Econ. Theory 6, 495–510 (1995)

Economic Theory

© Springer-Verlag 1995

Generic existence of competitive equilibria when the asset market is incomplete: A symmetric argument*

Herakles M. Polemarchakis[1] and Paolo Siconolfi[2]

[1] CORE, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, BELGIUM
[2] Graduate School of Business, Columbia University, New York, NY 10027, USA

Received: July 19, 1993; revised version November 21, 1994

Summary. The argument for the generic existence of competitive equilibria when the asset market is incomplete can be formulated with symmetric treatment of all individuals. The argument extends to a large economy.

1. Introduction

When the asset market is incomplete, competitive equilibria need not exist. This is due to the discontinuous dependence of the attainable reallocations of revenue on commodity prices and was noted first by Radner (1972) and Hart (1975). Nevertheless, competitive equilibria exist generically: it suffices to perturb the initial endowments or preferences of the individuals and the payoffs of the marketed assets. This was proved for a canonical model of an economy with incomplete asset markets by Duffie and Shafer (1985, 1986), and, later, Geanakoplos and Shafer (1990), Hirsch, Magill and Mas-Colell (1990) and Husseini, Lasry and Magill (1990) elaborated on the argument.

In order to circumvent the discontinuity in the attainable reallocations of revenue as commodity prices adjust to attain market clearing, it is convenient to introduce the notion of a pseudo-equilibrium, which requires that the reallocations of revenue attainable by individuals include, but do not necessarily coincide with the reallocations attainable by holding the marketed assets. A subsequent argument shows that generically a pseudo-equilibrium is a competitive equilibrium, which had been noted earlier by Mas-Colell (1982).

The proofs for the existence of pseudo-equilibria do not consider directly asset prices and equilibrium in the asset market. Circumventing the asset market is

* This text presents research results of the Belgian programme on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by its authors. The Commission of the European Communities provided additional support through the SPES grant CT91-0057. The second author acknowledges the generous support of the Graduate School of Business, Columbia University.
Correspondence to: H.M. Polemarchakis

essential for the argument, since feasible allocations of assets, unlike feasible allocations of commodities, are not compact; it is possible, since equilibrium in the commodity markets implies that the asset market clears as well. Circumventing asset prices is then necessary; it is possible, since it suffices to consider implicit prices for elementary securities or state-contingent revenue.

In the proof of existence of competitive equilibria, the normalization of prices, the elimination of redundant variations, serves to restrict the excess demand on a bounded domain with appropriate boundary behavior.

When the asset market is incomplete, elementary security prices which differ by an element of the null space of the matrix of asset payoffs are equivalent, since individuals are constrained reallocations of revenue in the span of this matrix.

One approach to the normalization of prices of elementary secutiries is to designate an individual to express excess demand under no constraint on the attainable reallocations of revenue. At a pseudo-equilibrium, since it coincides with the negative of the aggregate excess demand of all other individuals who do satisfy the subspace constraint on the attainable reallocations of revenue, the excess demand of the designated individual satisfies the constraint as well. This ingenious normalization was introduced by Cass (1985) and has been adopted by the literature on the existence of pseudo-equilibria.

Here, we normalize the prices of elementary securities by considering the quotient space which identifies prices which differ by an element of the null space of the matrix of asset payoffs. The proof of existence of pseudo-equilibria which we give treats individuals symmetrically.

The interest of a proof in which aggregate excess demand involves all individuals satisfying the subspace constraint on the attainable reallocations of revenue which characterizes an incomplete asset market goes beyond simple curiosity. The characterization of behavior out of equilibrium requires all individuals to be constrained in their attainable reallocation of revenue. More importantly, perhaps, as Shafer (1993) has pointed out, the symmetric argument without a designated individual is appropriate for a large economy, with a continuum of diverse, negligible individuals. In conclusion, we outline this argument.

2. The economy

Economic activity occurs over two periods, 1 and 2. Uncertainty is described by states of the world indexed by

$$s \in \mathbf{S} = \{1, \ldots, S\},$$

a finite, nonempty set, and is resolved all in the second period.

Commodities, abstracting from temporal and contingent characteristics, are indexed by

$$\ell \in \mathbf{L} = \{1, \ldots, L\},$$

a finite, nonempty set, and are traded in commodity spot markets in the first period and, following the resolution of uncertainty, in the second. Commodities are thus indexed by $(1, \ell)$ in the first period, and by $((2, s), \ell)$ in state s in the second.

