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# Nash–Walras equilibria

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**Summary**

At a Nash–Walras equilibrium, individuals exchange commodities competitively, and, simultaneously, they interact strategically. Under standard assumptions, Nash–Walras equilibria exist; equilibrium profiles of actions are, typically, determinate but pareto suboptimal, though not constrained pareto suboptimal: a transfer of revenue need not suffice for a pareto improvement in welfare.



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## 1. Introduction

Even in advanced market economies, the welfare of individuals, the allocation of resources more narrowly, are not determined exclusively by the market mechanism. Externalities and public goods are well-known instances of market failure. When the market for the transfer of purchasing power across time and contingencies is incomplete, there is scope for contracts that offer additional opportunities. Personal and social relations and interactions, though hardly insignificant for the welfare of individuals and the allocation of resources, are largely beyond the scope of the market. In economies at earlier stages of development, the limited role of the market is evident.

Within a competitive market, individuals take only implicit account of the choices of other individuals and interact through the intermediation of the price mechanism. When the intermediation of the price mechanism is not operative, outside the market, individuals take explicit account of the choices of other individuals and interact strategically.

For competitive market economies, the notion of equilibrium is

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that introduced by Walras (1950) and formalized by Arrow and Debreu (1954) and McKenzie (1954), while for situations of strategic interaction or games the notion of equilibrium is that introduced and formalized by Nash (1950).

At a Nash–Walras equilibrium, competitive markets operate alongside areas of strategic interaction.

Under standard assumptions, Nash–Walras equilibria exist; they are, typically, determinate but pareto suboptimal, though not constrained pareto suboptimal: a transfer of revenue need not suffice for a pareto improvement in welfare.

## 2. The economy

The economy consists of individuals who trade competitively in commodities and, simultaneously, choose strategies.

Individuals are $i \in \mathscr{I} = \{1, \ldots, I\}$, a finite, non-empty set.

Commodities are $l \in \mathscr{L} = \{1, \ldots, L\}$, a finite, non-empty set, and a trade in commodities is $z = (z_l: l \in \mathscr{L})$.

Dimensions of strategies for an individual are $k^i \in \mathscr{K}^i = \{1, \ldots, K^i\}$, a finite, non-empty set, and a strategy for the individual is $s^i = (s_{k^i}: k^i \in \mathscr{K}^i)$.

An action for an individual is a pair, $a^i = (z^i, s^i)$, of a trade in commodities and a strategy.

For $\mathscr{F} \subset \mathscr{I}$, a non-empty subset of individuals, an allocation of trades in commodities is $z^{\mathscr{F}} = (z^i: i \in \mathscr{F})$, a profile of strategies is $s^{\mathscr{F}} = (s^i: i \in \mathscr{F})$, and a profile of actions is $a^{\mathscr{F}} = (a^i: i \in \mathscr{F})$.†

The domain of actions of an individual is $\mathscr{A}^i$, the domain of trades in commodities is $\mathscr{Z}^i = \text{proj}_{z^i}\mathscr{A}^i$, and the domain of strategies is $\mathscr{S}^i = \text{proj}_{s^i}\mathscr{A}^i$. For $\mathscr{F} \subset \mathscr{I}$, a non-empty subset of individuals, $\mathscr{A}^{\mathscr{F}} = \times_{i \in \mathscr{F}}\mathscr{A}^i$, $\mathscr{Z}^{\mathscr{F}} = \times_{i \in \mathscr{F}}\mathscr{Z}^i$, and $\mathscr{S}^{\mathscr{F}} = \times_{i \in \mathscr{F}}\mathscr{S}^i$. The aggregate domain of trades in commodities is $\mathscr{Z}^a = \Sigma_{i \in \mathscr{I}} \mathscr{Z}^i$.

The preference relation $\mathscr{R}^i$, with domain $\mathscr{A}^i \times \mathscr{S}^{\mathscr{I} \setminus \{i\}}$, is the preference relation of the individual over actions of his and strategy profiles of others. Associated with the preference relation, there is a strict preference relation, $\mathscr{P}^i$, and an indifference relation, $\mathscr{I}^i$. For a given profile of strategies, $s^{\mathscr{I} \setminus \{i\}}$, the restriction of the preference relation is $\mathscr{R}^i(s^{\mathscr{I} \setminus \{i\}})$, with domain $\mathscr{A}^i$.

