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ASSET MARKETS AND INVESTMENT DECISIONS*

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In an incomplete asset market, firms assign values to investment plans by projecting their payoffs on the span of the payoffs of marketed assets. This is a criterion that does not require firms to possess information, such as the marginal valuation of revenue across date-events by shareholders, which is not directly observable; rather, it is based on the prices and payoffs of marketed assets. Under standard assumptions, competitive equilibria exist. However, even in the absence of nominal assets, competitive equilibrium allocations are generically indeterminate. The set of competitive equilibria is indexed by the price level at each state of the world, which has implications for the effectiveness of monetary policy.

1. INTRODUCTION

When all commodities are priced and exchanged in a universal, competitive market, where individuals optimize subject to one, overall budget constraint, the decision criterion of firms is unambiguous: Firms maximize profit. The profit associated with an investment plan[2] is well defined, whereas individual shareholders, their heterogeneity in preferences notwithstanding, unanimously favor profit maximization. The argument encompasses economies over time and under uncertainty, so long as there is a complete market in elementary securities (Arrow, 1953) or in contingent commodities (Debreu, 1960); prices of elementary securities serve to define the profit associated with investment plans across date-events, whereas the heterogeneity of shareholders extends to allow for heterogeneity in time preferences and beliefs. Importantly, prices of elementary securities may be only implicitly available; in a recent contribution, Cass and Rouzaud (1999) argue that there is a circularity in the definition of equilibrium: Firms must know their investment plans in order to determine the implicit prices of elementary securities employed in profit maximization.

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[2] The use of the term ''investment'' plan, as opposed to ''production'' plan indicates that the problem addressed is of interest, mostly, in an intertemporal setting.

When the economy is subject to uncertainty, and the asset market is incomplete, individuals optimize under an irreducible multiplicity of budget constraints; equivalently, they optimize under two constraints: the overall budget constraint and the constraint that their expenditure across date-events be the payoff of a portfolio of marketed assets, the shares of firms among them (Hart, 1975; Radner, 1972). As a consequence, marginal rates of substitution for revenue at different date-events typically differ across individuals; this poses a problem for firms in their attempt to value and choose among investment plans: The profit or market value of an investment plan not spanned by marketed assets is not defined.

Distinct, though related, approaches to the problem have been put forward. One restricts attention to technologies of firms such that any investment plan is indeed spanned by portfolios of marketed assets (Diamond, 1967; Ekern and Wilson, 1974; Radner, 1974); but this is excessively restrictive on the technologies of firms. Another supposes that firms value payoffs based on an average of the marginal valuations of individual shareholders (Drèze, 1974; Drèze et al., 1990; Grossman and Hart, 1979); but this is excessively demanding on the information available to and used by firms. The attribution of objective functions to firms not linked to the preferences of shareholders (Drèze, 1982; Leland, 1974) is arbitrary. More generally, firms employ any valuation of payoffs across date-events not incompatible with the prices of marketed assets and, hence, with the marginal valuations of individual shareholders (Duffie and Shafer, 1987); this employs only the prices and payoffs of marketed assets, but it leads, typically, to a nontrivial continuum of distinct equilibrium allocations.

Here, firms compute the value of investment plans by approximating their payoffs with the payoffs of portfolios of marketed assets: They project the payoff of shares on the span of the payoffs of marketed assets.

The appeal of the projection criterion is heuristic: It is simple, and it can be understood as an approximation. In addition, it does not require firms to possess information other than the payoffs and prices of marketed assets, and it avoids the circularity that Cass and Rouzaud (1999) object to.

If the technologies of firms satisfy the spanning condition of Ekern and Wilson (1974), profit maximization, which is well defined, coincides with the criterion here.

When a firm values the projection of the payoff of a production plan on the span of the matrix of payoffs of shares, it applies the generalized capital asset pricing model (CAPM): The classical CAPM of Lintner (1965), Sharp (1964), and Traynor (1961) imposes strong restrictions on the preferences and endowments of individuals, which, in particular, imply that the asset market is effectively complete; the generalized CAPM—Duffie (1992), Dutta and Polemarchakis (1991), Geanakoplos and Shubik (1990), Huang and Litzenberger (1988), Merton (1973), and Roll (1977), among others—does not require effective completeness, but, applied to assets that are not spanned, should be understood at best as an approximation.

Under standard assumptions, competitive equilibria exist. The only, minor, difficulty concerns the implicit prices of revenue across date-events deduced by firms from the prices and payoffs of marketed assets, which may not be positive; a decomposability condition on the technologies of firms eliminates the problem.

Firms do not adopt implicit prices for revenue across date-events that aggregate the marginal valuations of shareholders, as in Drèze (1974); even for economies with one, representative individual, a feasible production and consumption plan may be preferred to a competitive equilibrium allocation.

In the application of the projection criterion, an ambiguity arises: Firms can project, and approximate, investment plans in physical or value terms. Projection of the value of investment plans, or payoffs, is natural in an economy with multiple commodities; the simplifying assumption of a single commodity, here, still allows for different levels of prices across states of the world.

When firms project the payoff of shares on the span of the payoffs of marketed assets, competitive equilibrium allocations are, typically, indeterminate. This is a further instance of indeterminacy in an economy with an incomplete asset market that does not rely on nominal assets (Cass, 1985). Perhaps surprisingly, the set of equilibrium allocations coincides with the set of equilibrium allocations according to the definition of Duffie and Shafer (1987): Variations in the price level yield the same set of equilibrium allocations as the adoption by firms of alternative implicit prices of revenue across date-events.

When equilibrium allocations are indeterminate, policy may be effective by selecting among equilibrium allocations, and desirable, when competitive equilibrium allocations fail to be constrained Pareto optima. In an example, choices of the price level different across states of the world implement Pareto superior allocations. When firms compute the value of production plans by approximating the value of payoffs with the value of payoffs of portfolios of marketed assets, monetary policy, which determines the price level at alternative realizations of uncertainty, is effective and desirable.

2. THE ECONOMY

States of the world are $s \in \mathcal{S} = \{1, \dots, S\}$.

Investment decisions are made and assets are exchanged prior to the resolution of uncertainty. After the resolution of uncertainty, firms produce, shares pay off, and individuals consume. There is one commodity at each state of the world.

