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The Relevance of Extrinsic Uncertainty

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ABSTRACT. – When the asset market is incomplete extrinsic risk is effective at competitive equilibrium allocations; this is the case whether commodities are exchanged indirectly, through the exchange of assets, or whether assets serve to transfer revenue and commodities are exchanged in spot markets. Individuals bear extrinsic risk for the benefit of exchanging commodities or transferring revenue in the absence of complete markets for the allocation of intrinsic risk.

Risque intrinsèque et allocations d'équilibre

RÉSUMÉ. – Quand les marchés sont incomplets, le risque extrinsèque affecte les allocations à l'équilibre concurrentiel. Cela est vrai dans le cas où les biens sont échangés indirectement, moyennant l'échange de titres réels, ainsi que dans le cas où les titres sont utilisés pour transférer du revenu et où les agents peuvent échanger les biens sur des marchés au comptant.

En échangeant des titres contingents au risque extrinsèque les agents acceptent de supporter ce type de risque. En revanche, et sous certaines conditions, cela leur donne la possibilité de mieux s'assurer relativement au risque intrinsèque.

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1 Introduction

States of the world constitute extrinsic risk if no pareto optimal allocation varies across these states.[1]

Extrinsic risk is effective at an allocation if the allocation varies across states of extrinsic risk.

If competitive equilibrium allocations are pareto optimal, extrinsic risk is, *a fortiori*, ineffective at a competitive allocation, as was argued by CASS and SHELL [1983].

When structural constraints, such as the incompleteness of spot or asset markets, asymmetric information and incentive compatibility constraints or strategic behavior of agents with market power sever the link between pareto optimal and competitive equilibrium allocations, two questions arise:

1. Is extrinsic uncertainty effective at competitive equilibrium or constrained optimal allocations?
2. Can extrinsic uncertainty be effective at a competitive equilibrium allocation even though competitive equilibrium allocations at which extrinsic uncertainty is not effective exist?

The work on "*sunspots*" addresses the second questions: conditions for sunspot and sunspot – free equilibria to co-exist.

For economies that extend over a finite horizon, following CASS and SHELL [1983], work on sunspot equilibria has focused on economies with an incomplete asset market and has analyzed convexity,[2] the multiplicity of equilibria in the economy with a complete asset market,[3] nominal assets[4] and real assets and sufficiently many commodities at each spot market[5] as underlying the existence of sunspot equilibria.

For economies that extend over an infinite horizon, following AZARIADES [1981], work has focused on economies of overlapping generations or economies subject to finance constraints.[6] The failure of competitive markets to render extrinsic uncertainty ineffective has influenced the analysis of macroeconomic and monetary policy.

The work here addresses the first question: conditions for effective extrinsic uncertainty at competitive equilibrium or constrained optimal allocations.

The result is that extrinsic uncertainty is typically effective if the asset market is incomplete. Importantly, this is the case whether commodities are exchanged indirectly, through the exchange of assets, or whether assets serve to transfer revenue and commodities are exchanged in spot markets.

1. It is pedantic to distinguish between allocations that are invariant across states of the world and allocations welfare equivalent to them.
2. BALASKO [1983], CASS and POLEMARCHAKIS [1990], SHELL and WRIGHT [1993].
3. GUESNERIE and LAFFONT [1988] MAS-COLELL [1989], HENS [2000].
4. CASS [1989], CASS [1992], PIETRA [1992] 1993), SICONOLFI [1991], SUDA, TALLON and VILLANACCI [1992].
5. GOTTARDI and KAJII [1995].
6. CHIAPPORI and GUSNERIE [1991] survey the literature.

Individuals bear extrinsic risk for the benefit of exchanging commodities or transferring revenue in the absence of complete markets for the allocation of intrinsic risk.

The economy may well have a globally unique competitive equilibrium: therefore effective extrinsic uncertainty does not derive from the multiplicity of equilibria in the underlying economy.

Moreover, the structure of payoffs of assets may allow for non-trivial allocations invariant with respect to the extrinsic risk: the asset structure does not force effective extrinsic uncertainty in allocations with trade.

The intuition that underlies the result is straightforward: with an incomplete asset market, individuals bear the cost of extrinsic risk at equilibrium for the benefit of trading in commodities or sharing intrinsic risk; since the payoffs of assets are not invariant with respect to states of extrinsic uncertainty, it is not surprising that extrinsic risk is effective at competitive equilibrium allocations.

It is well established that randomization may improve the allocation of resources: mixed strategies in games or, in a walrasian framework, random taxation,[7] random transfers of revenue[8] in economies with asymmetric information, random monetary policy.[9] Effective extrinsic risk with incomplete asset market establishes this fact in an abstract setting.

