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Balance-of-Payments Equilibrium and the Determinacy of Interest Rates

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Abstract

In a world economy of overlapping generations with multiple countries, a competitive equilibrium need not satisfy the condition that the balance of payments be in equilibrium. The requirement that, in addition to market clearing, a balance-of-payments equilibrium condition be satisfied selects Pareto-optimal allocations and guarantees the determinacy of equilibrium.

1. Introduction

In economies with overlapping generations[1] the determinacy of competitive equilibrium allocations fails;[2] which is not the case for economies with finitely many individuals.[3]

For an elementary stationary economy of overlapping generations, with one commodity at each date and two dates in the lifespan of every individual, there is, generically, a continuum of distinct competitive equilibrium allocations.[4] With multiple commodities at each date or multiple dates in the lifespans of individuals, there are robust examples of economies with a continuum of distinct equilibrium allocations.[5] The failure of determinacy of competitive equilibrium allocations in economies of overlapping generations, even if not completely characterized, is, nevertheless, understood.[6]

It is of interest whether economically meaningful restrictions, beyond market clearing, can serve to select one or finitely many equilibria.

The structural features that account for the properties of competitive equilibria in economies of overlapping generations and their divergence from the properties that obtain in economies with finitely many individuals have long captured the curiosity of economic theorists; in particular, since the model of overlapping generations is an alternative to the infinitely lived individuals model for the analysis of macroeconomic and monetary policy. It is now well understood that the possible failure of an aggregate budget constraint is a defining property of economies of overlapping generations: the aggregate endowment may fail to have finite value at prices of commodities or interest rates at which the endowment of each individual has finite value, even at competitive equilibrium prices or interest rates. In a stationary economy of overlapping generations, the aggregate endowment fails to have finite value at the rate of interest that coincides with the rate of growth of population. The failure of aggregate valuation accounts for the failure of Pareto-optimality of competitive equilibrium allocations.[7]

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The condition that the balance of payments of a country be in equilibrium is equivalent to the condition that the country satisfy an aggregate budget constraint. Since each individual satisfies a budget constraint, the balance-of-payments equilibrium condition is *a fortiori* satisfied as long as the budget constraints of individuals aggregate; this is *a fortiori* the case in economies with finitely many, even infinitely lived, individuals, but not in economies of overlapping generations.

In a world economy of overlapping generations with multiple countries, a competitive equilibrium need not satisfy the condition that the balance of payments be in equilibrium.[8]

Evidently, it can be argued that there is no reason to impose a seemingly ad hoc balance-of-payments equilibrium condition. Nevertheless, a positive as well as a normative argument plead in its favor: the requirement that, at some point or in the long run, balance-of-payments equilibrium be satisfied is standard practice among policymakers; more importantly, as the argument that follows shows, the imposition of a balance-of-payments equilibrium condition can serve as a selection criterion that overcomes the indeterminacy and suboptimality of competitive equilibrium allocations in economies of overlapping generations.

Applied or empirical work is often hampered by the multiplicity of competitive equilibria. In economies that are not stationary or that are subject to stochastic shocks, a balance-of-payments equilibrium condition will not select a stationary equilibrium path, which is unlikely to exist, but should offer a way out of the conundrum of indeterminacy.

In a stationary economy of overlapping generations with heterogeneous countries, the requirement that, in addition to market clearing, a balance-of-payments equilibrium condition be satisfied restores the determinacy of equilibrium; and it selects Pareto-optimal allocations.

Since it is the failure of an aggregate budget constraint that accounts for the properties of competitive equilibrium allocations in economies of overlapping generations, it does not come as a surprise that these properties are affected nontrivially by the imposition of an additional budget constraint, a balance-of-payments equilibrium condition.

2. The Economy

Dates are $t = 1, \ldots$. At each date, one perishable commodity is available for trade and consumption: a quantity of the commodity z. Countries are $i = 1,2$. The store of value is money balances, in constant supply. There are no restrictions on the mobility of financial instruments, of money, or of the commodity across countries at any date.

In neither country does money bear interest. Interest rate parity requires that the exchange rate be constant across time[9] and, without loss of generality, equal to 1. Purchasing power parity requires that, at each date, the price of the commodity be equal across countries.