A commodity bundle is a column vector

$$x = (x_1, \ldots, x_{(2s)}, \ldots) \in \mathbf{X},$$

an element of the commodity space, $\mathbb{R}^{L(S+1)}$, where the column vector

$$x_1 = (\ldots, x_{(1,\ell)}, \ldots),$$

of dimension L, is a first period commodity bundle and the column vector

$$x_{(2,s)} = (\ldots, x_{((2,s)\ell)}, \ldots),$$

of dimension L, is a commodity bundle in state s in the second period, and a column vector

$$x_2 = (\ldots, x_{(2,s)}, \ldots),$$

of dimension SL, is a commodity bundle in the second period.
Assets are indexed by

$$a \in \mathbf{A} = \{1, \ldots, \mathbf{A}\},$$

a finite, nonempty set, and are traded in the first period and payoff in the second, after the resolution of the uncertainty but prior to trade in the commodity spot markets. The payoff of an asset is a column vector

$$r_a = (\ldots, r_{a,(2,s)}, \ldots),$$

of dimension LS, a second period commodity bundle. The asset structure is thus described by the matrix of asset payoffs

$$R = (\ldots, r_a, \ldots),$$

of dimension $LS \times A$.

Assumption 1: The matrix of asset payoffs, R, has full column rank.
It follows that $A \leq S$.
The asset structure can thus be identified as a point

$$R \in \mathbf{R},$$

as open set of $\mathbb{R}^{ALS}$.
Commodity prices are a row vector

$$p = (p_1, \ldots, p_{(2,s)}, \ldots) \in \mathbf{P},$$

an element of the space of commodity prices, $\mathbb{R}^{L(S+1)}_{++}$, where the row vector

$$p_1 = (\ldots, p_{(1,\ell)}, \ldots),$$

of dimension L, is the vector of first period, spot commodity prices, and the row vector

$$p_{(2,s)} = (\ldots, p_{((2,s),\ell)}, \ldots),$$

of dimension L, is the vector of spot commodity prices in state s, in the second period, and the row vector

$$p_2 = (\ldots, p_{(2,s)}, \ldots),$$

of dimension LS, are commodity prices in the second period.

At commodity prices p, the payoff of an asset in terms of revenue is

$$r_a(p) = (\ldots, p_{(2,s)} r_{a,(2,s)}, \ldots),$$

a column vector of dimension S, and the matrix of asset payoffs in terms of revenue is

$$R(p) = (\ldots, r_a(p), \ldots),$$

of dimension $S \times A$.

The matrix of asset payoffs in terms of revenue need not have full column rank at all commodity prices $p \in \mathbf{P}$, even if assumption 1 is satisfied by the asset structure. At commodity prices p, the attainable reallocations of revenue are described by the column span

$$[R(p)],$$

a subspace of dimension at most A of the space of reallocations of revenue, $\mathbb{R}^S$.

Grassmanians

For $J \leq S$, the set of subspaces of $\mathbb{R}^S$ of dimension J has a structure of a smooth, C^∞-manifold of dimension $J(S-J)$, the grassmanian manifold, $\mathbf{\Gamma}^{S,J}$.

An atlas for $\mathbf{\Gamma}^{S,J}$ is constructed as follows: Consider the set, $\mathbf{M}^{S,J}$, of matrices of dimension $S \times J$ of full column rank, J, which can be identified with an open set in $\mathbb{R}^{SJ}$. The column span, $[M]$, of a matrix $M \in \mathbf{M}^{S,J}$ defines a subspace of $\mathbb{R}^S$ of dimension J and, conversely, any subspace of dimension J in $\mathbb{R}^{SJ}$ can be so obtained. Nevertheless, distinct matrices in $\mathbf{M}^{S,J}$ may define the same subspace of dimension J. Thus, we consider the equivalence relation $\sim$ on $\mathbf{M}^{S,J}$ which identifies matrices whose column spans coincide: $\mathbf{M} \sim \mathbf{M}'$ if and only if $M = M'B$, for some invertible matrix B of dimension $J \times J$. The set $\mathbf{\Gamma}^{S,J}$ can be thus identified with the quotient space $\mathbf{M}^{S,J}/\sim$. The projection map $\text{proj}_{S,J}: \mathbf{M}^{S,J} \rightarrow \mathbf{\Gamma}^{S,J}$ associates with each element its equivalence class, $\text{proj}_{S,J}(M) = V \in \mathbf{\Gamma}^{S,J}$ such that $[M] = V$, and defines a topology on the quotient space: $\mathbf{V} \subseteq \mathbf{\Gamma}^{S,J}$ is open if and only if $\text{proj}_{S,J}^{-1}(V)$ is an open set of $\mathbf{M}^{S,J}$.