An economy is

$$\mathscr{E} = \{(\mathscr{A}^i, \mathscr{R}^i): i \in \mathscr{I}\}.$$

A profile of actions, $a^{\mathscr{I}}$, is feasible if and only if $a^i \in \mathscr{A}^i$, for

† For $\{\mathscr{F}_1, \mathscr{F}_2\}$, a non-trivial partition of $\mathscr{F}$, without ambiguity, $z^{\mathscr{F}} = (z^{\mathscr{F}_1}, z^{\mathscr{F}_2})$, $s^{\mathscr{F}} = (s^{\mathscr{F}_1}, s^{\mathscr{F}_2})$, and $a^{\mathscr{F}} = (a^{\mathscr{F}_1}, a^{\mathscr{F}_2})$.

every individual, and $z^a = \Sigma_{i \in \mathscr{I}} z^i = 0$: the aggregate net trade in commodities is feasible.

Prices of commodities are $p = (p_l: l \in \mathscr{L})$.

At $(p, s^{\mathscr{I} \setminus \{i\}})$, prices of commodities and a profile of strategies, individual $i$ solves the optimization problem

$$\begin{aligned} \max_{(z^i, s^i)} \quad & \mathscr{R}^i(s^{\mathscr{I} \setminus \{i\}}), \\ \text{s.t.} \quad & pz \leqslant 0, \\ & (z^i, s^i) \in \mathscr{A}^i. \end{aligned}$$

A transfer of revenue is $\tau^{\mathscr{I}} = (\tau^i: i \in \mathscr{I})$, such that $\Sigma_{i \in \mathscr{I}} \tau^i = 0$.

At $(p, s^{\mathscr{I} \setminus \{i\}}, \tau^i)$, prices of commodities, a profile of strategies, and transfer of revenue, individual $i$ solves the optimization problem

$$\begin{aligned} \max_{(z^i, s^i)} \quad & \mathscr{R}^i(s^{\mathscr{I} \setminus \{i\}}), \\ \text{s.t.} \quad & z \leqslant \tau^i, \\ & (z^i, s^i) \in \mathscr{A}^i. \end{aligned}$$

A Nash–Walras equilibrium is a pair, $(p^*, a^{\mathscr{I}*})$, of prices of commodities and a feasible profile of actions, such that $a^{i*}$ is a solution to the optimization problem of individual $i$ at $(p^*, s^{\mathscr{I} \setminus \{i\}*})$.

A Nash–Walras equilibrium with a transfer of revenue is a triple, $(p^*, a^{\mathscr{I}*}, \tau^*)$, of prices of commodities, a feasible profile of actions, and a transfer of revenue, such that $a^{i*}$ is a solution to the optimization problem of individual $i$ at $(p^*, s^{\mathscr{I} \setminus \{i\}*}, \tau^{i*})$.

This is an extension of the definition of Nash (1950) for games and of Arrow and Debreu (1954) and McKenzie (1954) for competitive economies.

For a set of economies identified with an open neighbourhood in euclidean space of finite dimension, Nash–Walras equilibria are *generically determinate* if and only if there exists an open subset of economies of full lebesgue measure in which equilibrium profiles of actions are, locally, the image of a non-empty, finite set of continuous functions.

This is an extension of the definition of Debreu (1970) for competitive economies.

A profile of actions, $a^{\mathscr{I}\prime}$, *pareto dominates* another, $a^{\mathscr{I}}$, if and only if, for every individual, $(a^{i\prime}, s^{\mathscr{I} \setminus \{i\}\prime}) \mathscr{R}^i (a^i, s^{\mathscr{I} \setminus \{i\}})$, with strict preference, $(a^{i\prime}, s^{\mathscr{I} \setminus \{i\}\prime}) \mathscr{P}^i (a^i, s^{\mathscr{I} \setminus \{i\}})$, for some.

A feasible profile of actions is *pareto suboptimal* if a feasible profile pareto dominates it.

A feasible profile of actions, $a^{\mathscr{I}}$, is *constrained pareto suboptimal* if and only if there exists in Nash–Walras equilibrium with a transfer of revenue, $(p^*, a^{\mathscr{I}*}, \tau^*)$, such that the profile of actions $a^{\mathscr{I}*}$ pareto dominates $a^{\mathscr{I}}$.

