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Economic Theory

Exchange and optimality★

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Summary. A feasible social state is irreducible if and only if, for any non-trivial partition of individuals into two groups, there exists another feasible social state at which every individual in the first group is equally well-off and someone strictly better-off. Competitive equilibria decentralize irreducible Pareto optimal social states.

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1 Introduction

The theorems of classical welfare economics (Arrow, 1951; Debreu, 1951) identify competitive equilibrium allocations with Pareto optimal allocations; this makes the case for the market as a mechanism for the allocation of resources.

Competitive equilibrium allocations are Pareto optimal if individuals are not satiated; Pareto optimal allocations are competitive equilibrium allocations if the economy is convex. For both, no external effects arise across individuals; this is the focus of this paper.

External effects do not arise when the objects of choice of each individual are the only arguments of his utility function, while his choice is restricted only by a budget constraint that reflects aggregate scarcity.

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Altruism or envy, which allow the consumption of one individual, chosen and paid for by him, to affect positively or adversely the utility of another, or public goods, for which the consumption of one individual is augmented by the consumption of another, constitute external effects and interfere with the identification of competitive equilibria with Pareto optima.

First, Lindahl (1919, 1928) realized that optimal allocations in the presence of external effects can be identified with competitive equilibrium allocations for a particular extension of the set of commodities and markets: it suffices to consider the arguments of the utility function of each individual as objects of choice distinct from the objects of choice of other individuals and price them correctly, requiring that at equilibrium, and possibly only there, the choices of individuals concerning an object of collective choice coincide. Individualized markets can hardly be competitive (Arrow, 1970), the revelation of the private informations of individuals necessary for the implementation of competitive outcomes is not evident (Groves and Ledyard, 1977), even the convexity of the economy is problematic (Starrett and Zeckhauser, 1974). Nevertheless, Lindahl equilibria remain an illuminating benchmark.

Here, in the framework of Lindahl, with a set of abstract social states as the domain of the preferences of individuals, one poses the question whether Pareto optimal states can obtain as competitive equilibria of an associated economy.

Importantly, there be any private goods. This is called for if social states are interpreted as complete descriptions of economic or social activity as is the case in social choice theory. Normative arguments often adopt a view point prior to any economic or social acitivity and, in this context, private goods are not appropriate.

In the economy which, using the construction of Lindahl, is associated with the set of social states, neither the free disposal of commodities nor the non-satiation of preferences of individuals are satisfied.

Two insights stand out:

1. For an arbitrary distribution of revenue, competitive equilibria need not exist.

2. Though standard conditions, importantly convexity, suffice for a Pareto optimal social state to be a quasi-equilibrium, the social state must be "irreducible" in order to be a competitive equilibrium.

In simple arguments (Coase, 1960), competitive equilibrium allocations are independent of the distribution of revenue. Here, the distribution of revenue may prevent the existence of a competitive equilibrium.

At a quasi-equilibrium, individuals minimize expenditure for the level of utility attained, while at a competitive equilibrium they maximize utility for the expenditure incurred. In the absence of external effects, a condition of irreducibility (McKenzie, 1959, 1961) guarantees the passage from quasi-equilibria to competitive equilibria and avoid the alternative, and less appealing, assumption that the Pareto optimum or the endowment are effectively interior to the consumptions sets of individuals (Arrow and Debreu, 1954).

An economy, with only private goods, is irreducible if and only if "[for any partition of] the consumers into two groups, if the first group receives an aggregate trade which is an attainable output for the rest of the market, the second group has within its feasible aggregate trades one which if added to the goods already obtained by the first group could be used to improve the position of someone in the group while damaging the position of none there" (McKenzie, 1959, pp. 58, 59). Irreducibility guarantees that if "some individual has income in the sense that he can trade with a budget whose value is below zero" (McKenzie, 1959, p. 58), which is the case by construction, all individuals do, and then, by a standard argument, a quasi-equilibrium is a competitive equilibrium. An economy with private and public goods or externalities can be defined to be irreducible if the private sector is irreducible, which suffices.

With preferences defined over social states and without recourse to private goods, which would simplify the argument (Foley, 1970; Mas-Colell, 1980; Milleron, 1972), the analogue of irreducibility may not seem evident; but it is : a feasible social state is irreducible if and only if, for any partition of the individuals into two groups, there exists an alternative feasible social state at which every individual in the first group is equally well off and someone strictly better off. So stated, the definition could apply equally well to an economy with private goods: the economy is irreducible if any, individually rational, feasible allocation is irreducible. Irreducible Pareto optimal social states are competitive equilibria of an associated economy.

2 Social states

Individuals are $i \in \mathscr{I} = \{1, \ldots, I\}$, a finite set.

Dimensions of a social state are $n \in \mathscr{N} = \{1, \ldots, N\}$, a finite set. A social states are $s \in \Re^N$, Euclidean space of finite dimension.

An individual, i, is described by the pair $(\mathscr{S}^i, u^i)$, where $\mathscr{S}^i \subset \Re^N$ is the domain of social states for the individual, and $u^i : \mathscr{S}^i \to \Re$ is his ordinal utility function[1].