Let $\mathbf{\Sigma}^J$, $1 \leq J \leq S$, be the collection of subsets of $\mathbf{S}$ of cardinality J. For $\sigma \in \mathbf{\Sigma}^J$ and a matrix $M \in \mathbf{M}^{S,J}$, let M_σ be the submatrix of M consisting of rows $s \in \sigma$, and let $M_{\setminus\sigma}$ be the complementary submatrix, the matrix M_σ is of dimension $J \times J$ and the matrix $M_{\setminus\sigma}$ of dimension $(\mathbf{S}-\mathbf{J}) \times J$.

Let $\mathbf{M}_\sigma^{S,J} \subseteq \mathbf{M}^{S,J}$ be the open set of matrices whose submatrix M_σ is invertible. Since every matrix $M \in \mathbf{M}^{S,J}$ has at least one invertible submatrix of order J, the collection $\{\mathbf{M}_\sigma^{S,J}: \sigma \in \mathbf{\Sigma}^J$ is an open cover of $\mathbf{M}^{S,J}$. It follows that, for $\mathbf{\Gamma}_\sigma^{S,J} = \mathbf{M}_\sigma^{S,J}/\sim$, the collection $\{\mathbf{\Gamma}_\sigma^{S,J}: \sigma \in \mathbf{\Sigma}^J\}$ is an open cover of $\mathbf{\Gamma}^{S,J}$.

With some abuse of notation, let $\sigma \in \mathbf{\Sigma}^J$ denote the canonical permutation of the elements of $\mathbf{S}$ such that $\sigma^{-1}(s) \in \sigma$, $1 \leq s \leq J$, and $\sigma^{-1}(s) > \sigma^{-1}(s')$, if $s > s'$ and either $1 \leq s, s' \leq J$ or $J+1 \leq s, s' \leq S$.

The permutation matrix associated with σ^{-1} is

$$\pi_\sigma = \begin{pmatrix} \pi_\sigma^1, & \pi_\sigma^2 \\ \pi_\sigma^3, & \pi_\sigma^4 \end{pmatrix},$$

where π_σ^1, π_σ^2, π_σ^3 and σ_σ^4 are matrices of dimension $A \times A$, $A \times (S-A)$, $(S-A) \times A$, and $(S-A) \times (S-A)$, respectively.

Moreover, by the definition of the canonical permutation $\sigma \in \mathbf{\Sigma}^J$, it follows that, for a matrix M of dimension $S \times J$,

$$\pi_\sigma \begin{pmatrix} M_\sigma \\ M_{\backslash\sigma} \end{pmatrix} = M.$$

For each integer N, let I_N be the identity matrix of dimension N. The function ψ_σ: $\mathbf{\Gamma}_\sigma^{S,J} \to \mathbb{R}^{J(S-J)}$ defined by $\psi_\sigma(V) = E$, where E is the unique matrix of dimension $(S-J) \times J$ such that $\left[\pi_\sigma \begin{pmatrix} I_J \\ E \end{pmatrix}\right] = V \in \mathbf{\Gamma}^{S,J}$, provides a chart and $\{(\mathbf{\Gamma}_\sigma^{S,J}, \psi_\sigma): \sigma \in \mathbf{\Sigma}^J\}$ an atlas on $\mathbf{\Gamma}^{S,J}$.

Finally, for an arbitrary matrix M of dimension $S \times J$, not necessarily of full column rank, and for $V \in \mathbf{\Gamma}_\sigma^{S,J}$, $\sigma \in \mathbf{\Sigma}^J$,

$$[M] \subseteq V \Leftrightarrow \phi_\sigma(V, M) = (-\psi_\sigma(V), I_{S-J})\pi_\sigma^{-1} M = 0.$$

Asset prices are described implicitly. Implicit prices for state contingent-revenue in the second period are a row vector

$$\lambda = (\ldots, \lambda_{(2,s)}, \ldots) \in \mathbf{\Lambda}$$

an element of the space of implicit asset prices, $\mathbb{R}^S$. Prices for state-contingent revenue can be interpreted as prices for elementary securities. They are not restricted to be positive.