The real interest rate across consecutive dates is $r > -1$; with no nominal interest, the interest factor, $1 + r$, coincides with the reciprocal of the inflation factor. At date t, the interest rate is r_t; across dates, a path of interest rates is $\mathbf{r} = (\ldots, r_t, \ldots)$. The path is stationary if and only if $r_t = r$, so that the interest rate is constant across dates. A competitive equilibrium path of interest rates, $\mathbf{r}^* = (\ldots, r_t^*, \ldots)$, is such that $(1 + r_{t-1}^*)z(r_{t-1}^*) - z(r_t^*) = 0$.

Equilibrium at date 1 determines the price level. A steady state is a competitive equilibrium path of interest rates that is stationary.

The economy is stationary, with no growth in population.

Savings, in country i, by the generation in the first period of its life are $z^i(r)$, a function of the interest rate; demand by the same generation in the second period of its life is $(1 + r)z^i(r)$. Across countries, aggregate savings by the generation in the first period of its life are $z(r) = z^1(r) + z^2(r)$; aggregate demand by the same generation in the second period of its life is $(1 + r)z(r)$.

Aggregate excess demand for the commodity at date $t = 2, \ldots$ is $(1 + r_{t-1})z(r_{t-1}) - z(r_t)$, a function of the interest rates r_{t-1} and r_t.

An equilibrium is monetary if $z(r_1^*) \neq 0$, and, as a consequence, $z(r_t^*) \neq 0$, at all dates. A positive supply of outside money serves as a store of value if and only if $z(r_1^*) > 0$, and, as a consequence, $z(r_t^*) > 0$, at all dates.

An equilibrium is autarkic if $z(r_1^*) = 0$, and, as a consequence, $z(r_t^*) = 0$, at all dates. At an autarkic equilibrium there can be intragenerational exchange, but no intergeneration exchange. Also, monetary mobility can be active across countries.

Competitive equilibrium paths of interest rates, $\mathbf{r}^{*1}$ and $\mathbf{r}^{*2}$, are distinct if and only if $z(r_t^{*1}) \neq z(r_t^{*2})$, for some t; equivalently, if and only if $r_t^{*1} \neq r_t^{*2}$ and either $z(r_t^{*1}) \neq 0$ or $z(r_t^{*2}) \neq 0$, for some date.

Along a path of interest rates, the balance of payments deficit of country i at date t is the aggregate excess demand for the commodity:

$$d_t^i = (1 + r_{t-1})z^i(r_{t-1}) - z^i(r_t).$$

The cumulative deficit, the external debt of country i at date t, is

$$D_t^i = Q_{t-1}z^i(r_1) - z^i(r_t),$$

where

$$Q_{t-1} = (1 + r_1) \times \ldots \times (1 + r_{t-1})$$

is the cumulative interest factor. The balance-of-payments equilibrium condition is satisfied at date t if and only if the external debt of each country vanishes or

$$\mathrm{D}_t^i = 0$$

The exchange rate at date 1 can be set to yield balance-of-payments equilibrium at some date, $\bar{t}$. But, this is not of interest: it amounts to a redistribution of purchasing power across countries, and it is simply a restatement of the failure of the balance-of-payments equilibrium condition along an equilibrium path. It should not be confused with the properties of equilibrium paths in economies in which aggregate valuation holds, and balance-of-payments equilibrium characterizes all equilibrium paths, with no reference to a redistribution of purchasing power across countries.

Example 1

In each country, each generation consists of one, representative individual. The utility functions of representative individuals over consumption in the two dates in their lives are

$$u^i = \alpha^i \ln x_1 + (1 - \alpha^i)\ln x_2, \quad 0 < \alpha^i < 1,$$

and their endowments are

$$e^i = (e_1^i, e_2^i) >> 0.$$

The parameters

$$\beta^i = (1 - \alpha^i)e_1^i \quad \text{and} \quad \gamma^i = \alpha^i e_2^i$$

are a sufficient description of preferences and endowments and are such that

$$\beta/\gamma > 1,$$

where

$$\beta = \beta^1 + \beta^2 \quad \text{and} \quad \gamma = \gamma^1$$

Savings by an individual in the first period of his life are

$$z^i(r) = \beta^i - \gamma^i \frac{1}{1+r},$$

a function of the interest rate, where $\beta^i = (1 - \alpha^i)e_1^i$ and $\gamma^i = \alpha^i e_2^i$. Demand by the same individual in the second period of his life is $(1 + r)z^i(r)$.

