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Prices, Asset Markets and Indeterminacy*

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Competitive equilibrium allocations are indeterminate when the net trades in commodities are constrained, while the asset market is incomplete.

The model encompasses economies with deferred payment for commodities, economies with nominal assets, and economies with bundling of commodities and assets. *Journal of Economic Literature* Classification Numbers: D50, D52. © 1998 Academic Press

1. INTRODUCTION

Individuals optimize under a multiplicity of constraints, of two types: constraints on expenditures and constraints on net trades.

Constraints on expenditures or budget constraints reflect the structure of assets for the relocation of revenue across markets [13]. At each market, net expenditure is constrained not to exceed the payoff of assets carried over from preceding markets.

Constraints on net trades reflect, most interestingly, the information available when net trades in different commodities are determined: The net trade in a commodity may be decided at a date–event prior to the date–event at which the economy is consumed or employed. Commodities may, then,

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be associated with multiple date–events, and thus, naturally, have nonzero prices at multiple spot markets: Relative to the consumption sets of individuals and the production sets of firms which capture the restrictions on net trades [12], the price system may be extended.

This is not only a theoretical possibility. Payment for goods, services and factors of production often extends to date–events past the date–event at which the good or service is consumed or the factor is employed. Pension plans, bonuses, downpayments and instalment plans, even alimony payments are examples of deferred payment, contingent on uncertainty realized after trade has occurred.

An alternative description of deferred contingent payment schemes considers trades in bundles of commodities and assets that pay off at multiple date–events, distinct from the date–event at which the trade in the commodity occurs. In exchange for the wage, a worker sells to a firm his labor and an asset whose (negative) payoff provides the health and pension benefits of the worker.

When the asset market is complete, an extension of the price system is of no consequence: via the implicit prices for elementary securities, the prices of a commodity at multiple date–events reduce to a price at the initial date–event and only then [1, 4]. The bundling of commodities and assets is of no effect, since access to a complete asset market allows individuals to undo the restrictions bundling imposes on their expenditures, as shareholders undo the implications of financial policies of firms [10].

When the asset market is incomplete, transfers of revenue across date–events are restricted. An extension of the price system need not reduce to a price system with a single price associated with each commodity; individuals may not be able to undo the restrictions bundling imposes on their expenditures. Indeterminacy, then, typically arises, as multiple, nonredundant prices effectively control each commodity market.

When the asset market is complete, competitive equilibrium allocations are typically locally unique or determinate [5]. Real assets, whose payoffs are denominated in commodities, preserve determinacy, even when the asset market is incomplete [8], though not when the cardinality of the state space is infinite [9].

Here, competitive equilibrium allocations are typically indeterminate when the asset market is incomplete, while the net trades in commodities are constrained; this is the case, even though assets are real.

Nominal assets, whose payoffs are denominated in abstract units of account, typically lead to indeterminacy when the asset market is incomplete [2, 3, 7, 11]. They can be understood as a particular extension of the price system.

In an aggregate model, the indeterminacy of competitive equilibrium allocations when the asset market is incomplete while the net trades in commodities are constrained accounts for macroeconomic regularities [6].

2. ECONOMY

States of the world are $S \in \mathscr{S} = \{1, ..., S\}$, a nonempty, finite set.

Commodities, consumption goods, are $l \in \mathscr{L} = \{1, ..., L\}$, a nonempty finite set, and are traded in spot markets following the resolution of uncertainty. A bundle of commodities in state s is $x_s = (..., x_{l,s}, ...)'$,[1] and a bundle of commodities, across states of the world, is $x = (..., x'_s, ...)'$, an element of the commodity space, $\mathscr{E}^{LS}$.[2]

Individuals, consumer–investors, are $i \in \mathscr{I} = \{1, ..., I\}$, a nonempty finite set. An individual is described by the triple of characteristics $(\mathscr{X}^i, u^i, e^i)$, where $\mathscr{X}^i \subset \mathscr{E}^{LS}$ is the consumption set, a subset of the commodity space, u^i: $\mathscr{X}^i \to \mathscr{E}$ is the utility function over consumption bundles, elements of the consumption set, and $e^i \in \mathscr{E}^{LS}$ is the endowment, a bundle of commodities.

Assumption 1. For every individual, (i) the consumption set coincides with the strictly positive orthant of the commodity space: $\mathscr{X}^i = \mathscr{E}^{SL}_{++}$; (ii) the utility function is continuous, strictly monotonically increasing and strictly quasi-concave; on the interior of its domain of definition, it is twice continuously differentiable, differentiably strictly monotonically increasing: $Du^i \gg 0$, and differentiably strictly quasi-concave: D^2u^i is negative definite on the orthogonal complement of the gradient, $[Du^i]^{\perp}$;[3] for any sequence, $(x_n \in \mathscr{X}^i$: $n = 1, ...)$, with $\lim_{n \to \infty} x_n = \bar{x} \in \text{Bd}$[4], $\mathscr{X}^i \backslash \{0\}$[5], $\lim_{n \to \infty}(\|Du^i(x_n)\|)^{-1} x'_n Du^i(x_n) = 0$; (iii) the endowment is a consumption bundle in the consumption set: $e^i \in \mathscr{X}^i$.

Strict monotonicity eliminates free goods. The boundary condition eliminates boundary solutions. Twice differentiability and the curvature condition guarantee the differentiability of demand.

The net trade of an individual in order to consume $x^i \in \mathscr{X}^i$ is

$$z^i = (x^i - e^i) \in \mathscr{X}^i - \{e^i\}.^6$$

An allocation is an array, $x^{\mathscr{I}} = (..., x^i, ...)$, such that, for every individual $x^i \in \mathscr{X}^i$, while $\sum_{i \in \mathscr{I}} x^i = \sum_{i \in \mathscr{I}} e^i$.

Assets are $a \in \mathscr{A} = \{1, ..., A\}$, a finite set, and are traded in the first period. They pay off in the second period after the realization of

[1] "$'$" denotes the transpose.

[2] "$\mathscr{E}^N$" denotes the euclidean space of dimension N; $\mathscr{E}^1 = \mathscr{E}$. The positive orthant is $\mathscr{E}^N_+$ and its interior, the strictly positive orthant, is $\mathscr{E}^N_{++}$.

[3] "[ ]" denotes the span of a collection of vectors or the column span of a matrix; "$\perp$" denotes the orthogonal complement.

[4] "Bd" denotes the boundary.

[5] "$\mathscr{A} \backslash \mathscr{B}$" denotes the set $\{a \in \mathscr{A}: a \notin \mathscr{B}\}$.