2 The Economy

Individuals are $i \in \mathcal{I} = \{1, \ldots, I\}$, a non-empty finite set.

States of the world are $(f, s) \in \times_{f \in \mathcal{F}}(f \times \mathcal{S}_f)$, where $\mathcal{F} = \{1, \ldots, F\}$, is a non-empty finite set of states of intrinsic risk, while $\mathcal{S}_f = \{1, \ldots, S_f\}$ is a non-empty, finite set of states of extrinsic risk conditional on a state of intrinsic risk.

The dependence of the set of states of extrinsic risk on the state of intrinsic risk allows for an intertemporal interpretation.

The distribution of states of extrinsic risk, conditional on a state of intrinsic risk, is described by the probability measure $\pi_s = (\ldots, \pi_{s|f}, \ldots)$, which is strictly positive.

Commodities are $l \in \mathcal{L} = \{1, \ldots, L\}$, a non-empty, finite set; a bundle of commodities at a state of the world is $x_{f,s} = (\ldots, x_{l,f,s}, \ldots)'$, and a bundle of commodities, across states of the world, is $x = (\ldots, x_{f,s}, \ldots)$.

A bundle of commodities across states of the world is invariant with respect to extrinsic risk if $x_{f,s} = x_f$.

7. WEISS [1976].

8. POLEMARCHAKIS [1979].

9. POLEMARCHAKIS and WEISS [1977] or hedonic preferences and product differentiation VENTURA [1995].

An individual is described by the utility function, u^i, over consumption plans, non-negative bundles of commodities across states of the world, and by the endowment, e^i, a bundle of commodities across states of the world.

Assumption 1: For every individual,

1. the utility function is additively separable across states of intrinsic risk, while, conditional on a state of intrinsic risk, the utility function has a von Neumann-Morgenstern representation, with respect to the probability measure π_s, over states of extrinsic risk:

$$u^i = \sum_{f \in \mathcal{F}} \mathrm{E}_{\pi_s} u^i_f;$$

2. the cardinal utility index is strictly concave.

3. the endowment is invariant with respect to extrinsic risk: $e^i_{f,s} = e^i_f$.

The aggregate endowment is $e^a = \sum_{i \in \mathcal{I}} e^i$.

An allocation of consumption plans is $x^{\mathcal{I}} = (\ldots, x^i, \ldots)$; it is feasible if the aggregate consumption plan, $x^a = \sum_{i \in \mathcal{I}} x^i$, coincides with the aggregate endowment: $x^a = e^a$.

An allocation of consumption plans, $x^{\mathcal{I}}$, pareto dominates another, $\tilde{x}^{\mathcal{I}}$, if, for every individual, $u^i(x^i) \geqslant u^i(\tilde{x}^i)$, with strict inequality, $u^i(x^i) > u^i(\tilde{x}^i)$, for some.

A feasible allocation of consumption plans is pareto optimal if no feasible allocation pareto dominates it.

Assumption 2: For every individual,

1. at every state of intrinsic risk, the cardinal utility index, u^i_f, is continuous; in the domain of strictly positive bundles of commodities, it is twice continuously differentiable, differentially strictly monotonically increasing: $Du^i_f \gg 0$, and strictly differentially concave: $D^2 u^i_f$ is negative definite; for a sequence of strictly positive bundles of commodities $(x_{f,n} \gg 0 : n = 1, \ldots)$, and a non-zero consumption, $x_f \neq 0$, on the boundary: $x_f \not\gg 0$, if $\lim_{n \to \infty} x_{f,n} = x_f$, then $\lim_{n \to \infty} \| Du^i_f(x_{f,n}) \| = \infty$, while $\lim_{n \to \infty} \| Du^i_f(x_{f,n}) \|^{-1} x_{n,f} Du^i_f(x_{f,n}) = 0$;
2. the endowment is strictly positive: $e^i \gg 0$.

This is strong, but standard.

It follows that Du^i is bounded away from 0 on any bounded subset of the interior of the domain of definition of the utility function.

A perturbation of the utility function of an individual is defined by

$$u^i_{\delta^i}(x) = u^i(x) + \sum_{f \in \mathcal{F}} \delta^i_f \mathrm{E}_{\pi_s} x_{f,s}$$

where $\delta^i_f = (\ldots, \delta^i_{l,f}, \ldots)$ and $\delta^i = (\ldots, \delta^i_f, \ldots)$ are preference parameters restricted to a neighborhood of the strictly positive orthant of dimension LF.

An economy is identified by the profile of preference parameters $\delta^{\mathcal{I}} = (\ldots, \delta^i, \ \ldots)$, and the set of economies is a neighborhood of the strictly positive orthant of dimension ILF. A property holds generically if, and only if, it holds for an open subset of economies of full lebesgue measure.