Across countries, aggregate savings by the generation in the first period of its life are

$$z(r) = \beta - \gamma \frac{1}{1+r}.$$

Aggregate demand by the same generation in the second period of its life is $(1 + r)z(r)$.

For equilibrium it is necessary and sufficient that $(1 + r_{t-1})z(r_{t-1}) - z(r_t) = 0$ or, by direct computation, that

$$r_t^* = \frac{\gamma}{\gamma - \beta r_{t-1}^*} - 1.$$

Along an equilibrium path:

$$Q_{t-1}^* = \frac{\dfrac{1}{1+r_t^*} - \dfrac{\beta}{\gamma}}{\dfrac{1}{1+r_1^*} - \dfrac{\beta}{\gamma}}.$$

Equilibrium at date 1 determines the price level. If a positive supply of outside money serves as a store of value, it is necessary and sufficient that $z(r_1^*) > 0$ or that

$$r_1^* > \frac{\gamma}{\beta} - 1.$$

There are two steady-state competitive equilibria: the monetary one, which obtains for $r^{**} = 0$, and the autarkic one, which obtains for $\bar{r} = (\gamma/\beta) - 1 < 0$; at the autarkic steady state there is no transfer of value across dates. And there is a continuum of distinct equilibrium paths of interest rates indexed by $r_1^* \in [\bar{r}, r^{**}]$. Along an equilibrium path, other than the monetary steady state, the interest rate decreases and tends towards $\bar{r}$, the interest rate at the autarkic steady state.

Along a path of interest rates, the balance-of-payments deficit of country i at date t is $d_t^i = (1 + r_{t-1})z^i(r_{t-1}) - z^i(r_t)$, the excess demand for the commodity, or

$$d_t^i = -\gamma^i + \beta^i r_{t-1} + \gamma^i \frac{1}{1+r_t}.$$

The cumulative deficit, the external debt of country i at date t, is $D_t^i = -Q_{t-1}z_1^i + z_t^i$ or

$$D_t^i = -Q_{t-1}\left(\gamma^i \frac{1}{1+r_1} - \beta^i\right) + \left(\gamma^i \frac{1}{1+r_t} - \beta^i\right),$$

where Q_t is the cumulative interest factor.

The balance-of-payments deficit of country i at date t is

$$d_t^{i*} = \gamma^i \left(\frac{\beta}{\gamma} - \frac{\beta^i}{\gamma^i} \right) r_{t-1}^*.$$

The cumulative deficit, the external debt of country i at date t, is

$$D_t^{i*} = \gamma^i \left[\left(\frac{1}{1+r_t^*} - \frac{\beta^i}{\gamma^i} \right) - \frac{\left(\frac{1}{1+r_t^*} - \frac{\beta}{\gamma} \right)}{\left(\frac{1}{1+r_1^*} - \frac{\beta}{\gamma} \right)} \left(\frac{1}{1+r_1^*} - \frac{\beta^i}{\gamma^i} \right) \right].$$

The balance-of-payments equilibrium condition is satisfied at date t if and only if $D_t^i = 0$.

If the time preference and endowments of the representative individuals in the two countries are such that $\beta^1/\gamma^1 = \beta^2/\gamma^2$, then $\beta^i/\gamma^i = \beta/\gamma$, and all equilibrium paths are characterized by balance-of-payments equilibrium: $D_t^{i*} = 0$, at all dates.