A quantity of the commodity at a state of the world is x_s; across states of the world, a bundle of commodities is[3] $x = (\dots, x_s, \dots)'$. The price of the commodity at a state of the world is p_s; across states of the world, prices of commodities are $p = (\dots, p_s, \dots)$. With one commodity at each spot market, no exchange occurs; the price of the commodity corresponds to the overall price level in a multicommodity world. The value, across states of the world, associated with the bundle x at prices of commodities p is $x(p) = Px$.[4]

Agents in the economy are individuals, consumer-investors, $i \in \mathcal{I} = \{1, \dots, I\}$, and firms, $j \in \mathcal{J} = \{1, \dots, J\}$.

[3] A prime sign denotes the transpose.

[4] For a vector $b = (\dots, b_s, \dots)$, $B = \text{diag}(b)$, where "diag(b)" denotes the diagonal matrix with elements $\text{diag}(b)_{s,s} = b_s$.

A firm is described by $(\mathcal{Y}^j, f^j)$: a production set, a set of investment plans, bundles of commodities; and an endowment, a bundle of commodities. The investment plan $y^j \in \mathcal{Y}^j$ is the commodity payoff of the shares of the firm.

Across firms, the allocation of endowments is $f^{\mathcal{J}} = (\ldots, f^j, \ldots)$, and the aggregate endowment in commodities of firms is $f^a = \sum_{j \in \mathcal{J}} f^j$. An allocation of investment plans is $y^{\mathcal{J}} = (\ldots, y^j, \ldots)'$, and the associated matrix, $Y = (\ldots, y^j, \ldots)$, is the matrix of commodity payoffs of shares of firms. The aggregate investment plan is $y^a = \sum_{j \in \mathcal{J}} y^j$.

The payoff of the shares of a firm is $y^j(p) = Py^j$, and the matrix of payoffs of shares is $Y(p) = PY$.

A portfolio of shares of firms is $z = (\ldots, z_j, \ldots)'$. The price of shares of a firm in the market for shares prior to the resolution of uncertainty is q_j; across firms, prices of shares are $q = (\ldots, q_j, \ldots)$. The value of the portfolio z at prices of shares of firms q is qz.

The shares of firms are the only marketed assets, available to investors for the transfer of revenue across states of the world. A transfer of revenue is attainable if and only if it is the payoff of a portfolio of shares. At prices of commodities p, if the allocation of investment plans is $y^{\mathcal{J}}$ and the matrix of payoffs of shares is $Y(p)$, the subspace of attainable transfers of revenue is[5] $[Y(p)]$.

Assumption 1. *There exists a partition of the set of states of the world*, $\{\mathcal{S}^1, \ldots, \mathcal{S}^j, \ldots, \mathcal{S}^J\}$, *into nonempty subsets such that*, *for every allocation of investment plans*, $y^{\mathcal{J}}$, *there exists an invertible matrix*, T, *such that*

$$YT = (r^1, \ldots, r^j, \ldots, r^J)$$

where

$$r_s^j = \begin{cases} 1, & \text{if } s \in \mathcal{S}^j \\ 0, & \text{if } s \in \mathcal{S} \backslash \mathcal{S}^j \end{cases}$$

This decomposability condition guarantees that, for prices of shares of firms that do not allow for arbitrage: $q = \pi' Y(p)$, for some $\pi \gg 0$, implicit prices of revenue for firms are strictly positive: $\pi^f = q(Y(p)'Y(p))^{-1} Y(p)' > 0$; it implies, in particular, that the matrix of commodity payoffs of shares, Y, has full column rank, and, also, that there exists a portfolio of shares with strictly positive payoffs: $Y(p)\bar{y} \gg 0$, for $\bar{y} = T\mathbf{1}_J$; and it is inherited by the matrix $Y(p)$, for strictly positive prices of commodities. The condition is one of decomposability of the aggregate technology across states of the world; it is satisfied whenever the asset market is complete; for an incomplete asset market, it is restrictive, but it is not a condition of decomposability of the aggregate technology across firms or, equivalently, of the matrix of payoffs across firms or assets.

Prices are a pair, (p, q), of prices of commodities and prices of shares.

An individual is described by $(\mathcal{X}^i, u^i, e^i, d^i)$: a consumption set, a set of consumption plans, bundles of commodities; a utility function representing a preference relation over consumption plans; an endowment in commodities, a bundle of commodities; and an endowment in shares, a portfolio of shares.

[5] "$[\cdot]$" denotes the span of the columns of a matrix or of a collection of vectors.

Across individuals, the allocation of endowments of commodities is $e^{\mathcal{I}} = (\ldots, e^i, \ldots)$, and the aggregate endowment of commodities of individuals is $e^a = \sum_{i \in \mathcal{I}} e^i$. An allocation of portfolios of shares is $z^{\mathcal{I}} = (\ldots, z^i, \ldots)'$. The aggregate portfolio of shares is $z^a = \sum_{i \in \mathcal{I}} z^i$. The aggregate endowment of portfolios of shares is[6] $\sum_{i \in \mathcal{I}} d^i = \mathbf{1}^{J\prime}$.

An allocation is $(z^{\mathcal{I}}, y^{\mathcal{J}})$, a pair of an allocation of portfolios of shares and an allocation of investment plans.[7]

An allocation is feasible if, for every individual, $x^i = (e^i + Yz^i) \in \mathcal{X}^i$, when $z^a = \mathbf{1}^J$. Associated with a feasible allocation, there is an allocation of consumption plans, $x^{\mathcal{I}}$.

At prices of shares q, the budget constraint of an individual is

$$qz \leq qd^i$$

at an allocation of investment plans $y^{\mathcal{J}}$, the consumption plan associated with a portfolio of shares is $x^i = e^i + Yz$; the demand for shares by the individual is $z^i(q, Y)$, and the demand correspondence is z^i.

At prices (p, q), if the matrix of commodity payoffs of shares is Y, the value of the projection of the payoff of a investment plan, y, on the span of the matrix of payoffs of shares is[8]

$$v(y, p, q, Y) = q\alpha(y, p, Y)$$

where

$$\alpha(y, p, Y) = \arg\min \{\|(y(p) - Y(p)\alpha\| : \alpha = (\ldots, \alpha_j, \ldots)'\}$$

is the portfolio of shares whose payoff best approximates the payoff $y(p)$; equivalently, $Y(p)\alpha(y, p, Y)$ is the projection of the payoff on the subspace, $[Y(p)]$, of attainable transfers of revenue.