Economies differ only with respect to the utility functions of individuals; all, evidently, satisfy assumptions 1 and 2.

At a pareto optimal allocation of consumption plans, the consumption plan of every individual is invariant with respect to extrinsic risk.

3 Inactive Spot Markets

There is no direct exchange of commodities: commodity spot markets are not operative, and commodities are exchanged indirectly, through the exchange of assets.

Assets are $a \in \mathcal{A} = \{1, \ldots, A\}$, a non-empty finite set.

The payoff of an asset, a, at a state of the world, (f, s), is a bundle of commodities $r_{a,f,s} = (\ldots, r_{a,l,f,s}, \ldots)'$, and the payoff of the asset, across states of the world, is a bundle of commodities, $r_a = (\ldots, r_{a,f,s}, \ldots)'$. The matrix of payoffs of assets in state of the world s is $R_{f,s} = (\ldots, r_{a,s}, \ldots)$, and the matrix of payoffs of assets, across states of the world, is $R = (\ldots, R_{f,s}, \ldots)'$. A portfolio of assets is $y = (\ldots, y_a, \ldots)'$. The payoff of a portfolio of assets, y, is Ry, a bundle of commodities across states of the world.

ASSUMPTION 3: There exists a portfolio of assets, $\overline{y}$, with positive payoffs in commodities:

$$R\overline{y} > 0.$$

This prevents satiation in the exchange of assets.

With no loss of generality, the portfolio of assets with positive payoffs coincides with the asset $a = 1 : r_a > 0$

The asset market is complete if the matrix of payoffs of assets is of full column rank, $L(\sum_{fin\mathcal{F}} S_f)$: for every bundle of commodities, x, there exists a portfolio of assets, y, such that $Ry = x$; otherwise it is incomplete.

The consumption of an individual associated with a portfolio of assets, y, is $x^i = e^i + Ry$.

An allocation of portfolios of assets is $y^{\mathcal{I}} = (\ldots, y^i, \ldots)$ such that, for every individual, x^i is a consumption plan; it is feasible if the aggregate portfolio of assets vanishes: $y^a = \sum_{i \in \mathcal{I}} y^i = 0$.

Associated with an allocation of portfolios of assets, there is an allocation of consumption plans and, similarly, for feasible allocations.

An allocation of portfolios of assets, $y^{\mathcal{I}}$, pareto dominates another, $\hat{y}^{\mathcal{I}}$, if and only if, for every individual, $u^i(e^i + Ry^i) \geqslant u^i(e^i + R\hat{y}^i)$, with strict inequality, $u^i(e^i + Ry^i) > u^i(e^i + R\hat{y}^i)$, for some.

A feasible allocation of portfolios is pareto optimal if no feasible allocation pareto dominates it.

The allocation of consumption plans associated with a pareto optimal allocation of portfolios is constrained pareto optimal; it is pareto optimal if the asset market is complete.

Prices of assets are $q = (1, \hat{q})$, where $\hat{q} = (q_2, \ldots)$: asset $a+1$, with positive payoffs, serves as *numéraire*.

For a portfolio of assets, $y = (y_1, \hat{y}')$, where $\hat{y} = (y_2, \ldots)'$.

At prices of assets q, the value of a portfolio of assets, y, is qy.

At prices of assets q, the optimization problem of an individual is

$$\max \sum_{f \in \mathcal{F}} \mathrm{E}_{\pi_s} u^i_s(e^i_f + R_{f,s} y) + \delta^i_f(e^i_f + R_{f,s} y),$$

$$\text{s.t. } qy \leqslant 0.$$

A competitive equilibrium is a pair, $(y^{\mathcal{I}*}, q^*)$, of a feasible allocation of portfolios and prices of assets, such that, for every individual, y^{i*} is a solution to the optimization problem at prices of assets q^*.

Competitive equilibria exist: with inoperative spot markets, the incompleteness of the asset market does not interfere with the existence of competitive equilibria, which involves only the exchange of assets.

Lemma 1: *If*

$$\dim[\sum_{f \in \mathcal{F}} \mathrm{E}_{\pi_s} R_{f,s}] = A, \qquad \text{and}$$

$$2 \leqslant A + 1 \leqslant I,$$

then, generically, at a competitive equilibrium,

$$\dim[\ldots, y^{i*}, \ldots] = A.$$

Proof. The function g is defined by

$$g(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q},\delta^{\mathcal{I}}) = \begin{cases} \sum_{f\in\mathcal{F}} \mathrm{E}_{\pi_s}(Du^i_f + \delta^i_f)R_{f,s} - \lambda^i q, & i \in \mathcal{I}, \\ qy^i, & i \in \mathcal{I}, \\ \sum_{i\in\mathcal{I}} \hat{y}^i, \end{cases}$$

where $\lambda^{\mathcal{I}} = (\ldots,\lambda^{\mathcal{I}},\ldots)$ is strictly positive.