Prices are a pair,

$$(p, \lambda) \in \mathbf{P} \times \mathbf{\Lambda},$$

of spot commodity prices and implicit asset prices.

At prices (p, λ), the prices of the marketed assets are a row vector

$$q = (\ldots, q_a, \ldots) = \lambda R(p).$$

They need not be positive.

Individuals are indexed by

$$h \in \mathbf{H} = \{1, \ldots, H\},$$

a finite, nonempty set.

An individual is described by the characteristics

$$(\mathbf{X}^h, u^h, w^h),$$

where

$$\mathbf{X}^h \subseteq \mathbf{X}$$

is the consumption set, a subset of the commodity space,

$$u^h \colon \mathbf{X}^h \to \mathbb{R}$$

is the utility function with domain the consumption set, and

$$w^h \in \mathbf{X}^h$$

is the initial endowment, a commodity bundle.

Assumption 2: For $h \in \mathbf{H}$,

i) the consumption set, $\mathbf{X}^h$, concides with the nonnegative orthant of the commodity space, $\mathbb{R}_+^{(S+1)L}$,
ii) the utility function, u^h: $\mathbf{X}^h \to \mathbb{R}$, is continuous, strictly quasi-concave and strictly monotonically increasing and, on the interior of its domain of definition, it is twice continuously differentiable $Du^h \gg 0$, differentiably strictly quasi concave, $\Delta x \neq 0$ and $Du^h(x)\Delta x = 0 \Rightarrow (\Delta x)^T D^2 u^h(x)(\Delta x) < 0$, and, along any sequence, $(x_n \in \text{Int}\mathbf{X}^h$: $n = 1, \ldots)$, of interior consumption bundles which converges to a consumption bundle on the boundary, $x_n \to x \in Bd\mathbf{X}^h$, $x_n^T(1/\|Du^h(x_n)\|)Du^h(x_n) \to 0$,
iii) the intial endowment is a consumption bundle in the interior of the consumption set.

The reallocation of revenue necessary to finance the purchase of a second period commodity bundle $z_2 = (\ldots, z_{(2,s)}, \ldots)$ at second period commodity prices $p_2 = (\ldots, p_{(2,s)}, \ldots)$ is

$$p_2 \otimes z_2 = (\ldots, p_{(2,s)} z_{(2,s)}, \ldots).$$

A feasible allocation is an array of consumption bundles,

$$x^H = (\ldots, x^h, \ldots) \in \mathbf{X}^H,$$

where

$$\mathbf{X}^H = \prod_{h \in H} \mathbf{X}^h,$$

such that

$$\sum_{h \in H} x^h = \sum_{h \in H} w^h.$$

The allocation of the initial endowments is a point

$$w^H = (\ldots, w^h, \ldots) \in \mathbf{X}^H.$$

an open subset of $\mathbb{R}_+^{(S+1)LH}$.

Economies are parametrized by the allocation of initial endowments and the matrix of asset payoffs

$$(w^H, R) \in \mathbf{X}^H \times \mathbf{R},$$

an element of an open subset of $\mathbb{R}^{HL(S+1)+ALS}$.

A property holds generically if and only if it holds for an open and dense set of economies of full Lebesgue measure.

3. Competitive equilibria

At prices (p, λ), the individual optimization problem is to

$$\begin{aligned} &\max u^h(z + w^h) \\ &s.t.\ p_1 z_1 + \lambda(p_2 \otimes z_2) = 0 \\ &p_2 \otimes z_2 \in [R(p)]. \end{aligned}$$

Definition 1: A competitive equilibrium is a pair

$$((p^*, \lambda^*), x^{*H}),$$

of prices, (p^*, λ^*), and a feasible allocation, x^{*H}, such that, for very individual, $z^{*h} = (x^{*h} - w^h)$ solves the individual optimization problem at (p^*, λ^*).

At prices and attainable reallocation of revenue (p, λ, V), the modified individual optimization problem is to

$$\max u^h(z + w^h)$$
$$s.t.\ p_1 z_1 + \lambda(p_2 \otimes z_2) = 0$$
$$p_2 \otimes z_2 \in V.$$

Definition 2: A pseudo-equilibrium is a triple

$$((p^*, \lambda^*), V^*, x^{*H}),$$

of prices, (p^*, λ^*), attainable reallocations of revenue, V^*, and a feasible allocation, $x^{*\mathrm{H}}$, such that (i) for every individual, $z^{*h} = (x^{*h} - w^h)$ solves the modified individual optimization problem at (p^*, λ^*, V^*), while (ii) $[R(p^*)] \subseteq V^*$.