A firm selects investment plans by maximizing the value of the projection of the payoff on the span of the matrix of payoffs of shares; the investment plan or the supply of the firm is

$$y^j(p, q, Y) = \arg\max \{v(y, p, q, Y) : y \in \mathcal{Y}^j\}$$

the supply correspondence is y^j, and the value of the firm is $\tilde{v}^j(p, q, Y)$.

If the matrix $Y(p)$ has full column rank,

$$\alpha(y, p, Y) = (Y(p)'Y(p))^{-1}Y(p)'y(p)$$

and

$$v(y, p, q, Y) = q(Y(p)'Y(p))^{-1}Y(p)'y(p)$$

[6] "$\mathbf{1}^K$" denotes the vector of 1s of dimension K.

[7] There is associated with a production set, a net production set, $\hat{\mathcal{Y}}^j = \mathcal{Y}^j - \{f^j\}$; with an investment plan, a net investment plan, $\hat{y}^j = y^j - f^j$; with a consumption plan, an excess consumption plan, $\hat{x}^i = x^i - e^i$; with a portfolio of shares, an excess portfolio of shares, $\hat{z}^i = z^i - d^i$; and similarly for payoffs of shares, aggregates, and allocations.

[8] "$\|\cdot\|$" denotes the eucleidean norm.

When a firm maximizes the value of the projection of the payoff on the span of the matrix of payoffs of shares, it maximizes the value of the payoff at implicit prices of revenue $\pi^f(y, p, Y) = q(Y(p)'Y(p))^{-1}Y(p)'$: Prices and the allocation of production plans determine implicit prices of revenue for firms.

If a firm selects investment plans by maximizing the value of the projection of the commodity payoff on the span of the commodity payoffs of marketed assets, the supply correspondence is defined by $y^j(q, Y) = \arg\max\{v(y, q, Y) : y \in \mathcal{Y}^j\}$, where $v(y, q, Y) = q\alpha(y, Y)$, and $\alpha(y, Y) = \arg\min\{\|(y - Y\alpha)\| : \alpha = (\ldots, \alpha_j, \ldots)'\}$. With one commodity in each state of the world, commodity prices do not enter the optimization problems of firms either; equivalently, prices of commodities are $p = \mathbf{1}^S$.

Implicit prices of revenue across states of the world are $\pi = (\ldots, \pi_s, \ldots)$.

As is well known, prices of shares, q, do not allow for arbitrage if and only if

$$Y(p)z > 0 \Rightarrow qz > 0^9$$

If the matrix of payoffs of shares has full column rank, prices of shares do not allow for arbitrage if and only if there exist implicit prices of revenue, such that

$$q = \pi Y(p), \quad \pi \gg 0$$

A competitive equilibrium is $((p^*, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a pair of prices and a feasible allocation, such that, for every firm, $y^{j*} \in y^j(p^*, q^*, Y^*)$, and, for every individual, $z^{i*} \in z^i(q^*, Y^*)$.

At prices of commodities $\bar{p}$, a competitive equilibrium is $(q^*, (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a pair of prices of shares and a feasible allocation, such that $((\bar{p}, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$ is a competitive equilibrium.

At a competitive equilibrium, the value of the objective function that a firm maximizes coincides with the price of its shares: $\tilde{v}^j(p^*, q^*, Y^*) = q_j^*$. Implicit prices of revenue for firms are $\pi^{f*} = q^*(Y^*(p^*)'Y^*(p^*))^{-1}Y^*(p^*)' \in \Pi(p^*, q^*)$.

Under standard assumptions, competitive equilibrium prices of shares do not allow for arbitrage.

If firms consider the commodity payoffs of shares, a competitive equilibrium is a pair, $(q^*, (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a competitive equilibrium at prices of commodities $\bar{p} = \mathbf{1}^S$.

As an alternative investment criterion, in Duffie and Shafer (1987), a firm selects production plans by maximizing the value of payoff of shares at some implicit prices of revenue $\pi \in \Pi(\mathbf{1}_S, q)$—without loss of generality, prices of commodities prices are $p = \mathbf{1}^S$; the supply correspondence is defined by $y^j(\pi) = \arg\max\{\pi y : y \in \mathcal{Y}^j\}$. A competitive equilibrium according to Duffie and Shafer (1987) is $((\pi^*, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a pair of implicit prices of revenue and prices of shares and a feasible allocation, such that $\pi^* \in \Pi(\mathbf{1}^S, q^*)$, for every firm, $y^{j*} \in y^j(\pi^*)$, and, for every individual, $z^{i*} \in z^i(q^*, Y^*)$.

Example. States of the world are $s = 1, 2, 3$. Bundles of commodities are $x = (x_1, x_2, x_3)$, and prices of commodities are $p = (1, p_2, p_3)$. Firms are $j = 1, 2$.

[9] Vector inequalities are "$\gg$," "$>$," and "$\geq$."

Firm 1 has endowment $f^1 = (1, 0, 0)$ and technology $\mathcal{Y}^1 = \{(1, 0, 0)\}$; firm 2 has endowment $f^2 = (0, 1, 0)$ and technology $\mathcal{Y}^2 = \{(y_1, y_2, y_3) : y_1 = -k, y_2 = 1, y_3 = k, k \geq 0\}$.

The matrix of payoffs of shares is

$$Y(p) = \begin{pmatrix} 1 & -k \\ 0 & p_2 \\ 0 & p_3 k \end{pmatrix}$$

The prices of shares are $q = (1, q_2)$. They do not allow for arbitrage if and only if $q_2 = (-1 + \pi_3 p_3)k + \pi_2 p_2$, for implicit prices of revenue $\pi = (1, \pi_2, \pi_3) \gg 0$; in particular, if prices of shares do not allow for arbitrage,

$$q_2 + k > 0$$

By direct computation,

$$(Y(p)'Y(p))^{-1}Y(p)' = (p_2^2 + p_3^2 k^2)^{-1} \begin{pmatrix} p_2^2 + p_3^2 k^2 & p_2 k & p_3 k^2 \\ 0 & p_2 & p_3 k \end{pmatrix}$$