The set of feasible social states is $\mathscr{S} \subseteq \Re^N$.

A society is

$$\sum = \{\mathscr{I}, (\mathscr{S}^i, u^i) : i \in \mathscr{I}, \mathscr{S}\} \ .$$

Assumption 1 (Compatibility) *The set of feasible social states is contained in the domain of social states of each individual*:

$$\mathscr{S} \in \mathscr{S}^i, \quad i \in \mathscr{I} \ .$$

[1] The representation of preferences by a utility function is not an essential loss of generality.

Social states not in the domain of social states for some individual can be ignored; it is, nevertheless, appropriate to realize that aggregate feasibility is a constraint beyond individual feasibility.

An individual, i, is satiated at a social state, $\bar{s}$, if and only if $u^i(\bar{s}) \geq u^i(s)$, for all $s \in \mathscr{S}^i$. An individual, i, is locally satiated at a social state, $\bar{s}$ if and only if there exists a social state $\bar{\bar{s}}$ and a neighborhood of $\bar{\bar{s}}, \mathscr{V}^i_{\bar{\bar{s}}} \subset \mathscr{S}^i$, such that $u^i(\bar{\bar{s}}) \geq u^i(s)$, for all $s \in \mathscr{V}^i_{\bar{\bar{s}}}$; this is stronger than the standard notion of local non-satiation, since the state $\bar{\bar{s}}$ may be different from the state $\bar{s}$.

At a social state s, for a set of individuals $\hat{\mathscr{I}} \subset \mathscr{I}, u^{\hat{\mathscr{I}}}(s) = (u^i(s) : i \in \hat{\mathscr{I}})$.

A social state, s, Pareto dominates another, s' if and only if $u^{\mathscr{I}}(s) > u^{\mathscr{I}}(s')$; it dominates it strongly if and only if $u^{\mathscr{I}}(s) \gg u^{\mathscr{I}}(s')$.

A feasible social state is Pareto optimal if and only if no feasible state Pareto dominates it; it is weakly Pareto optimal if and only if no feasible state Pareto dominates it strongly.

Assumption 2 (Continuity) *The set of feasible social states, $\mathscr{S}$, is compact. The utility function of every individual, $u^i, i \in \mathscr{I}$, is continuous.*

If the assumption of continuity fails, Pareto optimal social states, even weakly Pareto optimal ones, need not exist.

Lemma 1 *Under the assumptions of compatibility and continuity, Pareto optimal social states exist.*

Proof. For $\bar{s} \in \mathscr{S}$, a social state, the set $\mathscr{S}^0 = \{s \in \mathscr{S} : u^{\mathscr{I}}(s) \geq u^{\mathscr{I}}(\bar{s})\}$ is non-empty and compact.

The sets defined inductively by

$$\bar{\mathscr{S}}^i = \{s \in \mathscr{S}^{i-1} : u^i(s) \geq u^i(s'), s' \in \bar{\mathscr{S}}^{i-1}\}, \quad i \in \mathscr{I}$$

are non-empty and compact and form a nested sequence:

$$\bar{\mathscr{S}}^0 \supset \cdots \supset \bar{\mathscr{S}}^{i-1} \supset \bar{\mathscr{S}}^i \supset \cdots \supset \bar{\mathscr{S}}^I \ .$$

Social states in $\bar{\mathscr{S}}^1$ are weakly Pareto optimal; since the set $\bar{\mathscr{S}}^1$ is non-empty, weakly Pareto optimal social states exist.

Social states in $\bar{\mathscr{S}}^I$ are Pareto optimal.

If not, there exist social states $s^* \in \bar{\mathscr{S}}^I$ and $\hat{s} \in \mathscr{S}$, such that $u^{\mathscr{I}}(\hat{s}) > u^{\mathscr{I}}(s^*)$; in particular, there exists an individual $\hat{i} \in \mathscr{I}$, such that $u^{\hat{i}}(\hat{s}) > u^{\hat{i}}(s^*)$, while, for individuals $i \in \mathscr{I} \setminus \{\hat{i}\}, u^i(\hat{s}) \geq u^i(s^*)$.

Since $u^{\hat{i}}(s^*) \geq u^{\hat{i}}(s)$, for all $s \in \bar{\mathscr{S}}^{\hat{i}-1}, \hat{s} \notin \bar{\mathscr{S}}^{\hat{i}-1}$.

If $\hat{i} = 1$, then $\hat{s} \notin \bar{\mathscr{S}}^0$, a contradiction, since $u^{\mathscr{I}}(\hat{s}) > u^{\mathscr{I}}(s^*) \geq u^{\mathscr{I}}(\bar{s})$.

If $\hat{i} > 1$, since $\hat{s} \notin \bar{\mathscr{S}}^{\hat{i}-1}$, while $u^{\hat{i}-1}(\hat{s}) \geq u^{\hat{i}-1}(s^*) \geq u^{\hat{i}-1}(s)$, for all $s \in \mathscr{S}^{\hat{i}-2}, \hat{s} \notin \bar{\mathscr{S}}^{\hat{i}-2}$. Similarly, $\hat{s} \notin \bar{\mathscr{S}}^{\hat{i}-3}, \ldots, \hat{s} \notin \bar{\mathscr{S}}^0$, as above, a contradiction. □

The procedure in the proof of lemma 1 generates all Pareto optimal or weakly Pareto optimal social states by varying $\bar{s}$, the initial social state.