This is an extension of the definition of Arrow (1951) and Debreu (1951) for competitive economies and of Geanakoplos and Polemarchakis (1986) for competitive economies with an incomplete asset market.

The model encompasses alternative formulations of economic activity in which competitive and strategic elements interact. In particular, consumption externalities, as in Arrow (1970) or Pigou (1947), public goods, as in Samuelson (1954), and strategic market games, as in Shapley and Shubik (1977).†

## 3. Equilibria

An economy satisfies *standard assumptions* if and only if, for every individual,

- the preference relation, $\mathscr{R}^i$, is complete and transitive;
- the domain of actions, $\mathscr{A}^i$, is closed, and the preference relation, $\mathscr{R}^i$, is continuous;
- the domain of net trades in commodities is bounded below: there exists a trade in commodities, $\underline{z}^i$, such that $z^i \in \mathscr{Z}^i \Rightarrow z^i \geqslant \underline{z}^i$;
- the subset of actions $\mathscr{S}^i(\mathscr{Z}^i_c) = \{s^i: (z^i, s^i) \in \mathscr{A}^i$ for some $z^i \in \mathscr{Z}^i_c\} \subset \mathscr{S}^i$ is compact whenever the subset of the set of trades in commodities $\mathscr{Z}^i_c \subset \mathscr{Z}^i$ is compact;
- the domain of actions allows for the free disposal of commodities: $(z^i, s^i) \in \mathscr{A}^i$ and $z^{i\prime} \geqslant z' \Rightarrow (z^{i\prime}, s^i) \in \mathscr{A}^i$, and, for every profile of strategies $s^{\mathscr{I}\setminus\{i\}}$, the preference relation is weakly monotonically increasing in trades in commodities: $z^{i\prime} \geqslant z^i \Rightarrow (z^i, s^i)\mathscr{R}^i(s^{\mathscr{I}\setminus\{i\}})(z^{i\prime}, s^i)$;
- autarky in trade is possible: there exists a strategy, $s^i_0$, such that $(0, s^i_0) \in \mathscr{A}^i$;
- the domain of actions, $\mathscr{A}^i$, is convex, and, for every profile of strategies, $s^{\mathscr{I}\setminus\{i\}}$, and every action, $a^i$, the preferred set, $\mathscr{R}^i(a^i, s^{\mathscr{I}\setminus\{i\}}) = \{a^{i\prime} \in \mathscr{A}^i: a^{i\prime}\mathscr{R}^i(s^{\mathscr{I}\setminus\{i\}})a^i\} \subset \mathscr{A}^i$ is convex;
- the preference relation does not display local satiation: for every profile of strategies, $s^{\mathscr{I}\setminus\{i\}}$, for every $\varepsilon > 0$, and for every action, $a^i$, the strictly preferred set $\mathscr{P}^i_\varepsilon(a^i, s^{\mathscr{I}\setminus\{i\}}) = \{a^{i\prime} \in \mathscr{A}^i: a^{i\prime}\mathscr{R}^i(s^{\mathscr{I}\setminus\{i\}})a^i, \|a^{i\prime} - a^i\| < 0\} \subset \mathscr{A}^i$ is non-empty.

The standard assumptions do not suffice to guarantee the existence of Nash–Walras equilibria; this follows from the example of Arrow and Debreu (1954) for Walrasian equilibria. They do if autarky in trade is not a minimum wealth point for any individual.

**Proposition 1.** Nash–Walras equilibria exist if the economy

† Cournot–Walras equilibria, as in Gabszewicz and Vial (1972) require a sequential version of the model.

satisfies the standard assumptions and, for every individual and for any positive prices of commodities, $p>0$, there exists an action, $a^i=(z^i, s^i)$, such that $pz^i<0$.

**Proof.** The domain of normalized prices of commodities is $\mathscr{P}=\{p\colon p>0 \text{ and } \Sigma_{l\in\mathscr{L}}=1\}$.

For $n>0$, the truncated optimization problem of an individual is the optimization problem subject to the additional constraint $\|z\|\leqslant n$. A solution to the truncated optimization problem, $a^i_n(p, s^{\mathscr{I}\setminus\{i\}})$ exists; which defines the individual, truncated action correspondence, $a^i_n$. The aggregate, truncated action correspondence is $a^a_n=(z^a_n, \ldots, s^i_n, \ldots)$.

