteq \mathcal{P}^{f^{j^*}}.$$

⁴ In a game with rational expectations, the two ex - post formulations of the game discussed above (see footnote 2) coincide, and they yield a solution to the ex - ante optimization problem subject to a measurability constraint which takes into account the information revealed at equilibrium.

Corollary 1. *A Nash equilibrium where the information of individuals is essentially symmetric with respect to a partition $\hat{\mathcal{P}}$ is an equilibrium for the game with rational expectations as long as the partition $\hat{\mathcal{P}}$ is, for every individual, (i) at least as fine as the partition $\mathcal{P}^i \vee_{j \in \mathcal{I}} \mathcal{P}^{f^{j^*}}$ or (ii) at least as coarse as the partition \mathcal{P}^i .*

Proof. For every individual, the act f^{i^*} is a solution to the optimization problem with respect to the partition $\mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$, and thus f^* is a Nash equilibrium for the game with rational expectations.

For an individual, if $\hat{\mathcal{P}} \subseteq \mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$, the relevant partitions are ordered as

$$\hat{\mathcal{P}} \subseteq \mathcal{P}^i \vee_{j \in \mathcal{I}} \mathcal{P}^{f^{j^*}} \subseteq \mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}} \subseteq \mathcal{P}^i,$$

while, if $\mathcal{P}^i \subseteq \hat{\mathcal{P}}$, since $\hat{\mathcal{P}} \subseteq \mathcal{P}^{f^{j^*}}$,

$$\mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}} = \mathcal{P}^i.$$

In either case, since f^{i^*} is measurable and optimal with respect to the partition \mathcal{P}^i as well as the partition $\hat{\mathcal{P}}$, it is measurable and optimal with respect to the partition $\mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$. □

A Nash equilibrium where the information of individuals is symmetric, but with respect to a partition $\hat{\mathcal{P}}$ which, for some individuals, fails to be either as fine as the partition $\mathcal{P}^i \vee_{j \in \mathcal{I}} \mathcal{P}^{f^{j^*}}$ or as coarse as the partition \mathcal{P}^i , need not be an equilibrium with rational expectations.

Example 1. A game with private information is described by $\mathcal{I} = \{1, 2\}$, $\mathcal{A}^1 = \{T, B\}$, $\mathcal{A}^2 = \{L, R\}$, $\mathcal{S} = \{1, 2, 3\}$, $\mathcal{P}^1 = \{\{1\}, \{2, 3\}\}$, $\mathcal{P}^2 = \{\{1\}, \{2\}, \{3\}\}$, $\pi = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ and payoffs

$s = 1$	L	R
T	5, 1	5, 2
B	1, 1	1, 2

$s = 2$	L	R
T	2, 1	2, 2
B	3, 1	3, 2

$s = 3$	L	R
T	3, 2	3, 1
B	1, 2	1, 1

The profile in which player 1 plays T in every state, while player 2 plays R in states 1 and 2, and L in state 3 is a Nash equilibrium, and information is essentially symmetric with respect to the partition $\hat{\mathcal{P}} = \{\{1, 2\}, \{3\}\}$. It is not a rational expectations equilibrium; player 1, with information $\mathcal{P}^1 \vee \mathcal{P}^{f^2} = \{\{1\}, \{2\}, \{3\}\}$, would play T in states 1 and 3, and B in state 2. □

3 Common knowledge and rational expectations

If the information of an individual is described by the partition \mathcal{Q}^i , then, at a state of the world, \bar{s} , the individual knows an event, $\mathcal{E} \subseteq \mathcal{S}$ if

$$\mathcal{Q}^i(\bar{s}) \subseteq \mathcal{E}.$$

The states of the world at which the individual knows \mathcal{E} is

$$\mathcal{K}^i(\mathcal{E}) = \{s \in \mathcal{S}, \text{ such that } \mathcal{Q}^i(s) \subseteq \mathcal{E}\}.$$

If, $\bar{s} \notin \mathcal{K}^i(\mathcal{E})$, the individual does not know \mathcal{E} at \bar{s} : there exists a $s' \in \mathcal{Q}(\bar{s})$, such that $s' \notin \mathcal{E}$.