At (p, q) and allocation of investment plans (y^1, y^2), with $y^2 = (-\bar{k}, 1, \bar{k})$, implicit prices of revenue for firms are

$$\pi^f(p, q, Y) = (1, q_2)(Y(p)'Y(p))^{-1}Y(p)')$$

or

$$(p_2^2 + p_3^2\bar{k}^2)^{-1}((p_2^2 + p_3^2\bar{k}^2), p_2(q_2 + \bar{k}), p_3\bar{k}(q_2 + \bar{k}))$$

If prices of shares do not allow for arbitrage, implicit prices of revenue for firms are positive: $\pi^f(p, q, Y) > 0$, and, for $k > 0$, strictly positive: $\pi^f(p, q, Y) \gg 0$; this follows from the structure of the matrix of payoffs of shares: There exists an invertible matrix,

$$T = \begin{pmatrix} 1 & k/p_2 \\ 0 & 1/p_2 \end{pmatrix}$$

such that

$$Y(p)T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & p_3 k/p_2 \end{pmatrix}$$

is orthogonal and positive, and, for $k > 0$, $Y(p)T\mathbf{1}_2 \gg 0$.

The objective function of firm 2 is to maximize

$$\pi^f(p, q, Y)(-k, p_2, p_3 k)$$

or, after simplification,

$$-p_2^2 k + q_2 p_3^2 \bar{k} k$$

For $k > 0$, equilibrium in production requires that

$$\bar{k} = \frac{p_2^2}{q_2 p_3^2}$$

The optimization problem of firm 1 is trivial.

There is one individual, with preferences represented by the utility function $u = \ln x_1 + \ln x_2 + \ln x_3$, endowment of commodities $e = (0, 0, 0)$, and endowment of shares $d = (1, 1)$.

The budget constraint of the individual is

$$z_1 + q_2 z_2 = 1 + q_2$$

and his/her indirect utility function over shares of firm 2 is

$$\tilde{u} = \ln(1 + q_2(1 - z_2) - z_2 k) + \ln z_2 + \ln(z_2 k)$$

The individual's demand for shares of firm 2 is

$$z_2 = \frac{2(1 + q_2)}{3(q_2 + k)}$$

Equilibrium in the market for shares requires that $z_2 = 1$ or

$$k = \frac{2 - q_2}{3}$$

The family of equilibria is described by

$$0 < k^* < \frac{2}{3}$$

Competitive equilibrium prices are

$$q_2^* = 2 - 3k^*$$

and

$$\frac{p_2^*}{p_3^*} = \sqrt{k^*(2 - 3k^*)}$$

The family of equilibria coincides with the family of equilibria following the alternative formulation, in Duffie and Shafer (1987), according to which a firm selects production plans by maximizing the value of payoff of shares at some implicit prices of revenue $\pi \in \Pi(\mathbf{1}_S, q)$.

Commodity prices are normalized to $p = (1, 1, 1)$.

At prices of shares q and allocation of investment plans $y^{\mathcal{J}}$, implicit prices of revenue are $\pi = (\pi_1, \pi_2, \pi_3)$, such that $q = \pi Y$. Importantly, the incompleteness of the asset market prevents the prices of shares from determining, up to normalization, the implicit prices of revenue.

The optimization of individuals remains unchanged.

Equilibrium in the market for shares requires, as before, that

$$k = \frac{2 - q_2}{3}$$

A firm selects investment plans by maximizing the value of the payoff, πy.

The objective function of firm 2 is to maximize

$$-\pi_1 + \pi_2 + \pi_3 k$$

for some $\pi = (\pi_1, \pi_2, \pi_3)$, such that $\pi Y = q$ or

$$\pi_1 = 1, \qquad -\pi_1 + \pi_2 + \pi_3 = q_2$$

The optimization of firm 1 is, as before, trivial.

For $k > 0$, equilibrium in production requires that

$$\pi_1 = \pi_3 = 1, \quad \pi_2 = q_2$$

As before, the family of equilibria is described by

$$0 < k^* < \frac{2}{3}$$

Competitive equilibrium prices are

$$q_2^* = 2 - 3k^*$$

and

$$(\pi_1^*, \pi_2^*, \pi_3^*) = (1, q_2^*, 1)$$

In addition, there exists an essentially autarkic competitive equilibrium, with $k = 0$; it obtains, for either definition of the optimization problem of firms, for an implicit price of revenue $\pi_3^* = 0$.

The sets of competitive equilibrium allocations when firms approximate the value of investment plans with the payoffs of marketed assets and spot commodity prices vary coincides with the set of competitive equilibria when firms adopt alternative implicit prices of revenue across realizations of uncertainty compatible with the observed prices of marketed assets.

Among equilibrium allocations, one is Pareto optimal: It maximizes the utility of the individual over production allocations or

$$\hat{u} = \ln(1 - y) + \ln 1 + \ln y$$

it involves

$$y^{**} = \frac{1}{2}$$

Interestingly, it obtains for

$$q = \frac{1}{2}$$

The price level that yields the Pareto-optimal allocation involves different price levels across realizations of uncertainty, which can be interpreted as active monetary policy.

The example shows that, when firms compute the value of investment plans by approximating the value of payoffs with the value of portfolios of marketed shares, a competitive equilibrium allocation need not be constrained Pareto optimal in production.

3. COMPETITIVE EQUILIBRIA

If the economy satisfies the decomposability condition and standard assumptions, and if the endowments of individuals in commodities are strictly positive and in shares of firms nonnegative, and the endowments of firms are nonnegative, then, at prices of commodities $\bar{p} \gg 0$, competitive equilibria exist; at a competitive equilibrium, the implicit prices of revenue for firms are strictly positive: $\pi^{f*} \gg 0$.

The argument for the existence of competitive equilibria only requires that there exist a portfolio of shares with positive payoffs or that, for every allocation of investment plans, the matrix of payoffs of shares of firms, Y, has full column rank, whereas, for an invertible matrix, T, the matrix YT is orthogonal and positive. For the strict positivity of implicit prices of revenue for firms at equilibrium, it is necessary that $YT\mathbf{1}^S \gg 0$. In the example, the latter fails for $k=0$, and, indeed, the implicit prices of revenue at the autarkic equilibrium are not strictly positive.

The following proposition establishes an equivalence between equilibria when firms project the payoffs of shares on the span of the payoffs of marketed assets and equilibria as in Duffie and Shafer (1987).