3 An economy

Agents in the economy are individuals, $i \in \mathscr{I}$, and a firm.

Distinct commodities, objects of exchange, correspond to distinct dimensions of social states or distinct individuals; this is necessary for the variables that affect the preferences of each individual to be the objects of choice of the individual.

Commodities are $(n, i) \in \mathscr{N} \times \mathscr{I}$. A bundle of commodities is $z = (\dots, z_i, \dots)$, where $z_i = (\dots, z_{n,i}, \dots)$; the domain of bundles of commodities is $\Re^{IN} = \times_{i \in \mathscr{I}} \Re^N$.

The firm is characterized by its technology, $\mathscr{Y} \subset \Re^{IN}$, a set of $y = (\dots, y_i, \dots)$, production bundles.

An individual is described by the pair $(\mathscr{X}^i, u^i)$, where $\mathscr{X}^i \subset \Re^{IN}$ is the exchange set for the individual, a set of $z = (\dots, z_i, \dots)$, exchange bundles, and $u^i : \mathscr{X}^i \to \Re$ is his ordinal utility function.

The characteristics of individuals and of the firm reflect the set of social states and the preferences of individuals.

The technology of the firm is

$$\mathscr{Y} = \{y = (\dots, y_i, \dots) \in \Re^{IN} : y_i = y_0 \in \mathscr{S}, i \in \mathscr{I}\} \ .$$

The exchange set of individual i is

$$\mathscr{X}^i = \{z = (\dots, z_i, \dots) \in \Re^{IN} : z_i \in \mathscr{S}^i, \text{ while } \ z_{i'} = 0, i' \in \mathscr{I} \setminus \{i\}\} \ .$$

There is an unambiguous association of a social state with an exchange bundle, a bundle in the exchange set of the individual, and vice versa. The utility function of the individual over exchange bundles is defined by $u^i(z) = u^i(z_i)$[2].

The economy associated with the society Σ is

$$\mathscr{E}(\Sigma) = \{\mathscr{I}, \mathscr{Y}, (\mathscr{X}^i, u^i) : i \in \mathscr{I}\} \ .$$

A state of the economy is an array $a = (y, \dots, z^i, \dots)$ of a production plan for the firm, $y \in \mathscr{Y}$, and, for each individual, of $z^i \in \mathscr{X}^i$, an exchange bundle.

A state of the economy is feasible if and only if $\sum_{i \in \mathscr{I}} z^i = y$. From the structure of the technology of the firm and the exchange sets of individuals, an allocation is feasible if and only if there is a feasible social state, $s \in \mathscr{S}$, such that $z^i_i = s$, for every individual, while $y = (\dots, s, \dots)$, for the firm. There is an unambiguous association of a feasible social state with a feasible state of the economy, and vice versa.

4 Competitive equilibria

The domain of prices of commodities is $\Re^{IN}$, as dual to the space of bundles of commodities.

[2] It is pedantic to use different symbols for the utility functions on $\mathscr{X}^i$ and $\mathscr{S}^i$.

Prices of commodities are $p = (\ldots, p_i, \ldots)$, where $p_i = (\ldots, p_{n,i}, \ldots)$. The value of a bundle of commodities, $z = (\ldots, z_i, \ldots)$ at prices $p = (\ldots, p_i, \ldots)$ is $pz = \sum_{i \in \mathscr{I}} p_i z_i$.

A competitive equilibrium is a pair, (p^*, a^*), of prices of commodities and a feasible allocation, such that

1. $p^* y^* \geq p^* y$, for all $y \in \mathscr{Y}$;
2. $u^i(z) > u^i(z^{i*}) \Rightarrow p^* z > p^* z^{i*}$, for all $i \in \mathscr{I}$.

A quasi-equilibrium is a pair, (p^*, a^*), of prices of commodities and a feasible allocation, such that

1. $p^* y^* \geq p^* y$, for all $y \in \mathscr{Y}$;
2. $u^i(z) \geq u^i(z^{i*}) \Rightarrow p^* z \geq p^* z^{i*}$, for all $i \in \mathscr{I}$;
3. $u^i(z) > u^h(z^{h*}) \Rightarrow p^* z > p^* z^{i*}$, for some $i \in \mathscr{I}$.

At a competitive equilibrium, all individuals maximize utility, while at a quasi-equilibrium, all individuals minimize expenditure and some, simultaneously, maximize utility; at either, the firm maximizes profit.

At a competitive equilibrium, (p^*, a^*), such that the associated feasible social state, s^*, is an interior point of the set of feasible social states, $\sum_{i \in \mathscr{I}} p_i^* = 0$.

That, at a quasi-equilibrium, some individuals maximize utility is necessary to prevent the definition from being vacuous, with $p^* = 0$.

With one individual, a competitive equilibrium is a quasi-equilibrium; weak Pareto optimality and Pareto optimality coincide.