[6] "$\mathscr{A} - \mathscr{B}$" denotes the sets $\{c: c = a - b, a \in \mathscr{A}, b \in \mathscr{B}\}$.

uncertainty, before commodity spot markets open. A portfolio is $y = (..., y_a, ...)'$, an element of the portfolio space, $\mathscr{E}^A$.

Assets are real. The payoffs of assets are denominated in a "numeraire" commodity, $\hat{l}$. Across states of the world, $r_a = (..., r_{a,s}, ...)'$, a column vector of dimension S. The asset structures is described by the matrix of payoffs of assets, $R = (..., r_a, ...)$, of dimension $(S \times A)$.

It is not difficult to allow the numeraire commodity to vary across states of the world, even to be a bundle of commodities; what matters is that it coincide across assets.

Assumption 2. The matrix of payoffs of assets has full column rank, and $A < S$.

This eliminates redundant assets, with no loss of generality. That $A < S$ guarantees that the asset market is incomplete.

Economies are parametrized by the allocation of endowments, $e^{\mathscr{I}} = (..., e^i, ...)$. The space of economies, $\mathscr{O}$, can thus be identified with the interior of the positive orthant, $\times_{i \in \mathscr{I}} \mathscr{X}^i$. A property holds generically if, and only if, it holds for an open set of economies of full Lebesgue measure.

Individuals face constraints in the net trades in commodities $l \in \bar{\mathscr{L}} = \{\bar{l}_1, ..., \bar{l}_{\bar{L}}\} \subset \mathscr{L}$, while net trades in commodities $l \in \hat{\mathscr{L}} = \{\hat{l}_1, ..., \hat{l}_{\hat{L}}\} \subset \mathscr{L}$ are unconstrained.

Bundle of commodities, in particular, bundles of net trades, write as $z = (\hat{z}, \bar{z})$, where $\hat{z} = (..., \hat{z}_s, ...)$, $\hat{z}_s = (\hat{z}_{\hat{l},s}: \hat{l} \in \hat{\mathscr{L}})$, is an element of the spaces of unconstrained commodities, $\mathscr{E}^{\hat{L}S}$, while $\bar{z} = (..., \bar{z}_s, ...)$, $\bar{z}_s = (\bar{z}_{\bar{l},s}: \bar{l} \in \bar{\mathscr{L}})$, is an element of the space of constrained commodities, $\mathscr{E}^{\bar{L}S}$.

The set of attainable net trades in the constrained commodities is described by the column span of a matrix M of dimension $(\bar{L}S \times K)$: the set of attainable net trades for an individual is

$$\mathscr{Z}^i = \{z^i = (\hat{z}^i, \bar{z}^i) \in \mathscr{X}^i - \{e^i\}: \bar{z}^i \in [M]\}.$$

The matrix whose column span describes the attainable net trades in constrained commodities writes as $M = (M_s: s \in \mathscr{S})'$, where M_s is a matrix of dimension $(\bar{L} \times K)$.

Assumption 3. The matrix M is in general position; it is orthonormal: $M'M = I_K$[7]; $1 \leqslant K < \bar{L}S$.

That the constraint matrix M be in general position implies that it has full column rank, K; and it excludes the case in which net trades are constrained only within each spot market and which is essentially of no consequence. If the net trades of commodities are constrained only within

[7] "I_N" denotes the identity matrix of dimension $N \times N$.

each spot market, then $M_s = (..., 0, M_{s,s}, 0, ...)$, where $M_{s,s}$ is a matrix of dimension $\bar{L} \times K_s$, and $\sum_{s \in \mathscr{S}} K_s = K$. Since $K < \bar{L}S$, $K_s < \bar{L}$, for some s. But then, the matrix M_s has $(K_s + 1) \leqslant K$ linearly dependent rows, which contradicts the general position of the matrix M. The dimension of the set of attainable net trades for each individuals is of dimension at least K; and thus, in particular, if $K \geqslant \bar{L}$, all net trades are attainable within each spot market, but not independently across spot markets. That the matrix M be orthonormal is with no loss if generality. That $K < \bar{L}S$ guarantees that there are effective constraints on net trades, while that $1 \leqslant K$ allows for some trade in constrained commodities.

Prices of commodities in state s are $p_s = (..., p_{l,s}, ...)$, and prices of commodities, across states of the world, are $p = (..., p_s, ...) \in \mathscr{P}$, an element of the set of prices of commodities, a subset of $\mathscr{E}^{SL}$. Prices of commodities write as $p = (\hat{p}, \bar{p})$, where $\hat{p} = (..., \hat{p}_s, ...) \in \hat{\mathscr{P}}$, with $\hat{p}_s = (\hat{p}_{l,s} \colon l \in \hat{\mathscr{L}})$, is an element of the set of prices of unconstrained commodities, the strictly positive orthant, $\mathscr{E}^{\hat{L}S}_{++}$, while $\bar{p} = (..., \bar{p}_s, ...) \in \bar{\mathscr{P}}$, with $\bar{p}_s = (\bar{p}_{l,s} \colon l \in \bar{\mathscr{L}})$, is an element of the set of prices of constrained commodities, $\mathscr{E}^{\bar{L}S}$.

Prices of constrained commodities, $\bar{p} \in \bar{\mathscr{P}}$, decomposes uniquely as

$$\bar{p} = m(\bar{p})\, M' + n(\bar{p}),$$

where $m(\bar{p})$ is a row vector, an element of $\mathscr{E}^K$, while $n(\bar{p}) \in [M]^{\perp}$, and, moreover, since M is orthonormal, $m(\bar{p}) = \bar{p}M$.

At prices of commodities p, the matrix of revenue payoffs of assets is $R(p) = (..., r_a(p), ...)$, where $r_a(p) = (..., p_{\hat{l},s} r_{a,s}, ...)$ are the revenue payoffs of asset a, a column vector of dimension S.

Assumption 4. The numeraire commodity, $\hat{l}$, in which the payoffs of assets are denominated is unconstrained: $\hat{l} \in \hat{\mathscr{L}}$ and $R(p) = R(\hat{p})$; $[R(\hat{p})] \cap \mathscr{E}^S_+ \backslash \{0\} \neq \varnothing$.

In economies in which consumption in state $s = 1$ is interpreted as consumption in the first period, concurrently with the trade in assets, and asset $a = 1$ as revenue in the first period—$r_{1,1} > 0$, $r_{1,s} = 0$, for $s \in \mathscr{S} \backslash \{1\}$, while $r_{a,1} = 0$, for $a \in \mathscr{A} \backslash \{1\}$—the condition $[R(\hat{p})] \cap \mathscr{E}^S_+ \backslash \{0\} \neq \varnothing$ is satisfied.

Assumption 5. $I \geqslant K + 1$.

This allows for sufficient heterogeneity.