Evidently, if $((p^*, \lambda^*), V^*, x^{*\mathrm{H}})$ is a pseudo-equilibrium and $[R(p^*)] = V^*$, $((p^*, \lambda^*), x^{*H})$ is a competitive equilibrium.

Proposition: *Pseudo-equilibria exist.*

Corollary: *Generically, competitive equilibria exist.*

In the next section we develop the argument and prove the proposition. In the last section we extend the argument for the existence of a pseudo equilibrium to a large economy. The proof of the corollary we omit, since it was given in Duffie and Shafer (1985, theorem 2).

4. Proof

Let

$$\mathbf{D}^* = \{(p, \lambda, V) \in \mathbf{P} \times \mathbf{\Lambda} \times \mathbf{\Gamma}^{S,A}: \|p_1\| = 1, \|p_2(s)\| = 1, s \in \mathbf{S}\},$$

be the domain of prices and attainable reallocations of revenue.

Evidently, $\mathbf{D}^* = \mathbf{D}_1 \times \mathbf{D}_2^*$, where $\mathbf{D}_1 = \{p \in \mathbf{P}: \|p_1\| = 1, \|p_2(s)\| = 1, s \in \mathbf{S}\}$, and $\mathbf{D}_2^* = \mathbf{\Lambda} \times \mathbf{\Gamma}^{S,A}$. Observe that for distinct implicit asset prices and attainable reallocations, (p, λ, V) and (p', λ', V'), the solution to the modified optimization problem of each individual coincides if $p = p'$, $V = V'$ and $(\lambda - \lambda') \perp V$. In order to eliminate redundant implicit asset prices, consider the equivalence relation $\sim$ on $\mathbf{D}_2^*$, defined by:

$$(\lambda, V) \sim (\lambda', V') \text{ if and only if } V = V' \text{ and } (\lambda - \lambda') \perp V.$$

The restricted domain of implicit asset prices and reallocations of revenue is

$$\mathbf{D}_2 = \mathbf{D}_2^* / \sim.$$

An element of $\mathbf{D}_2$ is $([\lambda], V)$, and $(\lambda, V) \in ([\lambda], V)$ means that (λ, V) belongs to the equivalence class of $([\lambda], V)$. The projection mapping $\mathrm{proj}_{\mathbf{D}2}$: $\mathbf{D}_2^* \to \mathbf{D}_2$ associates with each element its equivalence class, $\text{proj}_{\mathbf{D}2}(\lambda, V) = ([\lambda], V)$. $\mathbf{D}_2$ is endowed with the quotient topology induced by the projection map: a set U is open in $\mathbf{D}_2$ if and only if $\text{proj}_{\mathbf{D}2}^{-1}(U)$ is an open set of $\mathbf{D}_2^*$.

The essential domain of prices and attainable reallocations of revenue is

$$\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2.$$

Lemma 1: **D** *is a smooth manifold of dimension* $(S+1)(L-1) + A + A(S-A)$ *without boundary.*

Proof: Evidently, $\mathbf{D}_1$ is a smooth manifold of dimension $(S+1)(L-1)$ without boundary. Thus, it remains to show that $\mathbf{D}_2$ is a smooth manifold of dimension $A + A(S-A)$ without boundary.

For $\sigma \in \Sigma^A$, let $\mathbf{D}_{2,\sigma} = \{([\lambda], V) \in \mathbf{D}_2 \colon V \in \Gamma_\sigma^{S,A}\}$. Since $\{\Gamma_\sigma^{S,A}, \sigma \in \Sigma^A\}$ is an open cover of $\Gamma^{S,A}$, $\{\mathbf{D}_{2,\sigma}, \sigma \in \Sigma^A\}$ is an open cover of $\mathbf{D}_2$.