If there exist finite sequences of states of the world, $s_n, s_{n-1}, \dots, s_1 = \bar{s}$, and individuals, i_n, i_{n-1}, \dots, i_1 , not necessarily distinct, such that $s_n \in \mathcal{Q}^{i_{n-1}}(s_{n-1})$, $\dots, s_2 \in \mathcal{Q}^{i_1}(\bar{s})$, while $s' \in \mathcal{Q}^{i_n}(s_n) \setminus \mathcal{E}$, then, since $s' \in \mathcal{Q}^{i_n}(s_n) \setminus \mathcal{E}$, $s_n \notin \mathcal{K}^{i_n}(\mathcal{E})$. But then, since $s_n \in \mathcal{Q}^{i_{n-1}}(s_{n-1}) \setminus \mathcal{K}^{i_n}(\mathcal{E})$, $s_{n-1} \notin \mathcal{K}^{i_{n-1}}(\mathcal{K}^{i_n}(\mathcal{E}))$. Continuing in this manner, one obtains we that $\bar{s} \notin \mathcal{K}^{i_1}(\mathcal{K}^{i_2}(\dots, \mathcal{K}^{i_n}(\mathcal{E})))$: individual i_1 does not know that individual i_2 knows that ... individual i_{n-1} knows that individual i_n knows \mathcal{E} . Thus, for any finite sequence of individuals i_n, i_{n-1}, \dots, i_1 , individual i_1 knows that i_2 knows that ... i_{n-1} knows that i_n knows \mathcal{E} at \bar{s} if and only if, for any sequence of states of the world, $s_n, s_{n-1}, \dots, s_2, s_1 = \bar{s}$, such that $s_n \in \mathcal{Q}^{i_{n-1}}(s_{n-1}), \dots, s_2 \in \mathcal{Q}^{i_1}(\bar{s}), \mathcal{Q}^{i_n}(s_n) \subseteq \mathcal{E}$; equivalently, the event \mathcal{E} contains $\underline{\mathcal{Q}}(\bar{s})$, the element of the meet or finest common coarsening of the individual partitions.

At a state of the world, \bar{s} , an event $\mathcal{E} \subseteq \mathcal{S}$ is common knowledge if

$$\underline{\mathcal{Q}}(\bar{s}) \subseteq \mathcal{E}, \quad \text{where } \underline{\mathcal{Q}} = \bigwedge_{i \in \mathcal{I}} \mathcal{Q}^i.$$

A function, f , with domain the set of states of the world, is common knowledge at \bar{s} if the event $f^{-1}(f(\bar{s})) = \{s \in \mathcal{S}, \text{ such that } f(s) = f(\bar{s})\}$ is common knowledge at \bar{s} . A function is common knowledge if it is common knowledge at all states of the world.

Corollary 2. *At a Nash equilibrium of a game with rational expectations, the strategies of all individuals are common knowledge.*

A Nash equilibrium where the strategies of all players are common knowledge is a Nash equilibrium for the game with rational expectations.

Proof. If $f^* = (\dots, f^{i^*}, \dots)$ is a Nash equilibrium profile of strategies for the game with rational expectations, the information partition of an individual at equilibrium $\tilde{\mathcal{P}}^i = \mathcal{P}^i \vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$, which is at least as fine as the partition $\vee_{j \in \{-i\}} \mathcal{P}^{f^{j^*}}$. It follows that the meet of the individuals partitions at the equilibrium is at least as fine as the partition $\mathcal{P}^* = \vee_{i \in \mathcal{I}} \mathcal{P}^{f^{i^*}}$. Since the act f^{i^*} is measurable with respect to the partition \mathcal{P}^* , the result follows.