At prices of commodities p, prices of assets are

$$q(p) = \mathbf{1}'_S R(\hat{p}).$$[8]

This is a normalization.

[8] "$\mathbf{1}_N$" denotes the column vector of 1's of dimension N.

The individual optimization problem at prices of commodities p and endowment e^i is

$$\begin{aligned} \max \quad & u^i(e^i + z), \\ \text{s.t.} \quad & pz \leqslant 0, \\ & \bar{z} \in [M], \\ & p \otimes z \in [R(\hat{p})]. \end{aligned}$$

A competitive equilibrium allocation for the economy $E^{\mathscr{I}}$ is an allocation, $x^{\mathscr{I}*} = (..., x^{i*}, ...)$, such that, for some prices of commodities, p^*, for every individual, $z^{i*} = (x^{i*} - e^i)$ solves the individual optimization problem.

Competitive equilibrium allocations are indeterminate of degree d if and only if the set of competitive equilibrium allocations contains the image, under a one-to-one and continuously differentiable function, of a neighborhood of dimension $d \geqslant 1$.

A simple economy illustrates the indeterminacy of equilibrium allocations which arises from the interaction of constraints on expenditures and constraints on net trades.

This example is analytically equivalent to the "leading example" introduced to display the indeterminacy of equilibrium allocations when the asset market is incomplete and assets are nominal [2]; which illustrates that the nominal denomination of the payoffs of assets is a special case of an extension of the price system.

Example 1. States of the world are $s \in \{1, 2, 3\}$, and commodities are $l \in \{1, 2\}$.

There are no assets for the transfer of revenue across states.

Net trades in commodity 1 are unconstrained: $\hat{\mathscr{L}} = \{1\}$, while net trades in commodity 2 are constrained: $\bar{\mathscr{L}} = \{2\}$. In particular, the net trade of commodity 2 is restricted to be equal across states of the world; equivalently, restrictions in net trades are described by the matrix

$$M = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It is pedantic to normalize so that $M'M = 1$. Commodity prices in state s are $p_s = (\hat{p}_{1,s}, \bar{p}_{2,s})$ and, across states, prices of commodities are $p = (..., p_s, ...)$.

At prices of commodities p, the constraints under which an individual optimizes are

$$\begin{aligned} p_s z_s &= 0, \qquad s \in \{1, 2, 3\}, \\ z_{2,1} &= z_2, \\ z_{2,2} &= z_2, \\ z_{2,3} &= z_2. \end{aligned}$$

Without loss of generality, $\bar{p}_{2,s} = 1$, which leaves $\hat{p}_{1,s}$, to attain market clearing.

The indeterminacy of equilibrium allocations follows by observing that, since $z_{2,s} = z_2$, while $p_s z_s = 0$, if one commodity market clears all do. Thus, $\hat{p}_{1,s}$, for $s \in \{2, 3\}$, can be set arbitrarily, and $\hat{p}_{1,1}$ employed to attain equilibrium in all markets. Different choices for the commodity prices which can be set arbitrarily typically yield distinct equilibrium allocations: typically, $\hat{p}_{1,s} \neq \hat{\hat{p}}_{1,s}$ prevents $z = (..., z_s, ...)$ from satisfying both $\hat{p}_s z_s = 0$ and $\hat{\hat{p}}_s z_s = 0$.

Claim 1. *An allocation, $x^{\mathscr{I}*} = (..., x^{i*}, ...)$, is a competitive equilibrium allocation if and only if there exist prices of commodities*

$$p^* = (\hat{p}^*, \bar{p}^*) = (\hat{p}^*, m(\bar{p}^*)\, M' + n(\bar{p}^*)),$$

such that, for $i \in \mathscr{I} \backslash \{1\}$, $z^{i*} = (x^{i*} - e^i)$ solves the individual optimization problem

$$\begin{aligned} \max \quad & u^i(\hat{e}^i + \hat{z}, \bar{e}^i + Mv), \\ s.t. \quad & \hat{p}^* \hat{z} + m(\bar{p}^*)\, v = 0, \\ & (\hat{p}^* \otimes \hat{z} + (m(\bar{p}^*)\, M' + n(\bar{p}^*)) \otimes Mv) \in [R(\hat{p}^*)], \end{aligned}$$

and, for $i = 1$, $z^{1*} = (x^{1*} - e^1)$ solves the individual optimization problem

$$\begin{aligned} \max \quad & u^i(\hat{e}^i + \hat{z}^i, \bar{e}^i + Mv), \\ s.t. \quad & \hat{p}^* \hat{z} + m(\bar{p}^*)\, v = 0. \end{aligned}$$

Proof. Evident. ∎

Here, v serves to obtain the excess demand for constrained commodities, via $\bar{z} = Mv$.

The set of nonarbitrage prices is

$$\mathscr{P}_{NA} = \{p = (\hat{p}, \bar{p}) \in \mathscr{P}\colon (\hat{z}, Mv) > 0 \Rightarrow \hat{p}\hat{z} + m(\bar{p})\, v > 0\};$$

equivalently

$$\mathcal{P}_{NA} = \{ p = (\hat{p}, \bar{p}) \in \mathcal{P} \colon \hat{p} \gg 0 \text{ and } Mv > 0 \Rightarrow m(\bar{p})\, v > 0 \}.$$

CLAIM 2. *For $(p, e^i) \in \mathcal{P}_{NA} \times \mathcal{X}^i$, a solution to the individual optimization problems, $z^i(p, e^i)$, exists, is unique and satisfies $z^i(p, e^i) + e^i \in \mathcal{X}^i$. As prices and the endowment vary, the excess demand function, $z^i\colon \mathcal{P}_{NA} \times \mathcal{X}^i \to \mathcal{E}^{SL}$, is continuously differentiable. For $i = 1$, along any sequence $((p_n, e_n^1) \in \mathcal{P}_{NA} \times \mathcal{X}^1 \colon n = 1, \ldots)$ with $\lim_{n \to \infty}(p_n, e_n^1) = (p, e^1) \in (\text{Bd}\mathcal{P}_{NA} \backslash \{(\hat{p}, m(\bar{p})) = 0\}) \times \mathcal{X}^1$, $\lim_{n \to \infty} \|z^1(p_n, e_n^1)\| = \infty$.*

Proof. Evident. ∎

The set of normalized nonarbitrage prices of commodities is

$$\mathcal{P}_N = \{ (\hat{p}, m) \colon (\hat{p}, mM') \in \mathcal{P}_{NA}, \text{ and } \|\hat{p}, m\| = 1 \}.$$

The normalization, in addition to fixing the absolute price level, $\|(\hat{p}, m)\| = 1$, eliminates the component of prices of constrained commodities normal to the transactions constraints: $n(\bar{p}) = 0$.