For fixed $\delta^{\mathcal{I}}$, the evaluation function, $f_{\delta^{\mathcal{I}}}$, is defined by

$$g_{\delta^{\mathcal{I}}}(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q}) = g(y^{\mathcal{I}},\lambda^{\mathcal{I}}\hat{q},\delta^{\mathcal{I}}).$$

If

$$(y^{\mathcal{I}*},\lambda^{\mathcal{I}*},\hat{q}^*) \in g^{-1}_{\delta^{\mathcal{I}}}(0),$$

$(y^{\mathcal{I}*},q^*)$ is a competitive equilibrium for the economy $\delta^{\mathcal{I}}$; this is also necessary. Since, for every economy, competitive equilibria exist, for every $\delta^{\mathcal{I}}$,

$$g^{-1}_{\delta^{\mathcal{I}}}(0) \neq \emptyset.$$

The augmented function, $\tilde{g}$, is defined by

$$\tilde{g}(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q},\theta,\delta^{\mathcal{I}}) = \begin{cases} g(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q},\delta^{\mathcal{I}}), \\ \theta\hat{y}^i, & i = 2,\ldots,A+1, \end{cases}$$

where $\|\theta\| = 1$.

For fixed $\delta^{\mathcal{I}}$, the augmented evaluation function, $\tilde{g}_{\delta^{\mathcal{I}}}$, is defined by

$$\tilde{g}_{\delta^{\mathcal{I}}}(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q},\theta) = \tilde{g}(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q},\theta,\delta^{\mathcal{I}}).$$

By direct computation, the augmented function, g, is transverse to 0: it suffices to consider the sub matrix of the jacobean matrix, $D\tilde{g}$, formed by the columns associated with the variables $\ldots,\delta^i,\ldots,y^2_1,\ldots,\hat{y}^1,\hat{y}^2,\ldots,\hat{y}^{A+1}$.

By the transversal density theorem and a standard argument, there exists an open set of economies of full lebesgue measure such that this augmented evaluation function, $\tilde{g}_{\delta^{\mathcal{I}}}$, is transverse to 0.

The dimension of the domain of the augmented evaluation function, $AI + I + (A-1) + (A-1)$, falls short of that of the range, $AI + I + (A-1) + A$. Transversality at 0 is thus equivalent to $\tilde{g}_{\delta^{\mathcal{I}}} \neq 0$ everywhere on its domain.

Equivalently, for every θ with $\|\theta\| = 1$, $g_{\delta^{\mathcal{I}}}(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q}) = 0 \Rightarrow \theta\hat{y}^i \neq 0$, for some $i \in \{2,\ldots,A+1\}$or $f_{\delta^{\mathcal{I}}}(y^{\mathcal{I}},\lambda^{\mathcal{I}},\hat{q}) = 0 \Rightarrow \dim[\hat{y}^2,\ldots,\hat{y}^{A+1}] = A$.

Effective extrinsic risk requires that the span of the matrix of payoffs of assets include bundles of commodities across states of the world not invariant with respect to extrinsic risk.

PROPOSITION 1: If

$$\dim[\sum_{f\in\mathcal{F}} \mathrm{E}_{\pi_s} R_{f,s}] = A,$$

$$2 \leqslant A+1 \leqslant I \text{ and}$$

$$[R] \not\subset \{x : x_{f,s} = x_f,\ (f,s) \in \times_{f\in\mathcal{F}} \mathcal{S}_f\},$$

then, generically, at every competitive equilibrium, extrinsic risk is effective.

Proof. Since the endowment of every individual is invariant with respect to extrinsic risk, it suffices to show that, generically, at every competitive equilibrium, for some individual, $Ry^{i*} \notin \{x : x_{f,s} = x_f,\ (f,s) \in \times_{f\in\mathcal{F}} \mathcal{S}_f\}$. But, by lemma 1, generically, $[\{\ldots, Ry^{i*}, \ldots\}] = [R]$, while $[R] \not\subset \{x : s \sim_{\mathcal{P}} s' \to x_s = x_{s'}\}$.

The condition $I \geqslant A+1 \geqslant 2$, together with the perturbations of the utility functions of individuals guarantee that the economy is sufficiently heterogeneous. They eliminate economies with a representative individual. If the incompleteness of the asset market accounts for effective extrinsic risk, a condition that eliminates an effectively complete asset market, and more, is not surprising.