The truncated domain of actions for an individual is $\mathscr{A}^i_n=\{a^i=(z^i, s^i)\colon \|z^i\|\leqslant n\}$, a compact set; the truncated domain of trades in commodities is $\mathscr{Z}^i_n=\text{proj}_{z^i}\mathscr{A}^i_n$, and the truncated domain of strategies is $\mathscr{S}^i_n=\text{proj}_{s^i}\mathscr{A}^i_n$, compact sets as well. Across individuals, $\mathscr{S}^{\mathscr{I}}_n=x_{i\in\mathscr{I}}\mathscr{S}^i_n$. The aggregate truncated domain of trades in commodities is $\mathscr{Z}^a_n=\Sigma_{i\in\mathscr{I}}\mathscr{Z}^i_n$.

The correspondence, on the domain $\mathscr{P}\times\mathscr{Z}^a_n\times\mathscr{S}^{\mathscr{I}}_n$, defined componentwise by $p'=\arg\max\{pz^a_n\colon p\in\mathscr{P}\}$, $z^{a\prime}_n=z^a_n(p, s^{\mathscr{I}})$, $a^{i\prime}_n=a^i_n(p, s^{\mathscr{I}\setminus\{i\}})$ has a fixed point, $(p^*_n, z^{a*}_n, s^{\mathscr{I}*}_n)$; associated with the fixed point, there is a profile of actions, $a^{\mathscr{I}*}_n=((z^{i*}_n, s^{i*}_n\colon i\in\mathscr{I})$, such that, for every individual, $(z^{i*}_n, s^{i*}_n)$ is a solution to the truncated optimization problem at $(p^*_n, s^{\mathscr{I}\setminus\{i\}}_n)$, and $\Sigma_{i\in\mathscr{I}} z^{i*}_n=z^{a*}_n\leqslant 0$.

The sequence $\{(p^*_n, z^{a*}_n, s^{\mathscr{I}}_n)\colon n=1,\ldots\}$ has a convergent subsequence. Associated with a limit $(p^*_n, z^{a*}, s^{\mathscr{I}})=\lim_{n\to\infty}(p^*_n, z^{a*}_n, s^{\mathscr{I}}_n)$, there is a profile of actions, $a^{\mathscr{I}*}=((z^{i*}, s^{i*}\colon i\in\mathscr{I})$, such that, for every individual, $(z^{i*}, s^{i*})$ is a solution to the truncated optimization problem at $(p^*, s^{\mathscr{I}\setminus\{i\}})$, and $\Sigma_{i\in\mathscr{I}} z^{i*}=z^{a*}=0$—the profile of actions $a^{\mathscr{I}*}$ is feasible. ■

For autarky in trade not to be a minimum wealth point at any positive prices of commodities, it is necessary and sufficient that a strictly negative net trade in all commodities be in the domain of trades in commodities. Alternatively, a condition akin to resource relatedness introduced in McKenzie (1959, 1961) suffices, in conjunction with the standard assumption to guarantee the existence of equilibria.

A profile of actions, $a^{\mathscr{I}}$, is *individually rational* if and only if, for every individual, $(z^i, s^i)\mathscr{R}^i(s^{\mathscr{I}\setminus\{i\}})(0, s^{i\prime})$, every action $(0, s^{i\prime})\in\mathscr{A}^i$. An equilibrium profile of actions is individually rational.

**Corollary 1.** Nash–Walras equilibria exist if the economy satisfies the standard assumptions and for every individually rational, feasible profile of actions, $a^{\mathscr{I}}=(\ldots, (z^i, s^i), \ldots)$, for every non-empty, proper subset of individuals, $\mathscr{I}'\subset\mathscr{I}$, $\mathscr{I}'\neq\emptyset$, $\mathscr{I}$, there exists a feasible profile of actions, $\hat{a}^{\mathscr{I}}=(\ldots, (\hat{z}^i, \bar{s}^i), \ldots)$, such that $(\hat{z}^i, \bar{s}^{\mathscr{I}})\mathscr{R}^i(z^i,$

$s^{\mathscr{I}}$), for all individuals, $i \in \mathscr{I}$, with strict preference, $(\hat{z}^i, \bar{s}^{\mathscr{I}})\mathscr{P}^i(z^i, s^{\mathscr{I}})$, for all individuals, $i \in \mathscr{I}$, for some individual $i \in \mathscr{I}'$, while for every profile of strategies, $s^{\mathscr{I}}$, 0 is an interior point of the set $Z^a(s^{\mathscr{I}}) = \{z = \Sigma_{i \in \mathscr{I}} z^i \colon z^i \in A^i(s)\}$.