On the set $\mathcal{P}_N \times [M]^{\perp} \times \mathcal{O}$, where $\mathcal{P}_N \times [M]^{\perp} = \{ (\hat{p}, m, n) \colon (\hat{p}, m) \in \mathcal{P}_N, n \in [M]^{\perp} \}$, the aggregate excess demand function is

$$z^a = \sum_{i \in \mathcal{I}} z^i \colon \mathcal{P}_N \times [M]^{\perp} \times \mathcal{O} \to \mathcal{E}^{LS}.$$

The essential excess demand function for constrained commodities is

$$v^a = \sum_{i \in \mathcal{I}} v^i \colon \mathcal{P}_N \times [M]^{\perp} \times \mathcal{O} \to \mathcal{E}^{K};$$

the essential excess demand function is

$$\tilde{z}^a = (\hat{z}^a_{\backslash 1}, v^a) \colon \mathcal{P}_N \times [M]^{\perp} \times \mathcal{O} \to \mathcal{E}^{\hat{L}S - 1 + K},$$

where $\hat{z}^a_{\backslash 1}$ is the aggregate excess demand function for unconstrained commodities other than $(\hat{l}_1, 1)$.

The normalized equilibrium manifold is

$$\mathcal{W}_N(M) = \{ (\hat{p}, m, e^{\mathcal{I}}) \colon (\hat{p}, m) \in \mathcal{P}_N \text{ and } z^a_N(p, m, e^{\mathcal{I}}) = 0 \},$$

where z^a_N is the restriction of the aggregate excess demand function to the set $\mathcal{P}_N \times \mathcal{O}$, and similarly for $\tilde{z}^a_N$.

CLAIM 3. *The normalized equilibrium set has the structure of a smooth manifold with* $\dim \mathcal{W}_N(M) = \dim \mathcal{O} = ILS$.

Proof. $\mathscr{W}_N(M) = (\tilde{z}_N^a)^{-1}(0)$, while, by a standard argument, $\tilde{z}_N^a$ is transverse to 0. ▮

The natural projection from the normalized equilibrium manifold to the space of initial endowments is

$$\pi\colon \mathscr{W}_N \to \mathscr{O}.$$

CLAIM 4. *The function π is proper and surjective.*

Proof. Properness follows from the boundary behavior of the excess demand of individual 1, as in Claim 2.

If $e^{\mathscr{I}*}$ is a pareto optimal allocation of endowments and p^* are walrasian supporting prices, which are unique by Assumption 1, the modified optimization problem for individual 1, implies that the unique normalized equilibrium prices are $(\hat{p}^*, m^*) = (\hat{p}^*, (M'M)^{-1} M'\bar{p}^*)$; which establishes that π is surjective.

Moreover, $e^{\mathscr{I}*}$ is a regular value of π, since the Jacobean matrix $D_{\hat{p},m}\tilde{z}_N^a(\hat{p}^*, m^*, e^{\mathscr{I}*})$ is invertible.

The invariance of degree mod 2 concludes the argument.[9] ▮

This claim implies the existence of competitive equilibria for every economy, $e^{\mathscr{I}} \in \mathscr{O}$. Alternatively, exploiting the boundary behavior of individual 1, a standard fixed point argument establishes the existence of equilibria. With the rank of the matrix of payoffs of assets in terms of revenue invariant with respect to the prices of commodities, the existence of competitive equilibria is not surprising. The argument, here, simply shows that the restrictions on net trades or, equivalently, the structure of the effective consumption sets of individuals does not pose a problem.

The demand of individuals $i \in \hat{\mathscr{I}} = \{2, ..., K+1\} \subset \mathscr{I}$ defines the matrix

$$V(\hat{p}, m, w^{\mathscr{I}}) = (..., v^i(\hat{p}, m, e^{\mathscr{I}}), ...),$$

of dimension $(K \times K)$.

CLAIM 5. *There exists an open, dense set of full Lebesgue measure of endowments, $\mathscr{O}^*$, such that, for $(\hat{p}, m, e^{\mathscr{I}}) \in \pi^{-1}(\mathscr{O}^*)$, the matrices*

$$V(\hat{p}, m, e^{\mathscr{I}})$$

and

$$D_{(\hat{p},m)}\tilde{z}^a(\hat{p}, m, e^{\mathscr{I}})$$

are invertible.

[9] See [14, Proposition X.X.X].

Proof. The function $\tilde{z}^a_N\colon \mathscr{P}_N \times \mathcal{O} \to \mathscr{E}^{S\hat{L}+K-1}$ is transverse to 0, since the Jacobean matrix $D_{e^1}\tilde{z}^a$ has rank equal to $\hat{L}S+K-1$. Therefore, the properness of π and the transversality theorem[10] imply that the function $\tilde{z}^a_{N,e^{\mathscr{I}}}\colon \mathscr{P}_N \to \mathscr{E}^{\hat{L}S+K-1}$ is transverse to 0 for $e^{\mathscr{I}} \in \mathcal{O}'$, an open subset of $\mathcal{O}$ of full Lebesgue measure.

The function[11] $\varphi = (\varphi_1, \varphi_2)\colon \mathscr{P}_N \times \mathcal{O} \times \mathscr{S}^{K-1} \to \mathscr{E}^{\hat{L}S+K-1} \times \mathscr{E}^K$ is defined by

$$\varphi(\hat{p}, m, e^{\mathscr{I}}, \xi) = \begin{pmatrix} \varphi_1(\hat{p}, m, e^{\mathscr{I}}) \\ \varphi_2(\hat{p}, m, e^{\mathscr{I}}) \end{pmatrix} = \begin{pmatrix} \tilde{z}^a_N(\hat{p}, m, e^{\mathscr{I}}) \\ \xi V(\hat{p}, m, e^{\mathscr{I}}) \end{pmatrix}.$$

The matrix $D_{e^1}\varphi_1 = D_{e^1} z^a_N$ has full row rank, $\hat{L}S-1+K$, while $D_{e^1}\varphi_2 = 0$.