A feasible allocation, a^*, is a competitive equilibrium allocation if and only if (p^*, a^*) is a competitive equilibrium for some prices, p^*. Similarly, a^* is a quasi-equilibrium allocation if and only if (p^*, a^*) is a quasi-equilibrium.

Competitive equilibria or quasi-equilibria as defined here do not fix the distribution of revenue across individuals. For this reason, the ownership of commodities or shares of firms by individuals is not part of the specification of the economy.

A distribution of revenue is $\tau = (\ldots, \tau^i, \ldots)$. A competitive equilibrium or a quasi-equilibrium for a distribution of revenue is such that, for every individual, $p^* z^i \leq \tau^i$, and, as a consequence, $p^* y^* \leq \sum_{i \in \mathscr{I}} \tau^i$.

With a fixed distribution of revenue, competitive equilibria need not exist.

Example 1 The set of feasible social states is $\mathscr{S} = \{s : -1 \leq s \leq 1\} \subset \Re$. Individuals are $i \in \mathscr{I} = \{1, 2\}$ and have utility functions $u^1 = s, u^2 = -(s - 1/2)^2$, with domain the set of feasible social states: $\mathscr{S}^i = \mathscr{S}$. The distribution of revenue is fixed and the revenue of each individual is 0. At prices $p = (p_1, p_2)$, the budget constraint of individual i is $p_i z_i^i \leq 0$. If $p_1 + p_2 = 0$, the budget constraints take the form $p_1 z_1^1 \leq 0$ and $p_1 z_2^2 \geq 0$, respectively. If $p_1 = 0$, then $z_1^1 = 1$, while $z_2^2 = 1/2$; if $p_1 > 0$, then $z_1^1 = 0$, while $z_2^2 = 1/2$; if $p_1 < 0$, then $z_1^1 = 1$, while $z_2^2 = 0$; all are contradictions. Alternatively, if $p_1 + p_2 > 0$, from the profit maximization of the firm, $y = (1, 1)$. If $z_1^1 = 1$ and $z_2^2 = 1$, then it follows from the budget constraints of individuals that

$p_1 \leq 0$ and $p_2 \leq 0$ or $p_1 + p_2 \leq 0$, a contradiction. Similarly, $p_1 + p_2 < 0$ leads to a contradiction. Thus, competitive equilibrium prices, where $z_1^{1*} = z_2^{2*} \in \mathscr{S}$, do not exist. □

Though competitive equilibria do not exist for some distribution of revenue, they do for other distributions. In particular, if $\tau^1 + \tau^2 = 0$, distributions of revenue with $1/2 \leq \tau^i \leq 1$ support the allocations associated with social states $s \in [1/2, 1]$ as competitive equilibrium allocations.

It is a consequence of proposition 1, below, that, under standard assumptions, there exist distributions of revenue for which competitive equilibria exist.

At the bundle $\bar{z} \in \mathscr{X}^i$, individual i is satiated if and only if he is satiated at the state $\bar{z}_i$. The individual is locally satiated if and only if he is locally satiated at $\bar{z}_i$.

A social state where all individuals are satiated is a competitive equilibrium, as well as a quasi-equilibrium, with $p^* = 0$, which is natural.

A competitive equilibrium, (p^*, a^*), is a quasi-equilibrium if no individual is locally satiated at z^{i*}.

A competitive equilibrium allocation or a quasi-equilibrium allocation is weakly Pareto optimal, but not necessarily Pareto optimal. A competitive equilibrium allocation which is also a quasi-equilibrium is Pareto optimal.

A feasible allocation, a^*, which is weakly Pareto optimal, need not be Pareto optimal even if no individual is locally satiated at z^{i*}.

Assumption 3 (Convexity) *The set of feasible social states, $\mathscr{S}$, is convex. For every individual, the domain of social states, $\mathscr{S}^i$, is convex and the utility function, u^i, is quasi-concave.*

If the assumption of convexity fails, competitive equilibria or quasi-equilibria need not exist.

At prices p, the bundle $\bar{z} \in \mathscr{X}^i$ is a minimum expenditure bundle for individual i if and only if $pz \geq pz^i$ or $p_i z_i \geq p_i z_i^i$, for all $z \in \mathscr{X}^i$.

Under the assumptions of continuity a convexity, a quasi-equilibrium, (p^*, a^*), is a competitive equilibrium if there is no individual for whom z^{i*} is a minimum expenditure bundle at prices p^*.

Assumption 4 (Full dimensionality) *The set of feasible social states has non-empty interior.*

This is a regularity assumption. Under the assumption of convexity, it is without loss of generality: it suffices to consider the lowest dimensional affine space that contains the set of feasible social states and then rely on convexity to guarantee that the set has non-empty interior.

5 Exchange and optimality

Under standard assumptions, the allocation associated with a social state which is weakly Pareto optimal is a quasi-equilibrium allocation.

Lemma 2 *Under the assumptions of compatibility, continuity, convexity and full dimensionality, the allocation associated with a social state which is weakly Pareto optimal and at which no individual is locally satiated is a quasi-equilibrium allocation with prices $p^* \neq 0$.*

Proof. The social state s^* is weakly Pareto optimal and a^* is the associated allocation. No individual, i, is locally satiated at s^* and, hence, at z^{i*}.