If the representative individuals are heterogeneous across countries, their time preference and endowments are such that $\beta^1/\gamma^1 \neq \beta^2/\gamma^2$, then

$$\min\left\{ \frac{\beta^1}{\gamma^1}, \frac{\beta^2}{\gamma^2} \right\} < \frac{\beta}{\gamma} < \max\left\{ \frac{\beta^1}{\gamma^1}, \frac{\beta^2}{\gamma^2} \right\},$$

and, along any equilibrium path other than the monetary steady state, $r_t^* < 0$. Hence $d_1^{i*} \neq 0$, and, at all dates, $D_t^{i*} > 0 \Leftrightarrow d_1^{i*} > 0$: balance-of-payments equilibrium, at any date, fails. Along the monetary steady-state equilibrium path, $Q_t^* = 1$; at all dates, $d_t^{i*} = D_t^{i*} = 0$ and the balance-of-payments equilibrium condition is satisfied.

With heterogeneous countries, the requirement that, in addition to market clearing, a balance-of-payments equilibrium condition be satisfied restores the determinacy of equilibrium; and it selects the monetary steady-state equilibrium.

3. Balance of Payments and Determinacy

The economy is well-behaved if (1) aggregate savings, $z(r)$, varies continuously with the interest rate; (2) $z(0) > 0$; and (3) $\limsup_{r \to -1} -z(r) = \limsup_{r \to \infty} (1+r)z(r) = \infty$. Condition (1), continuity, reflects standard properties of preferences and endowments; condition (2) implies that the monetary steady state is not autarkic; condition (3) guarantees that rates of interest along equilibrium paths stay bounded away from extreme values.

If the economy is well-behaved, there exists a continuum of distinct equilibrium paths of interest factors. There exists a monetary steady-state equilibrium path, with $r^{**} = 0$; and, possibly distinct, autarkic steady-state equilibrium paths, with $\bar{r} \neq 0$, such that $z(\bar{r}) = 0$. Along the monetary steady-state equilibrium, the balance-of-payments equilibrium condition is satisfied at every date.

Countries are heterogeneous if

1. $z^1(r) = 0 \Rightarrow z^2(r) \neq 0$ and $z^2(r) = 0 \Rightarrow z^1(r) \neq 0$;
2. for $z^1(r') \neq 0$ and $z^2(r') \neq 0$,

$$r \neq r' \Rightarrow \frac{z^1(r)}{z^1(r')} \neq \frac{z^2(r)}{z^2(r')}.$$

Heterogeneity excludes the unlikely configuration of parameters under which the savings functions of countries coincide.

Restrictions on fundamentals serve to guarantee heterogeneity: in the example, countries are heterogeneous if and only if $\beta^1/\gamma^1 \neq \beta^2/\gamma^2$.

If countries are heterogeneous, along an autarkic equilibrium path of interest factors, the balance-of-payments equilibrium condition is not satisfied at any date.

Proposition 1. *If countries are heterogeneous, the balance-of-payments equilibrium condition is satisfied at date t, along an equilibrium path of interest rates,* $\mathbf{r}^* = (\ldots, r_t^*, \ldots)$, *that is not autarkic if and only if*

$$r_t^* = r_1^*.$$

Proof. If $r_t^* = r_1^*$, then $z^i(r_t^*) = z^i(r_1^*)$ and $z(r_t^*) = z(r_1^*)$. The market-clearing condition, $(1 + r_{t'-1}^*)z(r_{t'-1}^*) - z(r_{t'}^*) = 0$, at dates $2, \ldots, t$, imply that $z(r_2^*) \times \ldots \times z(r_t^*) = Q_{t-1}^* z(r_1^*) \times \ldots \times z(r_{t-1}^*)$, where $Q_{t-1}^* = (1 + r_1^*) \times \ldots \times (1 + r_{t-1}^*)$ is the cumulative interest factor at $t - 1$ or, since, $z(r_2^*) \neq 0, \ldots, z(r_{t-1}^*) \neq 0$, $z(r_t^*) = Q_{t-1}^* z(r_1^*)$. Since $z(r_t^*) = z(r_1^*) \neq 0$, $Q_{t-1}^* = 1$, and, since $z^i(r_t^*) = z^i(r_1^*)$, $D_t^{i*} = -Q_{t-1}^* z^i(r_1^*) + z^i(r_t^*) = 0$: the balance-of-payments equilibrium condition at date t is satisfied.