**Proof.** For $n>0$, the modified economy, $\mathscr{E}_n = \{(\mathscr{A}^i_n, \mathscr{R}^i_n) \colon i \in \mathscr{I}\}$ is defined by modifying the domain of actions of each individual according to† $(z^i, s^i) \in \mathscr{A}^i_n \Leftrightarrow (z^i + (1/n)\mathbf{1}_L, s^i) \in \mathscr{A}^i_n$, and his preference relation according to $(z^{i\prime}, s^{i\prime})\mathscr{R}^i_n(s^{\mathscr{I}\backslash\{i\}})(z^i, s^i) \Leftrightarrow (z^{i\prime} + (1/n)\mathbf{1}_L, s^{i\prime})\mathscr{R}^i(s^{\mathscr{I}\backslash\{i\}})$ $(z^i + (1/n)\mathbf{1}_L, s^i)$.

The economy $\mathscr{E}_n$ has a Nash–Walras equilibrium, $(p^*_n, a^{\mathscr{I}}_n{}^*)$.

The sequence $\{(p^*_n, a{*}\mathscr{I}_n) \colon n = 1, \ldots\}$ has a convergent subsequence. A limit, $(p^*, a{*}\mathscr{I})$, is an equilibrium for the economy $\mathscr{E}$ as long as the set $\mathscr{I}_1 = \{i \colon p^* z^i \geqslant 0$, for all actions $a^i = (z^i, s^i) \in \mathscr{A}^i\}$ is empty.

Since 0 is an interior point of the set $Z^a(s^{\mathscr{I}}{}^*)$, the set $\mathscr{I}_2 = \mathscr{I} \backslash \mathscr{I}_1$ is non-empty.

If the set $\mathscr{I}_1$ is non-empty, there exists a feasible profile of actions, $\bar{a}^{\mathscr{I}} = (\ldots, (\bar{z}^i, s^{i*}), \ldots)$, such that $(\bar{z}^i, s^{i*})\mathscr{R}^i(s^{\mathscr{I}\backslash\{i\}*})(z^{i*}, s^{i*})$, for all individuals, $i \in \mathscr{I}_2$, with strict preference, $(\bar{z}^i, s^{i*})\mathscr{P}^i(s^{\mathscr{I}\backslash\{i\}*})(z^{i*}, s^{i*})$, for some. For individuals in $i \in \mathscr{I}_2$, since $(z^{i*}, a^{i*})$ is a solution to the optimization problem, $p^* \bar{z}^i \geqslant 0$, with strict inequality in the case of strict preference, while for individuals in $i \in \mathscr{I}_1$, since $p^* z^i \geqslant 0$, for all actions, in particular, $p^* \bar{z}^i \geqslant 0$. It follows that $p^* \bar{z}^a > 0$, which contradicts the feasibility of the profile $\bar{a}^{\mathscr{I}}$: $\bar{z}^a \neq 0$. ■

An economy satisfies *standard smoothness assumptions* if and only if, for every individual, there exist constraint functions, $f^i = (f^i_j \colon j \in \mathscr{F})$, $\mathscr{F} = \{1, \ldots, F\}$ and $g^i = (g^i_j \colon j \in \mathscr{G})$, $\mathscr{G} = \{1, \ldots, G\}$, that describe the domain of actions:

$$\mathscr{A}^i = \{a^i = (z^i, s^i) \colon \; f^i(z^i, s^i) \leqslant 0, g^i(z^i, s^i) \leqslant 0\},$$

and a utility function, $u^i$, that represents the preference relation:

$$u^i(a^i, s^{\mathscr{I}\backslash\{i\}}) \geqslant u^i(a^{i\prime}, s^{\mathscr{I}\backslash\{i\}\prime}) \Leftrightarrow (a^i, s^{\mathscr{I}\backslash\{i\}})\mathscr{R}^i(a^i, s^{\mathscr{I}\backslash\{i\}\prime});$$