The set of aggregate bundles of commodities associated with allocations that strictly Pareto dominate the allocation a^* is

$$\mathscr{D}(a^*) = \Big\{ x \in \Re^{IN} : x = \sum_{i \in \mathscr{I}} z^i - y, \text{such that } y \in \mathscr{Y}, \text{ while } z^i \in \mathscr{Z}^i \text{ and } u^i(z^i) > u^i(z^{i*}),\ i \in \mathscr{I} \Big\} .$$

By the assumption of convexity assumption, the set $\mathscr{D}(a^*)$ is convex.

Since, ata^*, no individual is satiated $\mathscr{D}(a^*) \neq \emptyset$.

Since the state s^* is weakly Pareto optimal, $0 \notin \mathscr{D}(a^*)$.

By the proper separating hyperplane theorem[3], there exists $p^* \neq 0$, such that $p^* x \geq 0$, for all $x \in \mathscr{D}(a^*)$.

If, for some individual, $\hat{i}, u^{\hat{i}}(z^{\hat{i}}) \geq u^{\hat{i}}(z^{\hat{i}})$, since individuals are not locally satiated at z^{h*}, there exist sequences, $(z^i_n : n = 1, \ldots)$, for $i \in \mathscr{I}$, such that $\lim_{n \to \infty} z^i_n = z^{i*}$, for $i \in \mathscr{I} \setminus \{\hat{i}\}$, while $\lim_{n \to \infty} z^{\hat{i}}_n = z^{\hat{i}}$, and $x_n = \sum_{i \in \mathscr{I}} z^i_n - y^* \in \mathscr{D}(a^*)$, for $n = 1, \ldots.$ It follows that $p^*(\sum_{i \in \mathscr{I}} z^i_n - y^*) \geq p^*(\sum_{i \in \mathscr{I}} z^{i*}_n - y^*) = 0$, for $n = 1, \ldots,$ and, hence $p^* z^{\hat{i}} \geq p^* z^{\hat{i}*}$. By a similar argument, $p^* y^* \geq p^* y$, for all $y \in \mathscr{Y}$.

If $p^* z^i \geq p^* z^{i*}$, for all $z^i \in \mathscr{Z}^i$ and all $i \in \mathscr{I}$, by the assumption of compatibility and profit maximization, it follows that $p^* y \geq p^* y^*$, for all $y \in \mathscr{Y}$. If $\sum_{i \in \mathscr{I}} p^*_i = 0$, since $p^* \neq 0$, without loss of generality, $p^*_1 \neq 0$. It follows that $p^*_1 s \geq p^*_1 s^*$ and $(\sum_{i \in \mathscr{I} \setminus \{1\}} p^*_i) s \geq (\sum_{i \in \mathscr{I} \setminus \{1\}} p^*_i) s^*$, for all $s \in §$; since $\sum_{i \in \mathscr{I}} p^*_i = 0$, $p^*_1 s = p^*_1 s^*$, for all $s \in \mathscr{S}$, which contradicts full dimensionality. If $\sum_{i \in \mathscr{I}} p^*_i \neq 0$, a contradiction follows by a similar argument. □

The allocation associated with a social state which is weakly Pareto optimal, and at which some individual is locally satiated, need not be a quasi-equilibrium allocation.

Example 2 The set of feasible social states is $\mathscr{S} = \{s : 0 \leq s \leq 3\} \subset \Re$. Individuals are $i \in \mathscr{I} = \{1, 2\}$ and have utility functions $u^1 = \min\{s, 1\}, u^2 = s$, respectively, with domain the set of feasible social states: $\mathscr{S}^i = \mathscr{S}$. The social state $s^* = 2$ is weakly Pareto optimal, but not Pareto optimal: it is dominated

[3] Rockafellar (1970, 97, theorem 11.3): There exists a hyperplane separating properly C_1 and C_2, non-empty sets, if and only if there exists a row vector, $b \neq 0$, such that $\inf\{bx : x \in C_1\} \geq \sup\{bx : x \in C_2\}$, while $\sup\{bx : x \in C_1\} > \inf\{bx : x \in C_2\}$. In order that there exist a hyperplane separating properly C_1 and C_2, non-empty, convex, sets, it is necessary and sufficient that the relative interior of C_1 and the relative interior of C_2 have no points in common.

by the states in $\mathscr{S}^* = \{s \in \mathscr{S} : 2 < s \leq 3\}$. If $p^* = (p_1^*, p_2^*)$ is such that $p_1^* s \geq p_1^* s^*$ whenever $u^2(s) \geq u^2(s^*)$, $p_1^* \geq 0$. It follows that $p_2^* \leq 0$. If $\sum_{i \in \mathscr{I}} p_i^* = 0$, as profit maximization requires, since s^* is an interior state, and $p^* \neq 0$, $p_2^* < 0$. But then, for $s \in \mathscr{S}^1 = \{s \in \mathscr{S} : 2 < s \leq 3\}, u^1(s) = u^1(s^*)$, while $p_2^* s < p_2^* s^*$. □

Under the assumption of convexity, the allocation associated with a social state, not necessarily a Pareto optimal one, at which some individual is satiated is a competitive equilibrium allocation.