The condition $\dim[\sum_{f\in\mathcal{F}} \mathrm{E}_{\pi_s} R_{f,s}] = A$ implies that the asset market is incomplete: the matrix $\sum_{f\in\mathcal{F}} \mathrm{E}_{\pi_s} R_{f,s}$ is of dimension $L \times A$, and, hence, $L \geqslant A$; as long as $\sum_{f\in\mathcal{F}} S_f \geqslant 2$, $A < L \sum_{f\in\mathcal{F}} S_f$. Also, it encompasses conditions that are known[10] to preclude effective extrinsic risk.

In the absence of intrinsic risk, one omits the subscript f: all risk is extrinsic and states of extrinsic risk are $s \in \mathcal{S} = \{1, \ldots, S\}$.

If the matrix of payoffs of assets is

$$R = \begin{pmatrix} D & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & D & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & D \end{pmatrix},$$

where D is a matrix of dimension $L \times K$ and $KS = A$, commodities in state s can be traded independently of commodities in state s' and bearing the risk of extrinsic risk is not justified in terms of trades in commodities;

$$\mathrm{E}_\pi R_s = (\pi_1 D, \ldots, \pi_s D, \ldots, \pi_S D),$$

10. GUESNERIE and LAFFONT [1988].

which is not of full column rank and the condition $\dim[\mathrm{E}_\pi R_s] = A$ fails; indeed, extrinsic risk cannot be effective.

Alternatively, if $[[R] \not\subset \{x : x_{f,s} = x_f, (f,s) \in \times_{f \in \mathcal{F}} \mathcal{S}_f\}$, the structure of payoffs of assets allows for effective extrinsic risk. This fails if the matrix of payoffs of assets takes the form

$$R = \begin{pmatrix} D \\ \vdots \\ D \\ \vdots \\ D \end{pmatrix};$$

by the construction of the matrix of payoffs of assets, extrinsic risk is ineffective.

An Example

Dates are 0, 1 and 2.

Assets are traded at date 0 and payoff at dates 1 and 2; assets are not retraded at date 1.

A single, perishable commodity is traded at dates 1 and 2.

States of the world 1 and 2 realize at date 2 with probability π_1 and π_2, respectively.

The matrix of payoffs of assets is

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The columns of this matrix represent the payoffs of the two assets at date 1 and at each state of the world at date 2.

Individuals are $i = 1,2$. An individual has a state-independent, inter temporal utility function $u^i = \ln x_1 + \delta^i E_\pi \ln x_{2,s}$ (with x_1 consumption in period 1 and $x_{2,s}$ consumption at date 2 and state s) and a state-independent endowment, $e^i = (e^i_1, e^i_2)$.

With a complete asset market, the competitive equilibrium allocation is unique.

The utility function of an individual over portfolios of assets is

$$u^i = \ln(e^i_1 + y_1 + y_2) + \delta^i \pi_1 \ln(e^i_2 + y_1) + \delta^i \pi_2 \ln(e^i_2 + y_2),$$

and the budget constraint is

$$y_1 + q y_2 = 0.$$

The consumption plan associated with a portfolio is state-independent if, and only if,

$$y_1 = y_2;$$

from the budget constraint in the asset market, state-independent consumption obtains only at autarky, when

$$y_1 = y_2 = 0.$$

For autarky at a competitive equilibrium it is necessary and sufficient that, for each individual,

$$\frac{1-q}{e_1^i} - \frac{\delta^i \pi_1}{e_2^i} + \frac{\delta^i \pi_2}{e_2^i} = 0.$$

The economy is equivalent to an atemporal economy with commodities $l = 1,2$ at each state of the world, assets with matrix of payoffs

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and individuals with state-independent utility functions $u^i = \mathrm{E}_\pi(\ln x_{1,s} + \delta^i \ln x_{2,s})$ and a state-independent endowment, $e^i = (e_1^i, e_2^i)$; it suffices to identify commodity 1 with the commodity at date 1, and commodity 2 with the commodity at date 2; commodity 1 is available in equal amounts in both states of the world, while the availability of commodity 2 depends on the states of the world.

For $\pi_1 = \pi_2 = 1/2$, the matrix

$$\mathrm{E}_\pi R_s = \begin{pmatrix} 1 & 1 \\ \pi_1 & \pi_2 \end{pmatrix},$$

whose generic element, (i, j), is the average amount of good i paid by asset j has rank $1 < A$, and autarky can indeed be sustained at equilibrium. For $\pi_1, \pi_2 \neq 1/2$, the matrix $\mathrm{E}_\pi R_s$ has rank $2 = A$, and autarky can be sustained at a competitive equilibrium only for non-generic values of the preference parameters.