If, at date t, the balance-of-payments equilibrium condition is satisfied, $D_t^{i*} = Q_{t-1}^* z^i(r_1^*) - z^i(r_t^*) = 0$; while market-clearing at dates $2, \ldots, t$ implies that $z(r_2^*) \times \ldots \times z(r_t^*) = Q_{t-1}^* z(r_1^*) \times \ldots \times z(r_{t-1}^*)$ or, since $z(r_2^*) \neq 0, \ldots, z(r_{t-1}^*) \neq 0$, $z(r_t^*) = Q_{t-1}^* z(r_1^*)$. It follows that $(z^i(r_t^*)/z^i(r_1^*)) = (z(r_t^*)/z(r_1^*))$, and, since countries are heterogeneous, $r_t^* = r_1^*$. □

If an equilibrium path of interest rates is cyclical and $r_t^* = r_{t'}^*$, for $t' = t \bmod_n$, where n is the length of the cycle, the balance-of-payments equilibrium condition is satisfied at all dates $t = t \bmod_n$.

This characterizes equilibrium paths along which the balance-of-payments equilibrium condition is satisfied at some date.

Corollary 1. *If countries are heterogeneous, the economy is well-behaved, and aggregate savings by a generation in the first period of its life increase with the interest rate:* $r' > r \Rightarrow z(r') > z(r)$, *then, along any equilibrium path of interest rates,* $\mathbf{r}^* = (\ldots, r_t^*, \ldots)$, *that is not autarkic, the balance of payments equilibrium condition is not satisfied at any date.*

Proof. Since $(1 + r_{t-1}^*)z(r_{t-1}^*) - z(r_t^*) = 0$, while $r_t^* < 0$, $z(r_{t-1}^*) > z(r_t^*)$. Since aggregate savings increase with the interest rate, $r_{t-1}^* > r_t^*$, and, as a consequence, $r_t^* < r_1^*$. □

This characterizes economies for which the requirement of balance-of-payments equilibrium selects the monetary steady-state equilibrium path. In particular, economies in which savings vary monotonically with the rate of interest have the monetary steady state as the only equilibrium for which balance-of-payments equilibrium is satisfied at some date. It remains to characterize equilibrium paths that satisfy the balance-of-payments condition in economies in which savings are not a monotonic function of the rate of interest. As it turns out, only local uniqueness or determinacy can be guaranteed; in particular, cyclical paths cannot be excluded.

Economies are indexed by $\theta \in \Theta$, an open set of finite dimension: associated with θ, there is an economy, described by the savings, in each country, by the generation in the first period of its life, $(z^1(r,\theta), z^2(r,\theta))$.

A property holds generically if it holds for an open set of economies of full measure. An arbitrarily small variation in parameters suffices to restore a generic property if it fails. If a generic property is satisfied, it is satisfied for all neighboring parameter values.

Economies are diverse if savings in each country vary across economies: $(\partial z^i(r,\theta)/\partial\theta)$

$\neq 0$, while the aggregate savings remain invariant: $z(r,\theta) = z(r)$. Diversity allows variation in economies that do not affect equilibrium paths.

Economies are smooth if, in each country, savings are continuously differentiable in r, the interest rate, and μ, the indexing parameter. It follows that aggregate savings are continuously differentiable in the interest rate, while their derivative with respect to the indexing parameter vanishes.

Economies are uniformly bounded if, in each country, savings are uniformly bounded: $z^i(r,\theta) \leq \bar{z}^i$. Since consumption is non-negative, this is the case if the endowments of individuals are uniformly bounded across economies; there is no contradiction with the assumption that $\lim\sup_{r\to -1} - z(r) = \lim\sup_{r\to\infty} (1 + r)z(r) = \infty$.