Example 3 The set of feasible social states is $\mathscr{S} = \{s = (s_1, s_2) : 0 \leq s_1 \leq 2,\ 0 \leq s_2 \leq 2\} \subset \Re^2$. Individuals are $i \in \mathscr{I} = \{1, 2, 3\}$ and have utility functions $u^1 = s_1 + s_2, u^2 = s_2 - s_1,\ u^3 = \min\{0, 1 - s_2\}$, respectively, with domain the set of feasible social states: $\mathscr{S}^i = \mathscr{S}$. The social state $s^* = (1, 0)$ is not Pareto optimal – it is dominated by states in $\mathscr{S}^* = \{s \in \mathscr{S} : s_1 + s_2 > 1,\ s_2 - s_1 > -1,\ s_2 \leq 1\}$, which is non-empty: $(1, 1) \in \mathscr{S}^*$. Nevertheless, the satiation of individual $i = 3$ at s^* allows the associated allocation to obtain as a competitive equilibrium allocation at prices $p^* = (p_1^*, p_2^*, p_3^*)$, where $p_1^* = (1, 1),\ p_2^* = (-1, 1),\ p_3^* = (0, -2)$. □

Under the assumptions of continuity and convexity, a social state at which no individual is satiated yields the same utility to all individuals as another state at which at least one individual is not locally satiated.

Corollary 1 *Under the assumptions of compatibility continuity, convexity and full dimensionality, the allocation associated with a social state which is Pareto optimal, and at which no individual is satiated and at least one individual is not locally satiated, is a quasi-equilibrium allocation.*

Proof. The social state s^* is weakly Pareto optimal and a^* is the associated allocation. Individual $i = 1$ is not locally satiated at s^* and, hence, at z^{1*}.

The argument proceeds as in the proof of Lemma 2, only with

$$\mathscr{D}(a^*) = \Big\{ x \in \Re^{IN} : x = \sum_{i \in \mathscr{I}} z^i - y, \text{ such that } y \in \mathscr{Y}, \text{ while } z^i \in \mathscr{Z}^i \text{ and } u^i(z^i) \geq u^i(z^{i*}), i \in \mathscr{I}, \text{ with } u^1(z^1) > u^1(z^{1*}) \Big\}.$$

It suffices to observe that, since no individual is satiated at s^*, the set $\mathscr{D}(a^*)$ is non-empty, while $0 \notin \mathscr{D}(a^*)$. □

6 Irreducibility

The allocation associated with a social state which is Pareto optimal, even in the interior of the set of feasible social states, and which, according to Lemma 2, is a quasi-equilibrium allocation, need not be a competitive equilibrium allocation.