Indeed, a sufficient condition for sunspots to matter at equilibrium generically refers to the matrix $\mathrm{E}_\pi R_s$.

4 Active Spot Markets

Assets are exchanged prior to the resolution of risk; following the resolution of risk, assets pay off, and commodities are exchanged in spot markets.

Assets pay off in revenue denominated in units of commodity $l = 1$, which serves as *numéraire* at every state of the world.

The payoff of an asset, a, at a state of the world, (f,s), is $r_{a,f,s}$, and the payoff of the asset, across states of the world, is $r_a = (\ldots, r_{a,f,s}, \ldots)'$. The payoffs of assets at a state of the world (f,s), are $R_{f,s} = (\ldots, r_{a,f,s}, \ldots)$, and the matrix of payoffs of assets, across states of the world, is $R = (\ldots, R_{f,s}, \ldots)'$. The payoff of a portfolio of assets, y, is Ry, a bundle of commodities across states of the world.

ASSUMPTION 4: There exists a portfolio of assets, $\overline{y}$, with positive payoffs in revenue:

$$R\overline{y} > 0.$$

With no loss of generality, the portfolio of assets with positive payoffs coincides with the asset $a = 1 : r_a > 0$.

Prices of assets are $q = (1, \hat{q})$, where $\hat{q} = (q_2, \ldots)$. For a portfolio of assets, $y = (y_1, \hat{y}')$, where $\hat{y} = (y_2, \ldots)'$.

The asset market is complete if the matrix of payoffs of assets is of full column rank, $\sum_{f in \mathcal{F}} S_f$: for every transfer of revenue across states of the world, $\tau = (\ldots, \tau_{f,s}, \ldots)$, there exists a portfolio of assets, y, such that $Ry = \tau$; otherwise it is incomplete.

Prices of commodities at a state of the world are $p_{f,s} = (1, \hat{p}_{f,s})$, where $\hat{p}_{f,s} = (p_{2,f,s} \ldots)$; across states of the world, $p = (\ldots, p_{f,s}, \ldots)$.

An allocation of portfolios of assets is $y^{\mathcal{I}} = (\ldots, y^i, \ldots)$; it is feasible if the aggregate portfolio of assets vanishes: $y^a = \sum_{i \in \mathcal{I}} y^i = 0$.

At prices of commodities and assets (p,q), the optimization problem of an individual is

$$\begin{aligned} \max \quad & \sum_{f \in \mathcal{F}} \mathrm{E}_{\pi_s} u^i_f(x_{f,s}) + \delta^i_f x_{f,s}, \\ \text{s.t.} \quad & qy \leqslant 0, \\ & p_{f,s} x_{f,s} \leqslant p_{f,s} e^i_{f,s} + R_{f,s} y \quad (f,s) \in \times_{f \in \mathcal{F}} (f \times \mathcal{S}_f). \end{aligned}$$

A competitive equilibrium is a pair, $((x^{\mathcal{I}*}, y^{\mathcal{I}*}), (p^*, q^*))$, of feasible allocations of consumption plans and portfolios and prices of commodities and assets, such that, for every individual, (x^{i*}, y^{i*}) is a solution to the optimization problem at prices (p^*, q^*).

Competitive equilibria exist: with payoffs of assets denominated in the numéraire commodity, the rank of the matrix of payoffs of assets does not vary with the relative prices of commodities in spot markets.

A feasible allocation of consumption plans and portfolios is constrained suboptimal if there exists a redistribution of portfolios that implements a pareto superior allocation of consumption plans after commodities are exchanged in competitive spot markets. Competitive equilibrium allocations may well be constrained suboptimal.[11]

11. GEANAKOPLOS and POLEMARCHAKIS [1986].

One defines a reduced economy without extrinsic risk: it suffices to define the payoff of an asset at a state of intrinsic risk, f, as the conditional expected payoff across states of extrinsic risk, $s \in \mathcal{S}_f$.

States of the world are $f \in \mathcal{F} = \{1,\ldots,F\}$.

A bundle of commodities at a state of the world is $x_f = (\ldots,x_{l,f},\ldots)'$, and a bundle of commodities, across states of the world, is $x = (\ldots,x_f,\ldots)$.

The utility function of an individual is

$$u^i = \sum_{f \in \mathcal{F}} u^i_f + \delta^i_f x_f,$$

and the endowment $e^i = (\ldots,e^i_f,\ldots)$.