If $f^* = (\dots, f^{i^*}, \dots)$ is a Nash equilibrium profile of strategies, and if the strategies of all individuals are common knowledge, then, for every individual, f^{i^*} is measurable with respect to the meet of private individual partitions, $\underline{\mathcal{P}} = \bigwedge_{i \in \mathcal{I}} \mathcal{P}^i$. Since f^{i^*} is optimal with respect to the partition $\mathcal{P}^i \subseteq \underline{\mathcal{P}}$, by corollary 1, the result follows. □

Corollary 2 and proposition 1 immediately imply that, at a Nash equilibrium in which acts happen to be common knowledge, information is essentially symmetric. This is an instance of the theorem that “common knowledge of actions negates asymmetric information about events” in Geanakoplos (1995).

Proposition 1 plays the same role with respect to this theorem that the result in Geanakoplos and Polemarchakis (1982) plays with respect to the theorem in Aumann (1976): even if, “to begin with” information is not symmetric and acts are not common knowledge, “eventually” acts are common knowledge and information is symmetric. The process of communication is not explicit. Rather, it is embedded in the definition of a game with rational expectations.

Example 2. The opinion game, (Geanakoplos and Polemarchakis (1982)), is a game with uncertainty and private information, $\mathcal{O} = \{\mathcal{I}, \mathcal{S}, (\mathcal{A}^i, u^i, \mathcal{P}^i) : i \in \mathcal{I}\}$, where $\mathcal{A}^i = (-\infty, \infty)$, and $u^i(f) = -\sum_{s \in \mathcal{S}} \pi(s) (f^i(s) - x(s))^2 = -E_\pi(f^i - x)^2$, for a common prior probability measure, π , on the set of states of the world: individuals are guessing the value of x , a random variable.

At a Nash equilibrium, each player chooses the conditional expectation of the random variable, given his private information at each state of the world: $f^{i*}(s) = E_\pi(x|\mathcal{P}^i(s))$.

Individuals may disagree, due to differences in their private information.

At a Nash equilibrium with rational expectations individuals choose a profile which, by proposition 1, is also a Nash equilibrium for the opinion game with symmetric information $\mathcal{O}_{\mathcal{P}^*}$, where $\mathcal{P}^* = \bigvee_{i \in \mathcal{I}} \mathcal{P}^{f^{i*}}$; at a Nash equilibrium with rational expectations $\varphi^{i*}(s) = E_\pi(x|\mathcal{P}^*(s))$. Individuals “agree” in the opinion game with rational expectations. \square

4 Speculation

In a veto game, each individual has a veto strategy, e^i , that guarantees a level of utility \bar{u}^i :

$$u^i(e^i, f^{-i}) \geq \bar{u}^i, \quad f^{-i} \in \mathcal{F}^{-i}, i \in \mathcal{I},$$

and such that the profile e is ex - ante pareto optimal: at any feasible profile $f \in \mathcal{F}$ that gives to each individual at least \bar{u}^i , each individual obtains exactly \bar{u}^i

$$u^i(f) \geq \bar{u}^i, \quad \Rightarrow \quad u^i(f) = \bar{u}^i, \quad i \in \mathcal{I}.$$

At a Nash equilibrium of a veto game, all individuals attain the level of utility associated with their veto strategies.

For a veto game with private information and additively separable utilities, there is speculation at a state of the world, s , for a profile of strategies, f , if

$$\sum_{s' \in \mathcal{P}^i(s)} v^i((f^i(s), f^{-i}(s')), s') \geq$$

$$\sum_{s' \in \mathcal{P}^i(s)} v^i((e^i(s), e^{-i}(s'))s'),$$

$i \in \mathcal{I}$, with some strict inequality;

there is speculation, if there is speculation at some state of the world or, equivalently, the event of speculation,

$$\Sigma_f = \{s \in \mathcal{S} : \text{there is speculation at } s\},$$

is not empty.

Speculation can occur at a rational expectations equilibrium, as in the following example.