For $V \in \Gamma_\sigma^{S,A}$, $\sigma \in \Sigma^A$, $(\lambda, V) \sim (\lambda', V)$ if and only if there is a $q \in \mathbb{R}^A$ such that

$$\lambda \pi_\sigma \begin{pmatrix} I_A \\ \psi_\sigma(V) \end{pmatrix} = \lambda' \pi_\sigma \begin{pmatrix} I_A \\ \psi_\sigma(V) \end{pmatrix} = q,$$

where, $\left[\pi_\sigma \begin{pmatrix} I_A \\ \psi_\sigma(V) \end{pmatrix} \right] = V$. It follows that, associated to each $([\lambda], V) \in \mathbf{D}_{2,\sigma}$, there is a $q([\lambda], V) \in \mathbb{R}^A$ and a matrix $\psi_\sigma(V)$ of dimension $(S-A)Xa$, such that

$$\left((q_\sigma([\lambda], V), 0)\pi_\sigma^{-1}, \left[\pi_\sigma \begin{pmatrix} I_A \\ \psi_\sigma(V) \end{pmatrix} \right] \right) \in ([\lambda], V),$$

where 0 is a row vector of dimension $S - A$ with all components equal to zero.

For $([\lambda], V) \in \mathbf{D}_{2,\sigma} \cap \mathbf{D}_{2,\sigma'}$, $\sigma \neq \sigma'$, there exist vectors $q_\sigma([\lambda], V)$, $q_{\sigma'}([\lambda], V) \in \mathbb{R}^A$ and matrices $\psi_\sigma(V)$ and $\psi_{\sigma'}(V)$ of dimension $(S-A)Xa$ such that

$$\left((q_\sigma([\lambda], V), 0)\pi_{\sigma'}^{-1}, \left[\pi_\sigma \begin{pmatrix} I_A \\ \psi_\sigma(V) \end{pmatrix} \right] \right) \in ([\lambda], V),$$

and

$$\left((q_\sigma([\lambda], V), 0)\pi_{\sigma'}^{-1}, \left[P_{\sigma'} \begin{pmatrix} I_A \\ \psi_{\sigma'}(V) \end{pmatrix} \right] \right) \in ([\lambda], V).$$

Moreover, a minor modification of the argument in Duffie and Shafer (1985, fact 2), implies that

$$\psi_\sigma(V) = (\pi^{(3)} + \pi^{(4)}\psi_{\sigma'}(V))(\pi^{(1)} + \pi^{(2)}\psi_{\sigma'}(V))^{-1},$$

and

$$q_\sigma([\lambda], V) = q_{\sigma'}([\lambda], V)(\pi^{(1)} + \pi^{(2)}\psi_{\sigma'}(V)),$$

where $\pi = \pi_\sigma \pi_{\sigma'}^{-1}$. Evidently, $\sigma = \sigma'$ implies that $q_\sigma([\lambda], V) = q_{\sigma'}([\lambda], V)$ and $\psi_\sigma(V) = \psi_{\sigma'}(V)$. Let

$$\beta_\sigma = (\psi_\sigma, q_\sigma) \colon \mathbf{D}_{2,\sigma} \to \mathbb{R}^A \times \mathbb{R}^{A(S-A)}$$

It follows that β_σ is an homeomorphism of $\mathbf{D}_{2,\sigma}$ onto $\mathbb{R}^A \times \mathbb{R}^{A(S-A)}$ and that the map

$$\beta_\sigma \circ \beta_{\sigma'}^{-1}: \beta_{\sigma'}(\mathbf{D}_{2,\sigma'} \cap \mathbf{D}_{2,\sigma}) \to \beta_\sigma(\mathbf{D}_{2,\sigma'} \cap \mathbf{D}_{2,\sigma})$$

is smooth for all σ, $\sigma' \in \mathbf{\Sigma}^A$. As indicated above, $\beta_\sigma \circ \beta_{\sigma'}^{-1}(q, E) = (q(\pi^{(1)} + \pi^{(2)}E), (\pi^{(3)} + \pi^{(4)}E)(\pi^{(1)} + \pi^{(2)}E)^{-1})$. Therefore, since $\{\mathbf{D}_{2,\sigma}: \sigma \in \mathbf{\Sigma}^A\}$ is an open cover, $\mathbf{D}_2$ is a smooth manifold of dimension $A + A(S - A)$ without boundary. □

When we perturb functions with respect to $([\lambda], V) \in \mathbf{D}_{2,\sigma}$, we take derivatives with respect to $(q, E) = \beta_\sigma([\lambda], V)$.

Consider the subset of the essential domain

$$\mathbf{D}_{NA} = \{(p, [\lambda], V) \in \mathbf{D}: \lambda b \leq 0, \text{ for } \lambda \in [\lambda], \text{ and } b \in V \Rightarrow b \not> 0\},$$

of prices and attainable reallocations of revenue which do not allow for arbitrage. Moreover, denote with d its representative element.