The payoff of an asset, a, at a state of the world, f, is

$$\overline{r}_{a,f} = \mathrm{E}_{\pi_s} r_{a,f,s},$$

and, across states of the world, $\overline{r}_a = (\ldots,r_{a,f},\ldots)'$. The payoffs of assets at a state of the world, f, are $\overline{R}_f = (\ldots,r_{a,f},\ldots)$, and the matrix of payoffs of assets, across states of the world, is

$$\overline{R} = (\ldots,R_f,\ldots)' = (\ldots,\mathrm{E}_{\pi_s} R_{f,s},\ldots)'.$$

Prices of commodities at a state of the world are $p_f = (1,\hat{p}_f)$, where $\hat{p}_f = (p_{2,f}\ldots)$; across states of the world, $p = (\ldots,p_f,\ldots)$.

At prices of commodities and assets (p,q), the optimization problem of an individual is

$$\begin{aligned} \max \quad & \sum_{f \in \mathcal{F}} u^i_f(x_f) + \delta^i_f x_f, \\ \text{s.t.} \quad & qy \leqslant 0, \\ & p_f x_f \leqslant p_f e^i_f + \overline{R}_f y \quad f \in \mathcal{F}. \end{aligned}$$

A competitive equilibrium is a pair, $((x^{\mathcal{I}*},y^{\mathcal{I}*}),(p^*,q^*))$, of feasible allocations of consumption plans and portfolios and prices of commodities and assets, such that, for every individual, (x^{i*},y^{i*}) is a solution to the optimization problem at prices (p^*,q^*).

Lemma 2: If

$$A \leqslant F \quad \text{and}$$
$$2 \leqslant A+1 \leqslant I,$$

then, generically, at every competitive equilibrium for the reduced economy without extrinsic risk,

$$\dim[\ldots,\hat{y}^{i*},\ldots] = A - 1$$

Proof. The argument is evident – a variation of the argument for lemma 1.

PROPOSITION 2: If

$$A \leqslant F,$$
$$2 \leqslant A+1 \leqslant I, \quad \text{and}$$
$$2 \leqslant \dim[R_f], \quad \text{for some } f \in \mathcal{F},$$

then, generically, at every competitive equilibrium, extrinsic risk is effective.

Proof. One restricts attention to the generic set of economies that, from lemma 2, have the property that at every competitive equilibrium for the reduced economy without extrinsic risk, $\dim[\ldots, \hat{y}^{i*}, \ldots] = A-1$.

A competitive equilibrium where extrinsic risk is not effective generates a competitive equilibrium for the reduced economy without extrinsic risk; in particular, $\dim[\ldots, \hat{y}^{i*}, \ldots] = A-1$, while $R_{f,s} y^{i*} = \overline{R}_f y^{i*}$, for every individual and for every state of the world.

Since the matrix $R_f = (\ldots, R_{f,s}, \ldots)'$ has rank at least equal to 2, so does the matrix $(\ldots, R_{f,s} - \overline{R}_f, \ldots)'$.

This leads to a contradiction: the rows, of dimension A, of the matrix $(\ldots, R_{f,s} - \overline{R}_f, \ldots)'$, of which at least 2 are linearly independent, are normal to the columns, of dimension A, matrix $(\ldots, \hat{y}^{i*}, \ldots)$, which has rank $A-1$.

Since $A \leqslant S$, assets suffice, at most, to span intrinsic risks; since, for some f, the rank of the matrix R_f is at least equal to 2, the asset market is incomplete.

The intuition that underlies effective extrinsic risk here is analogous to the intuition for inoperative spot markets: sharing intrinsic risk requires full use of the asset structure due to the heterogeneity of individuals; which prevents the economy from insulating competitive allocations from extrinsic risk.

An Example

The economy extends over two periods, 0 and 1. States of nature at date 1 are 1, 2 and 3. State 1 is a state of intrinsic uncertainty, whereas states 2 and 3 are states of extrinsic uncertainty.

There are two agents, $i = 1,2$, whose preferences can be represented by the following utility function:

$$U^i(.) = 7\,\mathrm{Log}[x_1^i] + \pi_1\sqrt{(x_{21}^i x_{22}^i)} + \pi_2\sqrt{(x_{31}^i x_{32}^i)}$$

and have endowments:

$$e^1 = ((1); (5,6); (5,6))$$

$$e^2 = ((5); (10,11); (10,11))$$

where the numbers in parenthesis represent, respectively, the quantity of good 1 and 2 initially owned by the agents in states 1 (only good 1), 2 and 3 at date 1. The meaning of the symbols appearing in the utility function is evident.

Two assets are traded, with payoffs contingent on the realization of extrinsic uncertainty; they are traded at date 0 and payoff at date 1. Agents can also trade two perishable commodities at date 1.