PROPOSITION 1. *A pair*, $((p^*, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, *of prices and a feasible allocation is a competitive equilibrium if and only if there exists a pair*, (π^{**}, q^{**}), *of strictly positive implicit prices of revenue*: $\pi^{**} \gg 0$, *and prices of shares*, *such that* $\pi^{**} \in \Pi(\mathbf{1}^S, q^{**})$, *for every firm*, $y^{j*} \in y^j(\pi^{**})$, *and*, *for every individual*, $z^{i*} \in z^i(q^{**}, Y^*)$.

PROOF. If

$$\pi^{**} = q^*(Y(p^*)'Y(p^*))^{-1}Y(p^*)' \quad \text{and} \quad q^{**} = \pi^{**}Y^*$$

then

$$q^{**} = q^* \quad \text{and} \quad \pi^{**}y = q^*(Y(p^*)'Y(p^*))^{-1}Y(p^*)'y$$

By construction, for every firm, $y^j(\pi^{**}) = y^j(p^*, q^*, Y^*)$, and, hence, $y^{j*} \in y^j(\pi^{**})$; since the technology satisfies decomposability, $\pi^{**} \gg 0$; for every individual, $z^i(q^{**}, Y^*) = z^i(q^*, Y^*)$, and, hence, $z^{i*} \in z^i(q^{**}, Y^*)$.

Since $Y^*\bar{y} \gg 0$, when $\pi^{**} \gg 0$, there exists $p^* \gg 0$, such that

$$\pi^{**} = \bar{y}'Y^{*\prime}P^{*\prime}$$

If

$$\pi^* = \bar{y}'Y^{*\prime}P^{*\prime} \quad \text{and} \quad q^* = \pi^* y^*(p^*)$$

then

$$q^* = q^{**} \quad \text{and} \quad q^*(Y(p^*)'Y(p^*))^{-1}Y(p^*)'y = \pi^{**}y$$

By construction, for every firm, $y^j(p^*, q^*, Y^*) = y^j(\pi^{**})$ and, hence, $y^{j*} \in y^j(p^*, q^*, Y^*)$; for every individual, $z^i(q^*, Y^*) = z^i(q^{**}, Y^*)$, and, hence, $z^{i*} \in z^i(q^{**}, Y^*)$. □

The previous proposition essentially states that competitive equilibrium allocations when firms compute the value of investment plans by approximating the value of payoffs with the value of portfolios of marketed shares $((p^*, q^*), z^{\mathcal{I}*}, y^{\mathcal{J}*})$ essentially coincide with competitive equilibrium allocations when, according to Duffie and Shafer (1987), firms adopt implicit prices of revenue compatible with the prices of marketed shares $((\pi^{**}, q^{**}), z^{\mathcal{I}**}, y^{\mathcal{J}**})$. However, the argument fails if the implicit prices of prices of revenue adopted by firms fail to be strictly positive at equilibrium.

Competitive equilibrium allocations are indeterminate of degree k if and only if the set of distinct competitive equilibrium allocations of commodities contains an open set of dimension k.

A property holds generically if and only if it holds for an open subset of full lebesgue measure.

If the economy is smooth, competitive equilibrium allocations are, generically, indeterminate of degree $(S - A)$; competitive equilibria at fixed prices of commodities are, generically, determinate.

A competitive equilibrium allocation need not be constrained Pareto optimal, as the example illustrates.

APPENDIX

A remark and two propositions complement and formalize results in the main body of the article.

A.1. *Projection Valuation and the CAPM.* When a firm values the projection of the payoff of a production plan on the span of the matrix of payoffs of shares, it applies the CAPM.

Implicit prices of revenue across states of the world are $\pi = (\ldots, \pi_s, \ldots)$.

Prices of shares, q, do not allow for arbitrage if and only if

$$Y(p)z > 0 \Rightarrow qz > 0^{10}$$

If the matrix of payoffs of shares has full column rank, prices of shares do not allow for arbitrage if and only if there exist implicit prices of revenue, such that

$$q = \pi Y(p), \quad \pi \gg 0$$

Alternatively, prices of shares, q, do not allow for arbitrage weakly if and only if

$$Y(p)z \gg 0 \Rightarrow qz > 0.$$

If the matrix of payoffs of shares has full column rank, and if there exists a portfolio of shares, $\bar{z}$, with strictly positive payoffs: $Y(p)\bar{z} \gg 0$, prices of shares do not allow for arbitrage weakly if and only if there exist implicit prices of revenue, such that

$$q = \pi Y(p), \quad \pi > 0$$

This elucidates the link between an economy with a complete market in elementary securities and an economy with a general asset market, in particular a market for shares (Ross, 1978).

[10] Vector inequalities are "$\gg$," "$>$," and "$\geq$."

The set of implicit prices of revenue compatible with prices of shares is $\Pi(p, q, Y) = \{\pi : q = \pi Y(p)\}$. When the market for shares is complete, the matrix of payoffs of shares, $Y(p)$, has full row rank, and the set $\Pi(p, q, Y) = \{\pi : q = \pi Y(p)\}$ is a singleton; when the market is incomplete, $\dim[Y(p)] < S$, and the set $\Pi(p, q, Y)$ is an affine subspace of dimension $(S - \dim[Y(p)]) > 0$.

Under standard assumptions on the characteristics of individuals, there exist, for every individual, implicit prices of revenue, $\pi^i \in \Pi(p, q, Y)$, $\pi^i > 0$, such that the solution to individual optimization under the constraints $x = Yz + e^i$, $qz \leq qd^i$ and the solution under the constraint $p^i x \leq p^i(e^i + Yd^i)$ coincide, where $p^i = \pi^i$; if the preferences of the individual are strictly monotonic, $\pi^i \gg 0$. Importantly, the implicit prices of revenue, π^i, such that the individual effectively optimizes under a single, overall budget constrain at prices $\pi^i P$, differ across individuals; the market for shares is effectively complete for $\hat{\pi}$ if every individual effectively optimizes under a single, overall budget constraint at prices $\hat{\pi} P$.

Shares of an investment plan, y, are redundant if and only if their payoff is spanned by the payoffs of marketed shares: $y(p) \in [Y(p)]$; equivalently, $y(p)$ is an attainable transfer of revenue.

Shares of an investment plan, y, that are redundant are priced by arbitrage at $\tilde{q} = \pi y(p)$, $\pi \in \Pi(p, q, Y)$, which is well defined—it is independent of the choice of $\pi \in \Pi(p, q, Y)$.