PROPOSITION 2. *If economies are diverse, smooth, and uniformly bounded, and the elasticity of aggregate savings with respect to the interest factor is different from 1:* $(1 + r)(\partial z(r)/\partial r) + z(r) \neq 0$*; then, generically, the balance-of-payments equilibrium condition at date t is satisfied for finitely many equilibrium paths of interest rates.*

PROOF. The function $f = (f_1, f_2, \ldots, f_t)$ is defined by

$$\begin{aligned} f_1(r_1,\ldots,r_t,\theta) &= Q_{t-1}z^i(r_1,\theta) - z^i(r_t,\theta),\\ f_2(r_1,\ldots,r_t,\theta) &= (1+r_1)z(r_1,\theta) - z(r_2,\theta),\\ &\vdots\\ f_t(r_1,\ldots,r_t,\theta) &= (1+r_{t-1})z(r_{t-1},\theta) - z(r_t,\theta), \end{aligned}$$

and the evaluation function, f_θ, is defined by

$$f_\theta(r_1,\ldots,r_t) = f(r_1,\ldots,r_t,\theta).$$

A path of interest rates for the economy θ is an equilibrium path and, in addition, satisfies the equilibrium condition at date t, only if

$$f_\theta\left(r_1^*,\ldots,r_t^*\right) = 0.$$

Since $\partial f_1/\partial\theta \neq 0$, while, for $t' = 2, \ldots, t$, $\partial f_{t'}/\partial r_{t'-1} = ((1 + r_{t'-1})\partial z(r_{t'-1})/\partial r_{t'-1}) + z(r_{t'-1}) \neq 0$, and $\partial f_{t'}/\partial\theta = \partial f_{t'}/\partial r_{t''} = 0$, for $t'' \neq t' - 1, t'$, the submatrix of the Jacobian of the function f associated with the variables $r_1, \ldots, r_{t-1}, \theta$ has full row rank: the function f is transverse to 0.

There exists an open set of full measure of regular economies,[10] for which the evaluation function f_θ is transverse to 0. Since the domain and the range of the evaluation function are equal, for a regular economy, $f_\theta(q_1, \ldots, q_t) = 0$ for finitely many values of $(q_1, \ldots, q_t)$. □

Generically, the requirement of balance-of-payments equilibrium restores the determinacy of equilibrium allocations and interest rates.

Economies with more than two countries typically do not have any equilibrium path of interest factors for which the balance-of-payments equilibrium condition, at some date, is satisfied for each country.

Economies with more than one commodity or, equivalently, more than two periods in the lifespans of individuals may have sets of distinct equilibrium paths of relative prices of commodities and interest factors of dimension more than one; in particular, equal to the number of commodities. Determinacy requires that the balance-of-payments equilibrium condition be imposed at more than one date.

Economies that are not stationary may not have any equilibrium path of interest factors for which the balance-of-payments equilibrium condition, at some date, is satisfied.

Example 2

This economy is a nonstationary modification of the economy in the previous example.

Generation $t = 1$ has preferences and endowments that differ from those of the succeeding generations; they are described by the parameters β_1^i and γ_1^i, for each representative individual, which yield the parameters $\beta_1 = \beta_1^1 + \beta_1^2$ and $\gamma_1 = \gamma_1^1 + \gamma_1^2$ for the aggregate. Generations $t = 2, \ldots$ have preferences and endowments as previously.

Savings by an individual in the first period of his life at date 1 are

$$z_1^i(r_1) = \beta_1^i - \gamma_1^i \frac{1}{1+r},$$

a function of the interest rate. Demand by the same individual in the second period of his life is $(1 + r_1)z_1^i(r_1)$. Across countries, aggregate savings by the generation in the first period of his life at date 1 is $z_1(r_1)$. Aggregate demand by the same generation in the second period of his life is $(1 + r_1)z_1(r_1)$. Savings and demand by generations $t = 2, \ldots$ are as previously.

For equilibrium it is necessary and sufficient that $(1 + r_1)z_1(r_1) - z(r_2) = 0$ and $(1 + r_{t-1})z(r_{t-1}) - z(r_t) = 0$, for $t = 3, \ldots$ or, by direct computation, that

$$r_2^* = \frac{\gamma}{\gamma_1 + \beta - \beta_1 - \beta_1 r_1^*} - 1,$$

$$r_t^* = \frac{\gamma}{\gamma - \beta r_{t-1}^*} - 1, \quad t = 3, \ldots$$

Along an equilibrium path of interest rates, the balance-of-payments deficit of country i at date t is

$$d_2^{i*} = \gamma^i \left[\left(\frac{\gamma_1}{\gamma} - \frac{\gamma_1^i}{\gamma^i} \right) + \left(\frac{\beta_1^i}{\gamma_1^i} - \frac{\beta^i}{\gamma^i} \right) + \left(\frac{\beta}{\gamma} - \frac{\beta_1^i}{\gamma_1^i} \right) r_1^* \right],$$

$$d_t^{i*} = \gamma^i \left(\frac{\beta}{\gamma} - \frac{\beta^i}{\gamma^i} \right) r_{t-1}^*, \quad t = 3, \ldots.$$