- the constraint functions, $f^i$ and $g^i$, and the utility function, $u^i$, are continuous;
- the domain of net trades in commodities is bounded below: there exists a net trade in commodities, $z^i$, such that $f^i(z^i, s^i) \leqslant 0$ and $g^i(z^i, s^i) \leqslant 0 \Rightarrow z^i \geqslant z^i$;
- the subset of strategies $\mathscr{S}^i(\mathscr{Z}^i_c) = \{s^i \colon f^i(z^i, s^i) \leqslant 0, g^i(z^i, s^i) \leqslant 0$, for some $z^i \in \mathscr{Z}^i_c\} \subset \mathscr{S}^i$ is compact whenever the subset of trades in commodities $\mathscr{Z}^i_c \subset \mathscr{Z}^i$ is compact;

† "$\mathbf{1}_K$" denotes the vector of 1's of dimension $K$.

- the domain of actions allows for the free disposal of commodities: $f^i(z^i, s^i) \leqslant 0, g^i(z^i, s^i) \leqslant 0$, and $z^{i\prime} \geqslant z^i \Rightarrow f^i(z^{i\prime}, s^i) \leqslant 0$, $g^i(z^{i\prime}, s^i) \leqslant 0$;
- autarky in trade is possible: there exists a strategy, $s^i_0$, such that $f^i(0, s^i_0) \leqslant 0$ and $g^i(0, s^i_0) \leqslant 0$;
- the constraint functions, $f^i$ and $g^i$, are quasi-convex, and the utility function, $u^i$, is quasi-concave;
- for every individually rational, feasible profile of actions, $a^{\mathscr{I}}$, with $g^i(z^i, s^i) \not\ll 0$, there exists a strategy, $s^{i\prime}$, such that $f^i(0, s^{i\prime}) \leqslant 0$, $g^i(0, s^{i\prime}) \ll 0$, and $u^i(0, s^{i\prime}, s^{\mathscr{I}\setminus\{i\}}) > u^i(z^i, s^i, s^{\mathscr{I}\setminus\{i\}})$;
- whenever $g^i(z^i, s^i) \ll 0$, the constraint function, $f^i$, and the utility function, $u^i$, are continuously differentiable;
- whenever $g^i(z^i, s^i) \ll 0$, for every commodity, $l \in \mathscr{L}$, for every dimension of strategy, $k^i \in \mathscr{K}^i$, and for every constraint function, $f^i_j$, $j \in \mathscr{F}$, $(\partial\ ^i_j/\partial s^i_{k^i}) \neq 0$ and $(\partial u^i/\partial z^i_l) - (\partial u^i/\partial s^i_{k^i})[(\partial f^i_j/\partial z^i_l)/(\partial f^i_j/\partial s^i_{k^i})] \neq 0$.

A profile of actions, $a^{\mathscr{I}}$, is interior if and only if $g^i(a^i) \ll 0$, for every individual.

A set of economies that satisfy standard smoothness assumptions is a *sufficiently diverse set of smooth economies* if and only if, for any economy and for any individually rational, feasible, interior profile of actions, $\bar{a}^{\mathscr{I}}$, it contains the economy with constraint and utility functions defined, in a neighbourhood of $\bar{a}^i$ by

$$f^i + \phi^i s^i$$

and

$$u^i + \zeta^i z^i + \sigma^{i,i} s^i + \sigma^{i,\mathscr{I}\setminus\{i\}} s^{\mathscr{I}\setminus\{i\}},$$

for parameters, $(\phi^i, \zeta^i, \sigma^{i,i}, \sigma^{i,\ \mathscr{I}\setminus\{i\}})$, in a neighbourhood of 0.

A property holds generically if and only if it holds for a set of economies of full lebesgue measure.

**Proposition 2.** For a sufficiently diverse set of smooth economies Nash–Walras equilibrium profiles of actions are generically determinate but pareto suboptimal.

**Proof.** Prices of commodities are $p = (1, \tilde{p})$, where $\tilde{p} = (p_l\colon l \in \mathscr{L} \setminus \{1\}) \gg 0$.