When the market for shares is complete, shares of any production plan are redundant and priced by arbitrage.

If the market for shares is effectively complete for $\hat{\pi}$, shares of any investment plan can be priced at $\tilde{q} = \hat{\pi} y(p)$.

The rate of return of shares of an investment plan with price $\tilde{q} \neq 0$ and payoff $y(p) = (\ldots, p_s y_s, \ldots)$ is $\rho = (\ldots, \rho_s, \ldots)$, where $\rho_s = (p_s y_s / \tilde{q}) - 1$.

Implicit prices of revenue that are positive: $\pi > 0$, and sum up to 1: $\sum_{s \in \mathcal{S}} \pi_s = 1$, can be interpreted as a probability measure on the set of states of the world.

With probability measure on the set of states of the world $\pi \in \Pi(p, q, Y)$, the expected rate of return of shares of any investment plan that are marketed vanishes: $E_\pi \tilde{\rho} = 0$; also, of any shares that are redundant. This is the martingale property satisfied by the price of any shares of investment plans that are marketed or redundant.

For π, a probability measure on the set of states of the world, there exist unique implicit prices of revenue, $\bar{\pi} \in \Pi(p, q, Y)$, such that $\bar{\pi}' \in [\Pi Y(p)]$. Alternatively, for $\bar{\pi} \in \Pi(p, q, Y)$, implicit prices of revenue, there exists a probability measure on the set of states of the world, π, not necessarily unique, such that $\bar{\pi}' \in [\Pi Y(p)]$; it is given, up to normalization, as the solution to the equation $\bar{\pi} = \Pi Y(p) \bar{z}$, where $\bar{z}$ is a portfolio of shares with strictly positive payoffs: $Y(p)\bar{z} \gg 0$, which is assumed to exist.

For $\pi \gg 0$, a strictly positive probability measure on the set of states of the world, the "market asset," m, is the asset with payoffs $y_m(p) = b\Pi^{-1}\bar{\pi}'$, for $b \neq 0$. Since $\bar{\pi}' \in [\Pi Y(p)]$, $y_m(p) \in [Y(p)]$, shares of the market asset are redundant. The price of the market asset is $q_m = b\bar{\pi}\Pi^{-1}\bar{\pi}' \neq 0$, and its rate of return is $\rho_m = b\Pi^{-1}\bar{\pi}' \times (\mathrm{diag}(\mathbf{1}_S q_m))^{-1} - \mathbf{1}_S$.

For shares of investment plans that are marketed or redundant,

$$\mathrm{E}_\pi \rho = \frac{\mathrm{Cov}_\pi(\rho, \rho_m)}{\mathrm{Var}_\pi \rho_m} \mathrm{E}_\pi \rho_m{}^{11}$$

This is the capital asset pricing model (CAPM) pricing equation.

By direct substitution, $\bar{\pi} y(p) = \bar{\pi} y(p)^{\pi, Y(p)}$, where $y(p)^{\pi, Y(p)} = Y(p)(Y(p)'\Pi \times Y(p))^{-1} Y(p)'\Pi y(p)$,[12] is the π-projection of payoffs $y(p)$ on $[Y(p)]$, the subspace of attainable transfers of revenue. The implicit prices of revenue $\bar{\pi}$ price approximately a nonredundant asset by pricing the associated redundant shares of the π-projection of the payoffs of the asset on the subspace of attainable transfers of revenue. In other words, applying the projection criterion is tantamount to applying CAPM to evaluate nonattainable assets. This is the generalized CAPM (Dutta and Polemarchakis, 1991; Geanakoplos and Shubik, 1990). The approximation is exact when the market for shares is effectively complete for $\bar{\pi}$.

If: (i) the preferences of every individual have a quadratic von Neumann–Morgenstern representation $u^i(x) = E_\pi(x_s - \alpha^i x_s^2)$, $\alpha^i > 0$, with common probability measure, π; (ii) the endowment of commodities of each individual lies in the span of the matrix of revenue payoffs of shares: $e^i \in [Y(p)]$; and (iii) there exists a "risk-free" firm, $f \in \mathcal{J}$, with investment plan $y^f = \mathbf{1}_S$, then, as long as the solution to the optimization problem of every individual satisfies $0 \leq x_s^i \leq \alpha^i/2$, the asset market is effectively complete for the probability measure on the set of states of the world common to the von Neumann–Morgenstern representations of the preferences of individuals, and the payoffs of the market asset coincide with the aggregate consumption. It suffices to observe that the gradient of the utility function of every individual lies in the span of the matrix of payoffs of shares: $Du^i(x^i) \in [\Pi Y]$. This is the classical CAPM (Lintner, 1965; Sharp, 1964; Traynor, 1961).

A.2. *The Existence of Competitive Equilibria.* The economy satisfies standard assumptions if

1. for every individual, the consumption set, $\mathcal{X}^i$, coincides with the set of bundles of commodities that are nonnegative; the preference relation, $\mathcal{R}^i$, is complete, transitive, continuous, convex, and strictly monotonically increasing: $x > \tilde{x} \Rightarrow x\mathcal{P}^i\tilde{x}$;
2. for every firm, the net production set, $\hat{\mathcal{Y}}^j$, is closed and convex;
3. for every firm, $0 \in \hat{\mathcal{Y}}^j$, and the endowment is nonnegative: $f^j \geq 0$;
4. across firms, the aggregate net production set, $\hat{\mathcal{Y}}^a$, is such that $\hat{\mathcal{Y}}^a \cap -\hat{\mathcal{Y}}^a = \{0\}$, whereas, whenever $y \in \hat{\mathcal{Y}}^a$ and $y \geq 0$, $y^a = 0$.

Proposition 2. *If the economy satisfies the decomposability condition and standard assumptions, and if, for every individual, the endowments of commodities is strictly positive*: $e^i \gg 0$, *and the endowment of shares nonnegative*: $d^i \geq 0$, *and, for*

[11] "E_π" denotes the expectation with respect to the probability measure π, and similarly for "Var_π" and "Cov_π."