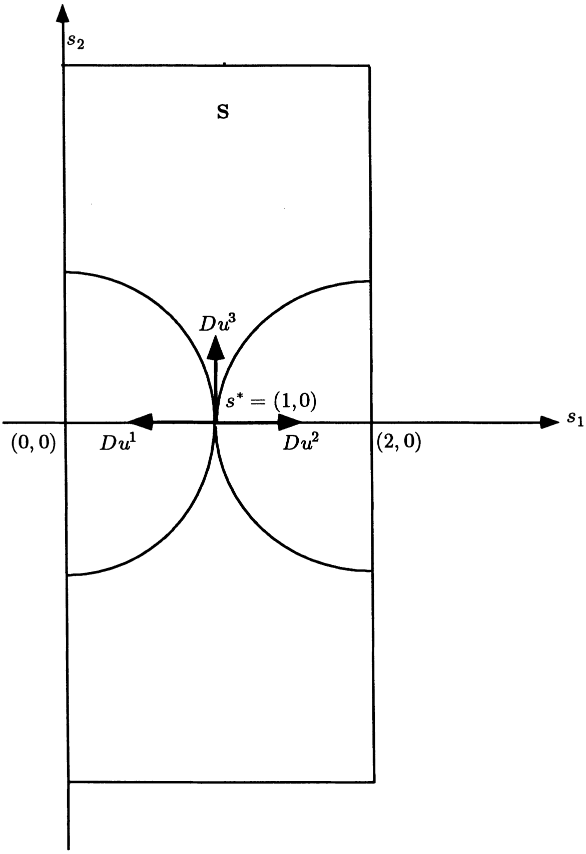

Figure 1

Example 4 The set of feasible social states is $\mathscr{S} = \{s = (s_1, s_2) : 0 \leq s_1 \leq 2, -2 \leq s_2 \leq 2\} \subset \Re^2$. Individuals are $i \in \mathscr{I} = \{1, 2, 3\}$ and have utility functions $u^1 = -\|s\|^2$, $u^2 = -\|s - (2,0)\|^2$, $u^3 = s_2$, respectively, with domain the set of feasible social states: $\mathscr{S}^i = \mathscr{S}$. The social state $s^* = (1,0)$ is a Pareto optimal social state where no individual is locally satiated. The associated allocation is a quasi-equilibrium allocation, but not an equilibrium allocation. At s^*, $Du^1 = (-2,0)$, $Du^2 = (2,0)$ and $Du^3 = (0,1)$. If $p^* = (p_1^*, p_2^*, p_3^*)$, with $\sum_{i \in \mathscr{I}} p_i^* = 0$, as profit maximization requires are competitive equilibrium prices, then, for individual $i = 1$, $Du^1 ds > 0 \Rightarrow p_{1,1}^* ds_1 + p_{1,2}^* ds_2 > 0$ or $p_{1,1}^* < 0$ and $p_{1,2}^* = 0$. Similarly, for individual $i = 2$, $p_{2,1}^* > 0$ and $p_{2,2}^* = 0$. For individual $i = 3$, $Du^3 ds > 0 \Rightarrow p_{3,1}^* ds_1 + p_{3,2}^* ds_2 > 0$ or $p_{3,1}^* = 0$ and $p_{3,2}^* > 0$. But this leads to a contradiction, since $\Sigma_{i \in \mathscr{I}} p_{i,2}^* > 0$. Nevertheless, the allocation is a quasi-equilibrium allocation at prices $p^* = (p_1^*, p_2^*, p_3^*)$, where $p_1^* = (-1,0), p_2^* = (1,0), p_3^* = (0,0)$. □

A feasible social state, s^* irreducible if and only if, for any non-empty proper subset of individuals, $\hat{\mathscr{I}} \subset \mathscr{I}, \hat{\mathscr{I}} \neq \emptyset, \mathscr{I}$, there exists a feasible social state, $\hat{s}$, such that $u^{\hat{\mathscr{I}}}(\hat{s}) > u^{\hat{\mathscr{I}}}(s^*)$.

The state s^* in example 5 is not irreducible: for $\hat{\mathscr{I}} = \{1, 2\}$, there is no state $\hat{s}$, such that $u^{\hat{\mathscr{I}}}(\hat{s}) > u^{\hat{\mathscr{I}}}(s^*)$.

Proposition 1 *Under the assumption of compatibility, convexity and full dimensionality, the allocation associated with a social state which is weakly Pareto optimal and irreducible, and at which no individual is locally satiated, is a competitive equilibrium allocation.*

Proof. The social state s^* is weakly Pareto optimal and irreducible and a^* is the associated social state; no individual is locally satiated at s^* and, hence, at z^{i*}.

By Lemma 2, there exists $p^* \neq 0$, such that (p^*, a^*) is a quasi-equilibrium.

In order to show that (p^*, a^*) is a competitive equilibrium, it suffices to show that the set of individuals, $i \in \mathscr{I}$, for whom z^{i*} is a minimum expenditure bundle at p^*,

$$\hat{\mathscr{I}} = \{i \in \mathscr{I} : p^* z \geq p^* z^{i*}, \text{ for all } z \in \mathscr{Z}^i\} \subset \mathscr{I} \quad ,$$

is empty.

By the concluding argument in the proof of Lemma 2, $\hat{\mathscr{I}} \neq \mathscr{I}$. If $\hat{\mathscr{I}} \neq \emptyset$, the set $\mathscr{I} \setminus \hat{\mathscr{I}}$ is a non-empty, proper subset of individuals. Since the state s^* is irreducible, there exists a state $\hat{s}$, such that $u^{\mathscr{I} \setminus \hat{\mathscr{I}}}(\hat{s}) > u^{\mathscr{I} \setminus \hat{\mathscr{I}}}(s^*)$. Since individuals in $\mathscr{I} \setminus \hat{\mathscr{I}}$ are not at a minimum expenditure bundle at (p^*, a^*), $\sum_{i \in \mathscr{I} \setminus \hat{\mathscr{I}}} p_i^* \hat{s} > \sum_{i \in \mathscr{I} \setminus \hat{\mathscr{I}}} p_i^* s^*$. Since individuals in $\hat{\mathscr{I}}$ are at a minimum expenditure at (p^*, a^*), $\sum_{i \in \hat{\mathscr{I}}} p_i^* \hat{s} \geq \sum_{i \in \hat{\mathscr{I}}} p_i^* s^*$. It follows that $\sum_{i \in \mathscr{I}} p_i^* \hat{s} > \sum_{i \in \mathscr{I}} p_i^* s^*$ or $p^* \hat{y} > p^* y^*$, for $\hat{y} = (\ldots, \hat{s}, \ldots)$, which contradicts profit maximization at p^*. □

As with quasi-equilibria, corollary 1, the requirement that no individual be locally satiated for the allocation associated with an irreducible social state to be a competitive equilibrium allocation can be relaxed to the requirement that at least one individual be locally non-satiated if the state is Pareto optimal.

7 Applications

In an exchange economy with no external effects, the condition of irreducibility coincides, essentially, with the condition studied in McKenzie (1959). Competitive equilibria for the economy associated with a society derived from an underlying economy coincide with the competitive equilibria for the underlying economy, which demonstrates the consistency of the construction.

With external effects across individuals, opening markets for externalities may not decentralize a Pareto optimal allocation if the associated social state fails to be a irreducible.