Example 3. A game with private information is described by $\mathcal{I} = \{1, 2\}$, $\mathcal{A}^1 = \{T, B\}$, $\mathcal{A}^2 = \{L, R\}$, $\mathcal{S} = \{1, 2\}$, $\mathcal{P}^1 = \{\{1\}, \{2\}\}$, $\mathcal{P}^2 = \{\{1, 2\}\}$, $\pi = \{\frac{1}{2}, \frac{1}{2}\}$ and payoffs

$s = 1$	L	R
T	1, -1	-1, 0
B	1, -1	0, 0

$s = 2$	L	R
T	-1, 1	-1, 0
B	-1, -1	0, 0

The choice of B in both states is a veto act for individual 1, and it guarantees $\bar{u}^1 = 0$; the choice of R in both states is a veto act for individual 2, and it guarantees $\bar{u}^2 = 0$. The profile in which individual 1 plays T in both states, while individual 2 plays L in both states is a Nash equilibrium and a rational expectations equilibrium; but, at state $s = 1$, there is speculation: individual 1 strictly prefers what he gets at equilibrium to what he obtains at the veto profile, while individual 2, given his information, is indifferent. □

In the example, the acts of individuals are common knowledge, but not the event of speculation: at state $s = 2$, individual 1 prefers the veto profile.

Indeed, for a profile of acts, $f = (\dots, f^i, \dots)$, not necessarily a Nash equilibrium, and for a state of the world, s , the event of speculation cannot be common knowledge. Common knowledge of speculation at a state s implies that $\sum_{s' \in P(s)} v^i(f(s'), s') \geq \sum_{s' \in P(s)} v^i(e(s'), s')$, for all $i \in \mathcal{I}$, with some strict inequality. But then, by choosing strategies that coincide with f^i on $P(s)$ and with e^i on the complement, all individuals would be at least as well off, and some strictly better off than at the veto profile, a contradiction. This is the argument of Milgrom and Stokey (1982) that speculation cannot be common knowledge. It is in the spirit, but stronger than the argument of Holmström and Myerson (1983) that ex-ante efficiency implies interim efficiency. As in the example, the occurrence of speculation at a given state of the world need not contradict the interim efficiency of the veto profile.

Corollary 3. *In a veto game, the event of speculation cannot be common knowledge.*

In a strong veto game, each individual has a strong veto act, i.e. a veto act $e^i \in \mathcal{F}^i$ such that the veto profile is the only ex - ante pareto optimal profile:

$$u^i(f) \geq \bar{u}^i, \Rightarrow f^i = e^i, \quad i \in \mathcal{I}.$$

At a Nash equilibrium of a strong veto game, all individuals play their strong veto acts. In particular, speculation never realizes at a Nash equilibrium profile. This is the argument of Geanakoplos (1995) that Nash equilibrium suffices to prevent speculation in strong veto games:

Corollary 4. *The event of speculation cannot realize at a Nash equilibrium of a strong veto game.*

If the veto strategies of all individuals are measurable with respect to some partition, \mathcal{P}^* , at least as coarse as their private information, $\mathcal{P}^i \subseteq \mathcal{P}^*$, the veto game has a Nash equilibrium, e , at which information is essentially symmetric and which, as a consequence is a rational expectations equilibrium. In a strong veto game, this equilibrium is unique.

It remains an open question to characterize the class of games with private information, broader than the class of strong veto games, such that at a Nash equilibrium information is essentially symmetric. For such games, the distinction between a Nash equilibrium and a rational expectations equilibrium vanishes. Kalai (2000) investigates this question for a class of large anonymous games.

5 Competitive equilibria

A competitive economy with uncertainty and private information is

$$\mathcal{E}_{\mathcal{P}} = \{\mathcal{I}, \mathcal{S}, \mathcal{L}, (\mathcal{Z}^i, \mathcal{P}^i, u^i) : i \in \mathcal{I}\}.$$

Commodities are $l \in \mathcal{L}$, a finite set, of cardinality L . An elementary net trade or an elementary act for an individual is $z^i \in \mathcal{Z}^i$, a subset of the commodity space, and a net trade, across states of the world, is $f^i = (\dots, z^i(s), \dots)$.

An allocation or a profile of acts, $f = (\dots, f^i, \dots)$, is feasible if $\sum_{i \in \mathcal{I}} f^i = 0$.

A feasible allocation is pareto optimal if and only if there does not exist another, feasible, allocation, $\hat{f} = (\dots, \hat{f}^i, \dots)$, such that $u^i(\hat{f}^i) \geq u^i(f^i)$, for every individual, with some strict inequality.