The function defined by

$$\varphi^i(p, z^i, s^{\mathscr{I}}, \lambda^i, \mu^i) = \begin{cases} D_{z^i}u^i - \mu^i D_{z^i} f^i - \lambda^i p, \\ D_{s^i}u^i - \mu^i D_{s^i} f^i, \\ pz^i, \\ f^i \end{cases}$$

characterizes the first-order conditions for a solution to the individual optimization problem: at prices of commodities and profile of strategies $(p, s^{\mathscr{I}\setminus\{i\}})$, it is necessary and sufficient for a solution that $\varphi^i(p, z^i, s^{\mathscr{I}}, \lambda^i, \mu^i) = 0$, for some lagrange multipliers, $(\lambda^i, \mu^i) \gg 0$, associated with the constraints on strategies and the budget constraint, respectively.

The function defined by

$$\varphi^0(\tilde{z}^{\mathscr{I}}) = \tilde{z}^a,$$

where $\tilde{z}$ denotes the deletion from a net trade in commodities of the net trade in commodity $l = 1$, characterizes feasibility of the aggregate net trades in commodities: in light of the budget constraint, it is necessary and sufficient for feasibility, $z^a = 0$, that $\varphi^0(\hat{z}^{\mathscr{I}}) = 0$.

The function $\varphi = (\varphi_0, \varphi_i\colon i \in \mathscr{I})$ characterizes Nash–Walras equilibria: up to the normalization of prices, a pair, $(p^*, a^{\mathscr{I}*})$, is a Nash–Walras equilibrium if and only if $\varphi(p^*, a^{\mathscr{I}*}, \lambda^{\mathscr{I}*}, \mu^{\mathscr{I}*}) = 0$, for some $(\lambda^{\mathscr{I}*}, \mu^{\mathscr{I}*}) = (\lambda^{i*}\colon i \in \mathscr{I}, \mu^{i*}\colon i \in \mathscr{I}) \gg 0$.

The dimension of the domain and the range of the function $\varphi$ are both equal to $(L-1) + \Sigma_{i\in\mathscr{I}} (L^i + K^i + F^i + 1)$. Nash–Walras equilibria are generically determinate for perturbations of the constraint and utility functions of individuals according to the parameters $(\phi^i, \zeta^i, \sigma^{i,i}, \sigma^{i,\mathscr{I}\setminus\{i\}})$ at individually rational, feasible, interior profiles of actions.

Perturbations of the utility functions of individuals according to the parameters $(\sigma^{i,\mathscr{I}\setminus\{i\}})$, do not affect the function $\varphi$, and hence the profile of actions at a Nash–Walras equilibrium; they do affect the utilities of individuals. The profile of actions at a Nash–Walras equilibrium is generically pareto suboptimal. ■

Though equilibrium profiles of actions are, generically, pareto suboptimal, a transfer of revenue need not suffice for a pareto improvement in welfare: they are not constrained suboptimal.

**Example.** There are two individuals, $i = 1, 2$, one commodity, $l = 1$, and one dimension of strategy for each individual, $k^i = 1$. A net trade in commodities is $z = (z_1, z_2)$, a strategy for an individual is $s^i$, and a profile of actions is $(z^1, z^2, s^1, s^2)$.

The preferences of an individual are represented by the utility function

$$u^i = z^i + (\beta^i/2)(z^i - s^i)^2 + \alpha^i s^{3-i};$$

lower bounds in the net trades in commodities and constraints on the strategies of individuals do not play an essential role, while $3 - i$ is the individual $i' \neq i$.

With only one commodity, the net trade in commodities is determined by the budget constraint, while the price of the commodity can be normalized to 1.

For a transfer of revenue $(\tau_1, \tau_2) = (\tau, -\tau)$, the equilibrium profile of actions is $(z^{1*}, z^{2*}, s^{1*}, s^{2*}) = (\tau, -\tau, \tau, -\tau)$, and the levels of utility of individuals at equilibrium are

$$u^{1*}(\tau) = -\alpha^1 \tau,\ u^{2*}(\tau) = \alpha^2 \tau.$$

As long as $\alpha^1\alpha^2 > 0$, a transfer of revenue does not yield a pareto improvement in welfare; it does, as long as $\alpha^1\alpha^2 < 0$: both cases are robust to perturbations, and neither case is generic. ■

## 4. Conclusion and extensions

The extension of the argument to economies with production is analytically straightforward, but conceptually problematic: beyond perfect competition, profit maximization is neither well defined nor unanimously agreed upon by shareholders.

More interestingly, preliminary results in Minelli and Polemarchakis (1995) indicate that the model encompasses economies with private information in which incentive compatibility constrains the allocation of resources.

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