The matrix $\Delta e^{\mathscr{I}} = (..., \Delta e^i, ...) = (..., \Delta \hat{e}^i, \Delta \bar{e}^i, ...)$ is defined by

$$\begin{aligned} \hat{p} \otimes \Delta \hat{e}^i &= mM'M^1, && i \in \hat{\mathscr{I}}, \\ \Delta \bar{e}^i &= -M^1, && i \in \hat{\mathscr{I}}, \\ \Delta e^1 &= -\sum_{i \in \hat{\mathscr{I}}} \Delta e^i, && \\ \Delta e^i &= 0, && i \in \mathscr{I} \backslash (\hat{\mathscr{I}} \cup \{1\}), \end{aligned}$$

where M^1 is the first column of the matrix M; it defines a perturbation which leaves the wealth of every individual in every spot market unchanged and, as a consequence,

$$D_{e^{\mathscr{I}}}\varphi_1 \Delta e^{\mathscr{I}} = 0;$$

since $\bar{x}^i(\hat{p}, m, e^i) = Mv^i(\hat{p}, nm, e^i) + \bar{e}^i$,

$$D_{e^{\mathscr{I}}}\varphi_2 \Delta e^{\mathscr{I}} = \operatorname{diag}_K(\xi_1);\text{[12]}$$

since $\xi \in \mathscr{S}^{K-1}$, without loss of generality, $\xi_1 \neq 0$.

It follows that the Jacobean matrix $D\varphi$ has full row rank, $\hat{L}S-1+2K$, and, hence, the function φ is transverse to 0. But then, since π is proper and $\mathscr{S}^{K-1}$ is compact, by the transversality theorem, there exists an open and dense subset of economies, $\mathcal{O}''$, of full Lebesgue measure, such that, for all $w \in \mathcal{O}''$, the function

$$\varphi_{e^{\mathscr{I}}}\colon \mathscr{P}_N \times \mathscr{S}^{K-1} \to \mathscr{E}^{\hat{L}S+K-1} \times \mathscr{E}^K,$$

[10] (See [14, Proposition 8.3.1, p. 320].

[11] "$\mathscr{S}^N$" denotes the sphere of dimension N.

[12] "$\operatorname{diag}_N(a)$" denotes the diagonal matrix of dimension $N \times N$, with all diagonal elements equal to a.

defined by $\varphi_{e^{\mathcal{I}}}(\hat{p}, m, \xi) = \varphi(\hat{p}, m, e^{\mathcal{I}}, \xi)$ is transverse to 0. Since the dimension of the set, $(S\hat{L} - 1 + 2K - 1)$, is one less than the dimension of the range, $(\hat{L}S + K - 1 + K)$, $\varphi_{e^{\mathcal{I}}}^{-1}(0) = \varnothing$, for all $e^{\mathcal{I}} \in \mathcal{O}''$. Hence, for $(\hat{p}, m) \in \pi^{-1}(e^{\mathcal{I}})$, $e^{\mathcal{I}} \in \mathcal{O}''$, $\xi V(\hat{p}, m, e^{\mathcal{I}}) \neq 0$, for all $\xi \in \mathcal{S}^{K-1}$; equivalently, $V(\hat{p}, m, e^{\mathcal{I}})$ is invertible. The subset of economies $\mathcal{O}^* = \mathcal{O}' \cap \mathcal{O}'' \subset \mathcal{O}$ is open and of full Lebesgue measure and, for $e^{\mathcal{I}} \in \mathcal{O}^*$, both $V(\hat{p}, m, e^{\mathcal{I}})$ and $D_{(\hat{p}, m)}\tilde{z}^a(\hat{p}, m, e^{\mathcal{I}})$ are invertible. ∎

3. INDETERMINACY

Variations of prices of constrained commodities in the orthogonal complement of the span of the matrix of attainable net trades do not affect the value of the net trade of an individual; but they do affect the distribution of expenditures across states of the world. With an incomplete asset market, reallocations of revenue across states of the world are restricted to the span of the matrix of payoffs of assets. Variations of prices of constrained commodities in the orthogonal complement of the span of the matrix of attainable net trades are associated with distinct allocations of commodities at equilibrium, as long as they require variations in the allocation of revenue across states of the world not in the span of the matrix of payoffs of assets. The argument that follows formalizes this intuition.

CLAIM 6. *For $e^{\mathcal{I}} \in \mathcal{O}^*$, there exists $(p^*, e^{\mathcal{I}}) = (\hat{p}^*, m^*, e^{\mathcal{I}}) \in \pi^{-1}(e^{\mathcal{I}})$, neighborhoods $\mathcal{B}(\hat{p}^*, m^*) \subset \mathcal{D}_N$ and $\mathcal{B}(0) \subset [M]^{\perp}$, and a continuously differentiable function g: $\mathcal{B}(0) \to \mathcal{B}(\hat{p}^*, m^*)$, such that*

$$(\hat{p}^*, m^*) = g(0),$$

and

$$\tilde{z}^a(\hat{p}, mM' + n, e^{\mathcal{I}}) = 0 \Leftrightarrow (\hat{p}, m) = g(n).$$

Proof. By Claim 5, for $e^{\mathcal{I}} \in \mathcal{O}^*$, $\tilde{z}^a(\hat{p}, m, 0, e^{\mathcal{I}}) = \tilde{z}^a_N(\hat{p}, m, e^{\mathcal{I}}) = 0$ has a solution, $(\hat{p}^*, m^*)$; equivalently, there exists $(\hat{p}^*, m^*, e^{\mathcal{I}}) \in \pi^{-1}(e^{\mathcal{I}})$.

By Claim 5, for $e^{\mathcal{I}} \in \mathcal{O}^*$, the Jacobean matrix $D_{(\hat{p}, m)}\hat{z}^a(\hat{p}^*, m^*, 0, e^{\mathcal{I}}) = D_{(\hat{p}, m)}\hat{z}^a_N(\hat{p}^*, m^*, e^{\mathcal{I}})$ is invertible.

The implicit function theorem then concludes the argument. ∎

For $e^{\mathcal{I}} \in \mathcal{O}^*$, the allocation function $x^{\mathcal{I}}$: $\mathcal{B}(0) \to \times_{i \in \mathcal{I}} \mathcal{X}^i$ is defined by

$$x^{\mathcal{I}}(n) = (..., e^i + z^i(\hat{p}^*(n), m(n)\, M' + n, e^i), ...),$$

where $(\hat{p}^*, (n), m(n)) = g(n)$. From Claim 6, the allocation function is continuously differentiable.

PROPOSITION 1. *Generically, competitive equilibrium allocations are indeterminate of degree* $d \geqslant 1$.

Proof. The argument is based on the properties of $[M]$ and of its orthogonal complement, $[M]^{\perp}$, which we state and prove in Claim 7.

The matrix N, of dimension $(\bar{L}S - K) \times S\bar{L}$, is such that

$$[N'] = [M]^{\perp}.$$

Since M is in general position, so is N. If n_j, for $j = 1, ..., (\bar{L}S - K)$, is the jth row of the matrix N,

$$N \otimes M = (..., n_j \otimes M, ...)$$

is a matrix of dimension $(S \times K(\bar{L}S - K))$.

CLAIM 7. *The rank of the matrix* $N \otimes M$ *is* $(S - 1)$.