If the no-trade allocation $e = 0$, is pareto optimal, there is speculation at an allocation of net trades, f , and, at some state of the world, s , if $\sum_{s' \in \mathcal{P}^i(s)} v^i(f^i(s), s') \geq \sum_{s' \in \mathcal{P}^i(s)} v^i(0, s')$, for all individuals, with some strict inequality.

Elementary or spot prices of commodities are $\pi \in \Delta^L$, the unit simplex, and commodity prices are $p : \mathcal{S} \rightarrow \Delta^L$.

In an economy with rational expectations, individuals refine their information with the information revealed by prices. The information revealed by prices is \mathcal{P}^p .

A competitive equilibrium for an economy with rational expectations, Radner (1979), is a pair, (p^*, f^*) , of prices and a feasible allocation, such that, for an individual, f^{i*} is a solution to the maximization of utility over the set

$$B^i(p) = \left\{ f : \mathcal{S} \rightarrow \mathcal{Z}^i : \begin{array}{l} f \text{ is measurable with respect to } \mathcal{P}^i \vee \mathcal{P}^p, \text{ and} \\ p(s)f^i(s) \leq 0 \text{ for all } s \in \mathcal{S} \end{array} \right\}.$$

Associated with an economy, Debreu (1952), there is a (generalized) Walrasian game with uncertainty and private information,

$$\mathcal{W}_{\tilde{\mathcal{P}}} = \{\tilde{\mathcal{I}}, \mathcal{S}, \mathcal{L}, (\mathcal{Z}^i, \mathcal{P}^i, u^i) : i \in \mathcal{I}, (\mathcal{A}^0, \mathcal{P}^0, u^0)\}.$$

Individuals are $i \in \tilde{\mathcal{I}} = \mathcal{I} \cup \{0\}$, and the profile of information partitions is $\tilde{\mathcal{P}} = \mathcal{P} \cup \{\mathcal{P}^0\}$; a profile of strategies is $\tilde{f} = (f, p)$. Individual $i = 0$ is the auctioneer; The set of elementary acts for the auctioneer is $\mathcal{A}^0 = \Delta^L$, the domain of elementary commodity prices, his utility function is $u^0(\tilde{f}) = \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} p(s)f^i(s)$, his information is complete: $\mathcal{P}^0 = \{\{s\} : s \in \mathcal{S}\}$.

In the Walrasian game with rational expectations, individuals refine their information with the information revealed jointly by the acts of all other players, not only of the auctioneer.

An equilibrium, for the walrasian game with rational expectations is revealing if and only if prices, the act of the auctioneer, reveal the information revealed jointly by the net trades, the acts of all other individuals.

A revealing competitive equilibrium for the economy $\mathcal{E}_{\bar{p}}$ is a Nash equilibrium for the walrasian game, $\mathcal{W}_{\mathcal{P}}$, with rational expectations, and, at such an equilibrium, the information of individuals is essentially symmetric.

Corollary 5. *If the utility functions of individuals are separable across states of the world, and strictly quasi concave, speculation cannot occur at a competitive equilibrium of the economy with rational expectations.*

It suffices to observe that a competitive equilibrium allocation is a Nash equilibrium allocation of a well defined (generalized) game, while, under the stated conditions, if the no - trade allocation is pareto optimal, it is also a strong veto profile.

In the economy without rational expectations, the budget correspondence need not be measurable with respect to the individual information partition; solutions of the optimization problem of an individual at every state need not yield a solution to the ex-ante optimization problem⁵. A competitive equilibrium need not be a Nash equilibrium of any well defined generalized game, and speculation might occur⁶.

The argument for no-speculation at a rational expectations equilibrium of an economy is thus the one in Geanakoplos (1995) and Tirole (1982), not the one in Milgrom and Stokey (1982): rational expectations are needed to guarantee that equilibrium allocations are Nash equilibria of a well defined (generalized) game; common knowledge, either of the acts, or of the event of speculation, is not the issue.

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⁵ See footnote 2.

⁶ Dubey, Geanakoplos and Shubik (1987) provide an example.

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