The cumulative deficit, the external debt of country i at date t, is

$$D_t^{i*} = \gamma^i \left[\left(\frac{1}{1+r_t^*} - \frac{\beta^i}{\gamma^i} \right) - \frac{\left(\frac{1}{1+r_t^*} - \frac{\beta}{\gamma} \right)}{\left(\frac{\gamma_1}{\gamma(1+r_1^*)} - \frac{\beta_1}{\gamma} \right)} \left(\frac{\gamma_1^i}{\gamma^i(1+r_1^*)} - \frac{\beta_1^i}{\gamma^i} \right) \right].$$

The balance-of-payments equilibrium condition is satisfied at date t if and only if $D_t^i = 0$.

If the time preferences and endowments of the representative individuals in the two countries are such that, after date 2, countries are heterogeneous,

$$\beta^1/\gamma^1 < \beta/\gamma,$$

while

$$\frac{\beta_1^1}{\gamma^1} \geq \frac{\beta_1}{\gamma}, \quad \frac{\gamma_1^1}{\gamma^1} \leq \frac{\gamma_1}{\gamma},$$

$$\frac{\gamma_1^1}{\gamma^i\left(1+r_1^*\right)}-\frac{\beta_1^1}{\gamma^1} < \frac{\gamma_1}{\gamma\left(1+r_1^*\right)}-\frac{\beta_1^1}{\gamma^1}, \quad \frac{1}{1+r_t^*}-\frac{\beta^1}{\gamma^1} > \frac{1}{1+r_t^*}-\frac{\beta}{\gamma},$$

and, along any equilibrium path, the cumulative deficit of country 1 at date *t* is

$$D_t^{1*} > 0:$$

balance-of-payments equilibrium, at any date, fails.

4. Conclusion

The requirement that the balance-of-payments equilibrium condition be satisfied "at $t = \infty$" is more appropriate, but more difficult to analyze. It is an open question what condition can substitute for the balance-of-payments equilibrium condition in economies with multiple countries or economies that are not stationary.

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Notes

1. Allais (1947) and Samuelson (1958) introduced the model.
2. Gale (1971) first pointed out the indeterminacy of competitive equilibrium allocations explicitly, though Samuelson (1958) was aware of the problem.
3. Debreu (1970) showed that, in smooth, finite, exchange economies, competitive equilibrium allocations are, generically, finitely many. Kehoe and Levine (1985) and Shannon (1995) extended the argument to economies with possibly countably infinitely many commodities or dates, but finitely many individuals.
4. Geanakoplos and Polemarchakis (1986) gave an elementary proof.
5. Geanakoplos and Polemarchakis (1986), Kehoe and Levine (1982), and Woodford (1984) gave such examples.
6. Geanakoplos and Polemarchakis (1984), Kehoe and Levine (1984), and Kehoe et al. (1989, 1991) provided characterizations. Brown and Geanakoplos (1982) argued that it can be understood as lack of market clearing at infinity. Brown and Geanakoplos (1985) and Burke (1990) provided a benchmark for comparative dynamics when determinacy fails.
7. Geanakoplos and Polemarchakis (1992) and Wilson (1981) developed the point.
8. Gale (1973) pointed out that imbalance of trade is possible at a competitive equilibrium of an economy of overlapping generations.
9. Kareken and Wallace (1981) pointed out the indeterminacy of the exchange rate at date 1; this is simply a normalization of the relative money supply in the two countries.
10. By the transversal density theorem, the set of regular economies has full Lebesgue measure. By the uniform boundedness of savings, it is open.