Proof. If $\lambda = (..., \lambda_s, ...)$ is a row vector of dimension S,

$$\lambda N \otimes M = \mathbf{1}'_S N\Lambda \otimes M = (..., n_j \Lambda M, ...),$$

where

$$\Lambda = \begin{pmatrix} \ddots & & \\ & \mathrm{diag}_{\bar{L}}(\lambda_s) & \\ & & \ddots \end{pmatrix}.$$

By the definition of the matrix N,

$$\mathbf{1}'_S N \otimes M = (..., n_j M, ...) = 0,$$

and, hence, $\mathrm{rank}(N \otimes M) \leqslant (S - 1)$. If $\mathrm{rank}(N \otimes M) < (S - 1)$. Then, there exists a vector $\lambda \neq 0$, of dimension S, such that $\lambda \neq \mathbf{1}_S$, and $\lambda(N \otimes M) = 0$; equivalently, $n_j \Lambda M = 0$, for $j = 1, ..., (\bar{L}S - K)$ or $N\Lambda M = 0$. Since M has full column rank,

$$[\Lambda N'] = [N'],$$

which contradicts the general position of N'. ∎

Since the set $\mathscr{B}(0)$ is connected, while the equilibrium allocation function is continuously differentiable, it is sufficient that

$$x^{\mathscr{I}}(n) \neq x^{\mathscr{I}}(0), \qquad \text{for some } n \in \mathscr{B}(0).$$

If $x^{\mathscr{I}}(n) = x^{\mathscr{I}}(0)$, for all $n \in \mathscr{B}(0)$, it follows from the optimization problem of individual 1, that $g(n) = g(0) = (\hat{p}^*, m^*)$ or

$$p(n) = (\hat{p}(n), m(n)\, M' + n) = (\hat{p}^*, m^* M' + n), \qquad n \in \mathscr{B}(0).$$

For individuals $i \in \mathscr{I} \backslash \{1\}$, if $(\hat{z}^{i*}, \hat{v}^{i*}) = (z^i(p^*, m^*, M', e^i), v^i(p^*, m^* M', e^i))$, and $(\hat{z}^i(n), v^i(n)) = (\hat{z}^i(\hat{p}^*, m^* M' + n, e^i), v^i(\hat{p}^*, m^* M' + n, e^i))$, for $n \in \mathscr{B}(0)$, it follows from the optimization problem of these individuals that

$$(\hat{p}^* \otimes \hat{z}^{i*} + m^* M' \otimes M v^{i*}) \in [R(\hat{p}^*)].$$

Therefore, $(\hat{z}^{i*}, v^{i*}) = (\hat{z}^i(n), v^i(n))$, and $n \in \mathscr{B}(0)$ only if

$$(\hat{p}^* \otimes \hat{z}^i(n) + (m^* M' + n) \otimes M v^i(n)) \in [R(p^*)], \qquad n \in \mathscr{B}(0).$$

By Claim 4, the matrix $V(\hat{p}^*, m^*, e^{\mathscr{I}})$ is invertible. Therefore,

$$[n \otimes M] \subset [R(\hat{p}^*)], \qquad \text{for all} \quad n \in \mathscr{B}(0).$$

Since $\mathscr{B}(0) \subset [M]^{\perp}$ is an open relative to $[M]^{\perp}$, there exist $(\bar{L}S - K)$ linearly independent vectors, $n_j \in \mathscr{B}(0)$, for $j = 1, ..., (\bar{L}S - K)$, than span $[M]^{\perp}$. Since $\mathbf{1}_S \notin [R(\hat{p}^*)]^{\perp}$, and $A < S$, it follows from Claim 7 that there exists at least one j, such that

$$[n_j \otimes M] \not\subset [R(\hat{p}^*)], \qquad n_j \in \mathscr{B}(0),$$

a contradiction. ∎

An estimate of the degree of indeterminacy is possible.

COROLLARY 1. *If $\bar{L} > K$, generically, equilibrium allocations are indeterminate of degree $d \geqslant \max\{1, (S - A - 1)\, K\}$.*

Proof. The argument is developped in steps.

If $(e^{\mathscr{I}}, \hat{p}^*, m^*) \in \pi^{-1}(e^{\mathscr{I}})$, there exists $\varepsilon > 0$, such that $V(\hat{p}, mM' + n, e^{\mathscr{I}})$ is invertible, for $n \in B^{\varepsilon}(0) \subset [M]^{\perp}$, where $(\hat{p}, m) = g(n)$.

STEP 1. If $n, n' \in B^{\varepsilon}(0)$, $n \neq n'$, $g(n) = (\hat{p}, m$, and $(n - n') \otimes M \not\subset [R(\hat{p}]$, then $x^{\mathscr{I}}(n) \neq x^{\mathscr{I}}(n')$.

Proof. If $x^{\mathscr{I}}(n) = x^{\mathscr{I}}(n')$, by the maximization problem of individual 1,

$$g(n) = g(n') = (\hat{p}, m),$$

while, by the maximization problems of individuals $i \in \mathscr{I} \setminus \{1\}$,

$$\hat{p} \otimes \hat{z}^i(n) + (mM' + n) \otimes Mv^i(n) \in [R(\hat{p})],$$

and

$$\hat{p} \otimes \hat{z}^i(n) + (mM' + n') \otimes Mv^i(n) \in [R(\hat{p})].$$

Hence, Claim 5 implies that

$$(n - n') \otimes M \subset [R(\hat{p})],$$

a contradiction. ∎

Step 1 establishes as a sufficient condition for indeterminacy that $[(n-n') \otimes M] \not\subset [R(\hat{p})]$ or, equivalently, that $\mathrm{proj}_{[R(\hat{p})]^\perp}[(n-n') \otimes M] \neq 0$, $n, n' \in B^\varepsilon(0)$, $n \neq n'$.

If

$$\mathscr{G} = \{G\text{: } G \text{ is a matrix of dimension } S \times K, \text{ and } \mathbf{1}'_S G = 0\},$$

since each column of a matrix $G \in \mathscr{G}$ is an element of $[\mathbf{1}_S]^\perp$, $\mathscr{G}$ is a linear subspace of dimension $(S-1)K$.