[12] $y(p)^{\pi, Y(p)} = Y(p)\gamma$, where $\gamma = \arg\min\{(y(p) - Y(p)\gamma)'\Pi(y(p) - Y(p)\gamma) \colon \gamma = (\ldots, \gamma_s, \ldots)\}$.

every firm, the endowment is nonnegative: $f^j \geq 0$, *then, at prices of commodities* $\bar{p} \gg 0$, *competitive equilibria exist.*

Proof. Without loss of generality, $\bar{p} = \mathbf{1}_S$; the payoffs and the commodity payoffs of shares coincide.

The domain of prices of shares, allocations of net investment plans of firms, and aggregate excess portfolios of shares is

$$\mathcal{D} = \left\{ (q, \hat{y}^{\mathcal{J}}, \hat{z}^a) \colon \begin{array}{c} q = \pi Y,\ \pi \gg 0,\ \|q\| = 1, \\ \hat{y}^j \in \hat{\mathcal{Y}}^j \colon j \in \mathcal{J} \end{array} \right\}$$

For $k > 0$, the closed, truncated domain of prices of shares, allocations of investment plans of firms, and aggregate portfolios of shares is

$$\mathcal{D}_k = \left\{ \begin{array}{c} q = \pi Y,\ \pi > 0,\ \|q\| = 1, \\ (q, \hat{y}^{\mathcal{J}}, \hat{z}^a) \colon \hat{y}^j \in \hat{\mathcal{Y}}^j,\ \|\hat{y}^j\| \leq k \colon j \in \mathcal{J}, \\ \|\hat{z}^a\| \leq kI \end{array} \right\}$$

It is nonempty, compact, and homeomorphic to a convex set.

On the domain $\mathcal{D}_k$, the truncated adjustment correspondence,

$$(q, \hat{y}^{\mathcal{J}}, \hat{z}^a) \rightarrow (q', \hat{y}^{J\prime}, \hat{z}^{a\prime}),$$

is defined by

$$\begin{aligned} q' &\in \arg\max \{ q z^a : q = \pi Y,\ \pi > 0,\ \|q\| = 1 \} \\ \hat{y}^{j\prime} &\in \arg\max \{ q (Y'Y)^{-1} Y y : y \in \mathcal{Y}^j,\ \|\hat{y}\| \leq k \} - \{ f^j \} \} \\ \hat{z}^{a\prime} &\in \sum_{i \in \mathcal{I}} \left(\arg\max \left\{ \mathcal{R}^i : \sum_{j \in \mathcal{J}} \tilde{v}^j (\mathbf{1}^S, q, Y) \hat{z}^i_j,\ \|\hat{z}\| \leq k \right\} - \{ d^i \} \right) \end{aligned}$$

The values of the correspondence are nonempty, compact, and homeomorphic to convex sets, and the correspondence is upper-hemi continuous. There exists a fixed point, $(q^*_k, \hat{y}^{\mathcal{J}*}_k, \hat{z}^{a*}_k)$; for every firm, $\tilde{v}^j(\mathbf{1}^S, q^*_k, Y^*) = q^*_{j,k}$.

As $k \rightarrow \infty$, the sequence of fixed points has a limit point, $(q^*, \hat{y}^{\mathcal{J}*}, \hat{z}^{a*})$: prices of assets lie in a compact set; allocations of net investment plans, such that the aggregate net production plan is bounded below: $\hat{y}^a \geq -(e^a + f^a)$, lie in a compact set; if, along a subsequence, $\|\hat{z}^{a*}_k\| \rightarrow \infty$, the normalized excess portfolios $\hat{\hat{z}}^{a*}_k = (\|\hat{z}^{a*}_k\|)^{-1} \hat{z}^{a*}_k$ have a limit point, $\hat{\hat{z}}^{a*} \neq 0$, with $Y^* \hat{\hat{z}}^{a*} \geq 0$, and $q^* \hat{\hat{z}}^{a*} \leq 0$; by the optimization of individuals with strictly monotonically increasing preferences, $Y^* \hat{\hat{z}}^{a*} = 0$, and by the full column rank of the matrix of payoffs of assets, $\hat{\hat{z}}^{a*} = 0$, a contradiction.

At a limit point, $\hat{z}^{a*} = 0$: by the local nonsatiation of the preference relations of individuals, at a limit point, $q^* \hat{z}^{a*} = 0$, and, by the definition of the adjustment correspondence for prices of shares, $Y^* \hat{z}^{a*} \leq 0$; if $Y^* \hat{z}^{a*} < 0$, the portfolio of shares $-\hat{z}^{a*}$ has positive payoffs, when it is valued at 0, which contradicts the optimization of individuals with strictly monotonically increasing preferences; hence, $Y^*\hat{z}^{a*}=0$, and, since the matrix of payoffs of shares has full column rank, $z^{a*}=0$.

Associated with the limit point, there is an allocation, $(z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, such that, for every firm, $y_j^* \in \arg\max\{q^*(Y^{*\prime}Y^*)^{-1}Y^{*\prime}y : y \in \mathcal{Y}^j\}$, for every individual, $z^{i*} \in \arg\max\{\mathcal{R}^i : q^*z^{i*} \leq q^*d^{i*}\}$, and $z^{a*} = \sum_{i\in\mathcal{I}} z^{i*}=0$: The allocation is feasible, and $(q^*, z^{\mathcal{I}*}, y^{\mathcal{J}*}))$ is a competitive equilibrium at prices of commodities $\bar{p}=\mathbf{1}_S$. □

A.3. *The Indeterminacy of Competitive Equilibria.* An economy that satisfies standard assumptions as well as the decomposability condition is smooth if

1. for every individual, for strictly positive consumption plans, the preference relation is represented by a utility function, u^i:$x\mathcal{R}^i\tilde{x} \Leftrightarrow u^i(x) \geq u^i(\tilde{x})$; the utility function, with domain the interior of the consumption set, is twice continuously differentiable, differentiably strictly monotonically increasing: $Du^i \gg 0$, and differentiably strictly quasiconcave: $\xi \neq 0$ and $Du^i\xi=0 \Rightarrow \xi' D^2u^i\xi < 0$; for consumption plans, $x \gg 0$ and $\bar{x} \not\gg 0$, $x\mathcal{P}^i\bar{x}$;
2. for every firm, the strict efficient boundary of the net production set, $\mathrm{Bd}_+\ \hat{\mathcal{Y}}^j=\{\hat{y} : \hat{y}= \arg\max \pi^f\hat{y},\ \pi^f \gg 0\}$, is described by a production function, $g^j : \hat{y} \in \mathrm{Bd}_+ \Leftrightarrow g^j(\hat{y})=0$; the production function, with domain an open set of investment plans, is twice continuously differentiable, differentiably strictly monotonically increasing: $Dg^j \gg 0$, and differentiably strictly quasiconcave: $\xi \neq 0$ and $Dg^j\xi=0 \Rightarrow \xi' D^2g^j\xi < 0$;
3. for every firm, the endowment of commodities is strictly positive: $f^j \gg 0$.