States of the world 2 and 3 realize with probability π_1 and π_2, respectively (we ignore, for simplicity, the relative probability of state 1). The matrix of payoffs of assets is:

$$R = \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The columns of this matrix represent the payoffs of the two assets at each state of nature at date 1. Asset 1, therefore, pays minus one unit of good 1 in state 1, one unit of good 1 in state 2 and nothing in state three. Asset 2, on the other hand, will payoff minus one unit of good 1 in state 1 and one unit of good 1 in state 3.

The optimization problem of agent i will be:

$$\text{Max}_{y_1,y_2,x} U^i(.)$$

subject to the following budget constraints:

$$\begin{aligned} y_1^i + q y_2^i &= 0. \\ x_{11}^i &= e_{11}^i - y_1^i - y_2^i \\ x_{21}^i + p_2 x_{22}^i &= e_{21}^i + p_2 e_{22}^i + y_1^i \\ x_{31}^i + p_3 x_{32}^i &= e_{31}^i + p_3 e_{32}^i + y_2^i \end{aligned}$$

The first budget constraint relates to time 0, and states that agent i's portfolio, (y_1^i, y_2^i), is self-financed (q is the relative price of asset 2 with respect to asset1). The other three budget constraints relate to the three states of nature at date 1.

Given that preferences in states 2 and 3 are essentially of a Cobb-Douglas type, the optimization problems of the two agents can be easily solved.

Setting and solving the equilibrium equations:

$$y_2^1 + y_2^2 = 0$$

$$x_{22}^1 + x_{22}^2 = 17$$

$$x_{32}^1 + x_{32}^2 = 17$$

we obtain the equilibrium prices and quantities, for $\pi = 0.8$:

$$q = 1.0829;\ p2 = p3 = 0.8823;$$

$$y_1^i = -y_1^j = -8.30;\ y_2^i = -y_2^j = 7.67;$$

$$x_1^i = 1.63;\ x_{22}^i = 1.12;\ x_{32}^i = 10.18;$$

$$x_1^j = 4.36;\ x_{22}^j = 15.87;\ x_{32}^j = 6.82.$$

This equilibrium allocation is clearly sunspot dependent, and the same would hold true for any value of π_1 different from 0.5 (in which case the allocation is sunspot free, as agents do not buy any quantity of either asset).

We can easily check that the conditions of Lemma 2 and Proposition 2 are satisfied in this example.

Once again, the intuition underlying the example is quite simple: markets are incomplete (two assets for three states of the world) and agents have a strong (and opposite) incentive to redistribute wealth from state 1 and states 2 and 3. In order to effect this redistribution, they are willing to use the only available assets, with sunspot contingent payoffs. In so doing, agents will be inevitably exposed to sunspot uncertainty, which is the price they pay to hedge against intrinsic uncertainty.

5 Conclusion

In this note, we have introduced sufficient conditions under which extrinsic uncertainty is, generically, relevant at equilibrium. The class of economies to which we can apply this condition is quite broad, in that it encompasses both economies with uncertainty and multiple goods per period, without spot trading (*ie*, only assets are traded in the first period and no other trade occurs), and economies stretching over many periods and/or states of nature and multiple goods per period.

Our result does not depend on the existence of multiple equilibria in the corresponding economy with complete markets and holds even if the structure of payoffs of assets allows for non-trivial allocations invariant with respect to the extrinsic uncertainty ($[R] \cap \{x : s \sim_{\mathcal{P}} s' \Rightarrow x_s = x_{s'}\} \neq \{0\}$ in the terminology of the previous sections).

The economic intuition underlying this result is very simple: when markets are incomplete, agents are (generically) willing to trade assets whose payoffs (also) depend on realizations of extrinsic uncertainty. By doing so they can smooth consumption over dates and events more efficiently, even at the cost of getting exposed to the effects of extrinsic uncertainty. ■

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APPENDIX

Notation

- "$'$" denotes the transpose of a vector or a matrix.
- "[ ]" denotes the span of a set of vectors or the column span of a matrix.
- "E_μ" denotes the expectation with respect to the probability measure μ.
- For vectors, $a = (\ldots, a_k, \ldots)$, and $b = (\ldots, b_k, \ldots)$, $a \geqslant b$ if $a_k \geqslant b_k$, for every k, $a > b$ if $a_k \geqslant b_k$, for every k, with strict inequality for some k, and $a \gg b$ if $a_k > b_k$, for every k, and analogously for $a \leqslant b$, $a < b$, and $a \ll b$. A vector, a, is strictly positive if $a \gg 0$, positive if $a > 0$, non-negative if $a \geqslant 0$, non-positive if $a \leqslant 0$, negative if $a < 0$, and strictly negative if $a \ll 0$.