Since $M = (..., M_s, ...)'$ is in general position and $\bar{L} \geqslant K$, M_s, a matrix of dimension $\bar{L} \times K$, has rank K. Hence,

$$\bar{N} = \{n = (..., n_s, ...) \in [M]^\perp\text{: } n_s \in [M_s]\}$$

is a subspace of dimension $(S-1)K$. For each $G \in \mathscr{G}$, there exists a unique $n \in \bar{N}$, such that $n \otimes M = G$, and, for each $n \in \bar{N}$, there exists a unique $G \in \mathscr{G}$, such that $G = n \otimes M$. If $e^{\mathscr{I}} \in \mathscr{O}^*$ and $(e^{\mathscr{I}}, \hat{p}, m) \in \pi^{-1}(e^{\mathscr{I}})$, and if $\mathbf{1}_S \notin [R(\hat{p})]$, then, each matrix $\bar{R}^\perp(\hat{p})$, of dimension $S \times (S-A-1)$, that solves

$$\bar{R}^\perp(\hat{p})' \, (\mathbf{1}_S, R(\hat{p})) = 0$$

has rank equal to $(S-A-1)$. Moreover, for any matrix, C, of dimension $(S-A-1) \times K$, $\bar{R}^\perp(\hat{p})\, C$ is an element of G and can be, therefore, generated as $n \otimes M$, for some $n \in \bar{N}$.

Step 2. For an open and dense and full Lebesgue measure subset, $\bar{\mathscr{O}}$ and $(e^{\mathscr{I}}, \hat{p}, m) \in \pi^{-1}(e^{\mathscr{I}})$, $e^{\mathscr{I}} \in \bar{\mathscr{O}}$, $\mathbf{1}_s \notin R(\hat{p})$.

Proof. Since the payoffs of assets are denominated in an unconstrained numeraire commodity, $\hat{l}_1 \in \hat{\mathscr{L}}$,

$$R(\hat{p}) = \mathrm{diag}_s(\hat{p}_{\hat{l}_1})\, R.$$

The function $\psi\colon \mathscr{P}_N \times \mathcal{O} \times e^A \to E^{\hat{L}S+K-1} \times E^A$ is defined by

$$\psi(\hat{p}, m, e^{\mathscr{I}}, y) = (\hat{z}^a(\hat{p}, m, e^{\mathscr{I}}), R(\hat{p})\, y - \mathbf{1}_S);$$

it is smooth. If $(\hat{p}^*, m^*, e^{\mathscr{I}*}, y^*) \in \psi^{-1}(0)$, since $R(\hat{p}^*)\, y^* = \mathbf{1}_S$ or $\hat{p}^*_{\hat{l}_1, s} \sum_{a \in \mathscr{A}} r^*_{(a, s)}\, y_a = 1$, for $s \in S$,

$$D_{\hat{p}_{\hat{l}_1}}(R(\hat{p})\, y - \mathbf{1}_S)|_{\hat{p}^*, y^*} = (\mathrm{diag}_s(\hat{p}_{\hat{l}_1}))^{-1},$$

and

$$D_{e^{\mathscr{I}}, \hat{p}_{\hat{l}_1}} \psi|_{(e^{\mathscr{I}*}, \hat{p}^*, m^*, y^*)} = \begin{pmatrix} D_{e^{\mathscr{I}}} \tilde{z}^a & D_{\hat{p}_{\hat{l}_1}} \tilde{z}^a \\ 0 & (\mathrm{diag}_s(\hat{p}_{\hat{l}_1}))^{-1} \end{pmatrix}.$$

It follows that ψ is transverse 0. The properness of π and the transversality theorem imply that $\psi_{e^{\mathscr{I}}}$ is transverse to 0, for $e^{\mathscr{I}} \in \bar{\mathcal{O}}$, an open, dense subset of $\mathcal{O}$ of full Lebesgue measure. But $\psi_{e^{\mathscr{I}}}\colon \mathscr{P}_N \times \mathscr{E}^A \to \mathscr{E}^{\hat{L}S+K-1+S}$, and, since $A < S$, the dimension of the set of $\psi_{e^{\mathscr{I}}}$ is less than $(\hat{L}S + K - 1 + S)$, the dimension of the range. Hence, for $e^{\mathscr{I}} \in \bar{\mathcal{O}}$, $\psi^{-1}_{e^{\mathscr{I}}}(0) = \varnothing$. ▌

The set $\mathcal{O}^{**} = \bar{\mathcal{O}} \cap \mathcal{O}^*$ is an open, dense subset of $\mathcal{O}$ of full Lebesgue measure. For $(e^{\mathscr{I}}, \hat{p}^*, m^*) \in \pi^{-1}(e^{\mathscr{I}})$, $e^{\mathscr{I}*} \in \mathcal{O}^*$, and for $\varepsilon > 0$,

$$\bar{B}^\varepsilon(0) = \{n \in B^\varepsilon(0) \cap \bar{N}\colon n \otimes M = \bar{R}^\perp(\hat{p}^*)'\, C, \text{ for some matrix } C \text{ of dimension } (s - A - 1) \times K\}.$$

By the definition of $\bar{B}^\varepsilon(0)$, there exists $\eta > 0$, such that, to each matrix, C, of dimension $(S - A - 1) \times K$, with $\|C\| < \eta$, there corresponds a unique $n \in \bar{B}^\varepsilon(0)$, with $n \otimes M = \bar{R}^\perp(\hat{p}^*)\, C$, and vice versa.

It follows that $\bar{B}^\varepsilon(0)$ is diffeomorphic to an open set in $\mathscr{E}^{(S-A-1)K}$. Furthermore, ε can be chosen so as to guarantee that the matrix $V(\hat{p}, mM' + n, e^{\mathscr{I}})$, defined by analogy with $V(\hat{p}, m, e^{\mathscr{I}})$, is invertible, for $n \in \bar{B}^\varepsilon(0)$ and $(\hat{p}, m) = g(n)$.

Step 3. If $S - A > 1$ and $e^{\mathscr{I}} \in \mathcal{O}^{**}$, then, for n and $n' \in \bar{B}^\varepsilon(0)$, $n \neq n' \Rightarrow x^{\mathscr{I}}(n) \neq x^{\mathscr{I}}(n')$.