Example 6 There is a single commodity. Individuals are $i \in \mathscr{I} = \{1, 2, 3\}$ and have utility functions $u^1 = z_1 + 1$, $u^2 = (z_2 + \frac{1}{2}) - (z_3 + \frac{1}{2})$, and $u^3 = (z_3 + \frac{1}{2}) - (z_2 + \frac{1}{2})$. For all individuals, $S^i = \{(z_1, z_2, z_3) \in \Re^3 : \ z_1 \geq \ -1, \ z_2 \geq -\frac{1}{2}, z_3 \geq -\frac{1}{2}\}$, while the set of feasible social states is $\mathscr{S} = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0, \ \ z_1 \geq -1, z_2 \geq -\frac{1}{2}, \ \ z_3 \geq -\frac{1}{2}\} \subset \Re^3$. The social state $s^* = (0, 0, 0)$ is Pareto optimal, but it is not irreducible which can be seen by setting $\mathscr{I}^1 = \{1\}$ and $\mathscr{I}^2 = \{2, 3\}$. Indeed, s^* can be supported as a quasi-equilibrium, but not as a competitive equilibrium. As s^* is in the interior of $\mathscr{S}$, to support s^* as a competitive equilibrium, from the profit maximization of the firm, $\sum_{i \in \mathscr{I}} p_i^* = 0$. At s^*, $Du^1 = (1, 0, 0)$, $Du^2 = (0, 1, -1)$, and $Du^3 = (0, -1, 1)$. For individual $i = 1$, $Du^1 ds > 0 \Rightarrow p_{1,1}^* ds_1 + p_{1,2}^* ds_2 + p_{1,3}^* ds_3 > 0$, requires that $p_{1,1}^* > 0, p_{1,2}^* = p_{1,3}^* = 0$. For individual $i = 2$, $Du^2 ds > 0 \Rightarrow p_{2,1}^* ds_1 + p_{2,2}^* ds_2 + p_{2,3}^* ds_3 > 0$ requires that $p_{2,1}^* = 0$, $p_{2,2}^* > 0$, and $p_{2,3}^* < 0$. For individual $i = 3$, $Du^3 ds > 0 \Rightarrow p_{3,1}^* ds_1 + p_{2,2}^* ds_2 + p_{2,3}^* ds_3 > 0$ requires that $p_{3,1}^* = 0$, $p_{3,2}^* < 0$, $p_{3,3}^* > 0$. But this leads to a contradiction, as $\sum_{i \in \mathscr{I}} p_{i,1}^* > 0$. Nevertheless, s^* can be supported as a quasi-equilibrium at prices $p^* = (p_1^*, p_2^*, p_3^*) = (0, 0, 0, 0, 1, -1, 0, -1, 1)$. □

In a game in normal form the prisoners' dilemma, players are $i \in \mathscr{I} = \{1, 2\}$, strategies for each player are $\{C^i, NC^i\}$, to cooperate or not cooperate, and payoffs are

	C^1	NC^1
C^2	$(2, 2)$	$(-1, 3)$
NC^2	$(3, -1)$	$(1, 1)$

Figure 2

Associated with this game there is a society with the players as individuals, the probability distributions over payoffs, $s = (s_1, s_2, s_3)$, with $s_1, s_2, s_3 \geq 0$, $s_1 + s_2 + s_3 \leq 1$, according to

	C^1	NC^1
C^2	s_1	s_2
NC^2	s_3	$1 - (s_1 + s_2 + s_3)$

Figure 3

as social states, and the expected payoffs of players as the utility functions of individuals,

$$u^1 = 2s_1 - s_2 + 3s_3 + 1 - (s_1 + s_2 + s_3) = s_1 + 2(s_3 - s_2) + 1,$$
$$u^2 = 2s_1 + 3s_2 - s_3 + 1 - (s_1 + s_2 + s_3) = s_1 + 2(s_2 - s_3) + 1 \ .$$

A Nash equilibrium (Nash, 1949), social state is $\bar{s} = (0, 0, 0)$, which is not Pareto optimal.

Probability distributions over payoffs define social states instead of mixed strategy tuples. In the economy associated with the set of social states, each player chooses a social state. Therefore, even though a player's utility is linear in his own choice of mixed strategies (given the choice of mixed strategy by other players), it is not quasi-concave in the mixed strategy profile. Using mixed strategy tuples to define social states implies that utility functions fail to be quasi-concave in the economy associated with the set of social states. Probability distributions over payoffs can be justified if we assume, as in the analysis of correlated equilibria, that players correlate their choice of strategies by using a correlation device whose recommendations are based on publicly observed signals.

In this game, the only correlated equilibrium based on public signals is the social state $\bar{s} = (0, 0, 0)$.

Pareto optimal social states are characterized by $s_1^* + s_2^* + s_3^* = 1$, and $s_2^* s_3^* = 0$. Each can be supported as a competitive equilibrium with competitive equilibrium prices $p^* = (p_1^*, p_2^*) = (0, 2, -2, 0, -2, 2)$.

By trading in probabilities or strategies, individuals can attain Pareto optimality.

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