Proposition 3. *If the economy is smooth, competitive equilibrium allocations are, generically, indeterminate of degree* $(S - A)$; *competitive equilibria at fixed prices of commodities are, generically, determinate.*

Proof.[13] If the function F of $(e^{\mathcal{I}}, f^{\mathcal{J}}, p, z^{\mathcal{I}}, \lambda^{\mathcal{I}}, y^{\mathcal{J}}, \mu^{\mathcal{J}})$ is defined by

$$F = \begin{pmatrix} (Du^i(e^i + Yz^i)) - \lambda^i q & \\ & i \in \mathcal{I} \\ q(z - d^i) & \\ \sum_{i\in\mathcal{I}} \tilde{z}^i - \mathbf{1}_{(A-1)} & \\ q(Y(p)'Y(p))^{-1}Y(p)' - \mu^j g^j(\hat{y}^j) & \\ & j \in \mathcal{J} \\ g^j(\hat{y}^j) & \end{pmatrix}$$

where $\lambda^{\mathcal{I}}=(\ldots, \lambda^i, \ldots) \gg 0$, $\mu^{\mathcal{J}}=(\ldots, \mu^j, \ldots) \gg 0$, $\tilde{z}=(z_2, \ldots, z_A)$, and $q=(1, q_2, \ldots, q_A)$—without loss of generality, $y^1 \gg 0$. Competitive equilibria are identified with $F^{-1}(0)$.

[13] The argument is that of Duffie and Shafer (1987); a simpler version is possible with one commodity per state of the world and constant rank of the matrix of payoffs of shares.

By a standard argument, the function F is transverse to 0. For a subset of economies and prices of commodities, $(\Omega \times \mathcal{P})^*$, the function $F_{(e^{\mathcal{I}}, f^{\mathcal{J}}, p)}$ is transverse to 0. The set of competitive equilibria for the economy $(\ldots, e^i, \ldots, f^j, \ldots)$ at prices of commodities p is finite and varies continuously with the parameters $(\ldots, e^i, \ldots, f^j, \ldots, p)$: at fixed prices of commodities, competitive equilibrium allocations are, generically, determinate.

The subset, Ω^*, of economies, ω, such that $(\omega, p) \in (\Omega \times \mathcal{P})^*$, for some prices of commodities, p, is open and of full lebesgue measure. For a fixed economy, $\omega \in \Omega^*$, there exists an open subset of prices of commodities, $\mathcal{P}(\omega)$, such that $\omega \times \mathcal{P}(\omega) \in (\Omega \times \mathcal{P})^*$. It remains to show that, as prices of commodities vary in $\mathcal{P}(\omega)$, the associated set of competitive equilibrium allocations contains an open set of dimension $(S - A)$.

The subset of states of the world $\bar{\mathcal{S}} = \{s^1, \ldots, s^j, \ldots, s^J\}$ is chosen such that $s^j \in \mathcal{S}^j$, where $\{\mathcal{S}^1, \ldots, \mathcal{S}^j, \ldots, \mathcal{S}^J\}$ is the partition associated with the decomposition of the matrix of payoffs of shares.

Eucleidean space, of dimension S is the direct sum of subspaces $\mathcal{E}^A$ and $\mathcal{E}^{(S-A)}$, of dimension A and $(S - A)$, respectively, where $\mathcal{E}^A = \{k : k_s = k_{s'}, \text{ for } s, s' \in \mathcal{S}^j\}$ and $\mathcal{E}^{(S-A)} = \{k : k_s = 0, \text{ for } s \in \bar{\mathcal{S}}\}$.

For prices of commodities p and p', such that $(p - p') \in \mathcal{E}^A$, the associated competitive equilibrium allocations coincide. If $Y(p) = PY$ is the matrix of payoffs of shares at a competitive equilibrium at prices of commodities p, and if the matrix $Y(p)T = (r^1, \ldots, r^j, \ldots, r^J)$ is such that $r^j_s = 1$, if $s \in \mathcal{S}^j$, and if $s \in \mathcal{S} \backslash \mathcal{S}^j$, the matrix of payoffs of shares at prices of commodities p', for the same allocation of investment plans is $Y(p') = P'Y$, and $Y(p')T = Y(p)T\Delta$, where $\delta = (\delta_{s^1}, \ldots, \delta_{s^j}, \ldots, \delta_{s^J})$.

For prices of commodities p and p', such that $(p - p') \in \mathcal{E}^{(S-A)}$, the associated competitive equilibrium allocations are distinct: If not, there exist prices of commodities p and p', such that $(p - p') \in \mathcal{E}^{(S-A)}$, and an allocation of investment plans with matrix of commodity payoffs of shares, Y, prices of shares, q, and implicit prices of revenue for firms, π^f, associated with competitive equilibria at both p and p'. In particular, $\pi^f = qT(T'Y(p)'Y(p)T)^{-1}T'Y(p)' = qT(T'Y(p')'Y(p')T)^{-1}T'Y(p')'$. Since $qT = \pi Y(p)T = \pi' Y(p')T$, for some $\pi \gg 0$, $\pi' \gg 0$. By the structure of the matrix $Y\,T$, since $p_s = p'_s$, for $s \in \bar{\mathcal{S}}$, $(T'Y(p)'Y(p)T)^{-1} = (T'Y(p')'Y(p')T)^{-1}$. But then, for $s \in \mathcal{S} \backslash \bar{\mathcal{S}}$, such that $p_s \neq p'_s$, it is not possible that the implicit price of revenue for firms, π^f_s, at p_s and p'_s coincide.

Since, for prices of commodities p and p', such that $(p - p') \in \mathcal{E}^{(S-A)}$, the associated competitive equilibrium allocations are distinct, whereas, for prices of commodities p and p' such that $(p - p') \in \mathcal{E}^A$, the associated competitive equilibrium allocations coincide, competitive equilibrium allocations are, generically, indeterminate of degree $(S - A)$. □

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