Proof. If $x^{\mathscr{I}}(n) = x^{\mathscr{I}}(n')$, then $g(n) = g(n') = (\hat{p}, m)$. Since assets are denominated in a numeraire commodity there exists a diagonal matrix of dimension $(S \times S) = \mathrm{diag}_s(\hat{p}_{\hat{l}_1})(\mathrm{diag}_s(\hat{p}^*_{\hat{l}_1}))^{-1}$, with strictly positive elements and such that

$$\Lambda R(\hat{p}^*) = R(\hat{p}), \qquad (\hat{p}^*, m^*) = g(0).$$

Since $g(n) = g(n')$, $x^{\mathscr{I}}(n) = x^{\mathscr{I}}(n')$ implies that $(n - n') \otimes M \subset [R(\hat{p})]$ or, equivalently,

$$(n - n') \otimes M = R(\hat{p})\, B,$$

for some matrix, B, of dimension $(A \times K) - R(\hat{p})\, B \neq 0$, since $n, n' \in \bar{N}$. But then, since $n, n' \in \bar{B}(0)$ and $n \neq n'$,

$$(n - n') \otimes M = \bar{R}^{\perp}(\hat{p}^*)'\, C, \qquad C \neq 0.$$

Hence

$$\Lambda R(\hat{p}^*)\, B = \bar{R}^{\perp}(\hat{p}^*)\, C \Rightarrow R(\hat{p}^*)'\, \Lambda R(\hat{p}^*)\, B = R(\hat{p}^*)'\, \bar{R}^{\perp}(\hat{p}^*)\, C.$$

But, by the definition of $\bar{R}^{\perp}(\hat{p}^*)$, $R(\hat{p}^*)'\, \bar{R}^{\perp}(\hat{p}^*)' = 0$, while $R(\hat{p}^*)'$ $\Lambda R(p^*) \mu B \neq 0$, since $B' R(\hat{p}^*)'\, \Lambda R(p^*) B \neq 0$, a contradiction. ∎

By Proposition 1, $d \geqslant 1$, while, by Step 3, if $(S - A) > 1$ and $\bar{L} > K$, $d \geqslant (S - A - 1)\, K$. Hence, $d \geqslant \max\{1, (S - A - 1)\, K\}$. ∎

Remark 1. In the special case in which there are no assets, $A = 0$, the exact degree of indeterminacy, generically, coincides with the dimension of $[M]^{\perp}$: $d = S\bar{L} - K$.

Remark 2. The model encompasses economies with bundling of commodities and assets.

Commodity $l = 1$ is consumed only concurrently with the exchange of assets, in the first period. It is exchanged bundled with an asset with payoffs $\bar{r} = (..., \bar{r}_s, ...)$; a quantity of the bundle is $\bar{z}$ and the price of the bundle at is $\bar{p}$.

In states of the world $s \in \mathscr{S}$, commodities $l \in \hat{\mathscr{L}} = \mathscr{L} \backslash \{1\}$, are traded with no constraints; the payoffs of assets are denominated in the numeraire commodity, $\hat{l} \in \hat{\mathscr{L}}$.

The budget constraints take the form

$$qy + \overline{pz} = 0,$$

$$\hat{p}_s \hat{z}_s = p_{\hat{l}, s} r_s y + p_{\hat{l}, s} \overline{rz}, \qquad s \in \mathscr{S}.$$

If the structure of payoffs of assets allows for the consumption and exchange of commodities concurrently with the exchange of assets, the set of constrained commodities is $\bar{\mathscr{L}} = \{1\}$, trading constraints are defined by

$$\bar{z} = \begin{pmatrix} \vdots \\ z_{1, s} \\ \vdots \end{pmatrix} \in \left[\begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \right],$$

and the price of the constrained commodity $l=1$ is

$$\bar{p}_s = p_{1,s} = -p_{\hat{l},s}\bar{r}_s \qquad s \in \mathscr{S},$$

competitive equilibrium allocations in the economy with trading constraints coincide with competitive equilibrium allocations in the economy with bundling of commodities and assets.

If the matrix of payoffs of assets, R, is invertible, every bundle, $\bar{l}$, can be undone, as trading constraints are ineffective.

An economy with even $S-1$ freely traded assets and, in addition, assets with linearly independent payoffs but bundled with commodities in $\bar{\mathscr{L}}$ is not equivalent to an economy with a complete asset market: by Proposition 1, the introduction of assets bundled with commodities can never complete the asset market.

Remark 3. The model encompasses economies in which assets are nominal: their payoffs are denominated in abstract units of account, and there are no constraints on net trades.

Commodities are exchanged along with assets; bundles and prices of commodities are indexed by x_0 and p_0, respectively. The matrix of asset payoffs is, without loss of generality, $R=(R_1, I_A)'$, where R_1 is a matrix of dimension $(S-A)\times A$. A net trade is $z=(z_0, z_1, z_2)'$, where z_0 is the net trade in commodities in the first, asset trading period, z_1 is the net trade in the second period in states $s=1, ..., (S-A)$, and z_2 is the net in the second period in states $s=(S-A+1), ..., S$. Similarly, commodity prices are $p=(p_0, p_1, p_2)$. The price of units of account are $\delta=(\delta_1, \delta_2)$ and, by an abuse of notation, $\Delta=(\Delta_1, \Delta_2)$ is the associated diagonal matrix. The constraints in the individual optimization problem are

$$pz \leqslant 0, \qquad p \otimes z \in [\Delta R];$$

equivalently,

$$p_0 z_0 + (\mathbf{1}'_{(S-A)}\Delta_1 R_1 \Delta_2^{-1} + \mathbf{1}'_A)\, p_2 \otimes z_2 = 0,$$

$$p_1 \otimes z_1 - (\Delta_1 R_1 \Delta_2^{-1})\, p_2 \otimes z_2 = 0.$$

If $\hat{z}=(z_0, z_1)'$, $\hat{p}=(p_0, p_1)$, $M'=(S-A+1)^{-1/2}\,(..., L_{AL}, ...)'$, $m=(S-A+1)^{-1/2}\,(p_{2,(S-A+1)}..., p_{2,S})$, and $v=(S-A+1)^{1/2}\, z_2$, the constraints take the form

$$\hat{p} \otimes \hat{z} + (mM' + n) \otimes Mv = 0,$$

where

$$n_s = -\delta_s \left(\frac{r_{(s,1)}}{\delta_{(S-A+1)}} p_{(2,S-A+1)}, \ldots, \frac{r_{(s,A)}}{\delta_S} p_{(2,S)} \right) - (S-A+1)^{-1/2} m,$$

$$s = 1, \ldots, (S-A),$$

and

$$n_0 = \left(\left(\sum_{s=1}^{(S-A)} \frac{r_{(s,1)}\delta_s}{\delta_{(S-A+1)}} + 1 \right) p_{(2,(S-A+1))}, \ldots, \left(\sum_{s=1}^{(S-A)} \frac{r_{(s,A)}\delta_s}{\delta_S} + 1 \right) p_{(2,S)} \right) + (S-A)(S-A+1)^{-1/2} m.$$

This is a special case of the model with constraints on net trades and no assets, in which case the constraint $pz \leqslant 0$ is redundant, since the constraint $p \otimes z \in [R(p)]$ reduces to $p \otimes z = 0$. In order to obtain the model with nominal assets, variation of prices are restricted to a subset of the set of restricted commodity prices.

Remark 4. The restriction of prices to the dual of the subspace on which net trades are restricted, $[M]$, which eliminates the indeterminacy, is ad hoc: such a restriction is not possible when the constraints on net trades differ across individuals, a generalization of the model.

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