Lemma 2: $\mathbf{D}_{NA}$ *is a nonempty, open subset of* $\mathbf{D}$ *and hence is a smooth manifold of dimension* $(S + 1)(L - 1) + A + A(S - A)$ *without boundary.*

Proof: For any $d = (p, [\lambda], V) \in \mathbf{D}$, such that $(\lambda^*, V) \in ([\lambda], V)$ for some $\lambda^* \gg 0$, $d \in \mathbf{D}_{NA}$. Therefore $\mathbf{D}_{NA}$ is nonempty.

To show that $\mathbf{D}_{NA}$ is open, we argue by contradiction. Suppose that there is a sequence $(d_n: d_n \in \mathbf{D}/\mathbf{D}_{NA}, n = 1, \ldots)$, and $d_n \to d \in \mathbf{D}_{NA}$. Then there exists a sequence $(b_n, n = 1, \ldots)$ such that

$$b_n > 0,\ \lambda_n \mathrm{b}_n \leq 0 \text{ and } b_n \in V_n,\ n = 1, \ldots, \tag{2.1}$$

where $(\lambda_n, V_n) \in ([\lambda]_n, V_n)$. Moreover since the constraints in (2.1) are homogenous of degree 1 in b, we can assume that $\|b_n\| = 1$, for $n = 1, \ldots$. The definition of the sequence implies, without loss of generality, that

$$b_n \to b > 0,\ \lambda b \leq 0,\ b \in V \text{ and } (\lambda, V) \in ([\lambda], V),$$

which contradicts that $d \in \mathbf{D}_{NA}$. □

The following example illustrates the construction of the essential domain of prices and attainable reallocations of revenue which do not allow for arbitrage, and its structure.

Example 1: There is one commodity, $L = 1$, two states of the world, $S = 2$, and one asset, $A = 1$. The domain of commodity prices $\mathbf{D}_1 = \{p = (p_1, p_{2,1}, p_{2,2}) \in \mathbf{P}: p = (1, 1, 1)\}$, a singleton. The domain for implicit prices for state-contingent revenue and attainable reallocations of revenue is $\mathbf{D}_2^* = \mathbf{\Lambda} \times \mathbf{\Gamma}^{2,1}$, where $\mathbf{\Lambda} = \mathbb{R}^2$, while $\mathbf{\Gamma}^{2,1}$ is the grassmanian manifold of lines in the plane. An open cover of $\Gamma^{2,1}$ is $\{\Gamma_1^{2,1}, \Gamma_2^{2,1}\}$, where $\Gamma_1^{2,1} = \{V \in \Gamma^{2,1}: V = [\binom{1}{\alpha}], \alpha \in \mathbb{R}\}$ and $\Gamma_2^{2,1} = \{V \in \Gamma^{2,1}: V = [\binom{\alpha}{1}], \alpha \in \mathbb{R}\}$. The equivalence relation $\sim$ on $\mathbf{D}_2^*$ is defined by $(\lambda, V) \sim (\lambda', V')$ if and only if $V = V'$ and $(\lambda - \lambda') \perp V$. The restricted domain of implicit asset prices and reallocations of revenue is the quotient domain $\mathbf{D}_2 = \mathbf{D}_2^* \backslash \sim$ a smooth manifold of dimension 2, without boundary. An open cover for $\mathbf{D}_2$ is $\{\mathbf{D}_{2,1}, \mathbf{D}_{2,2}\}$, where $\mathbf{D}_{2,1} = \{([\lambda], V) \in \mathbf{D}_2: V \in \Gamma_1^{2,1}\}$, and similarly for $\mathbf{D}_{2,2}$. A chart on $\mathbf{D}_{2,1}$ is defined by $\beta_1^{-1}(\alpha, q) = \{([\lambda], V) \in \mathbf{D}_{2,1}: [\lambda] = \{\lambda: \lambda[\binom{1}{\alpha}] = q\}, V = [\binom{1}{\alpha}]\}$ and similarly for $\mathbf{D}_{2,2}$.

Note that the essential domain $\mathbf{D}=\mathbf{D}_1\times\mathbf{D}_2$ does not involve a restriction of noarbitrage.

The subset of the essential domain of prices and attainable reallocations of revenue which do not allow for arbitrage is

$$\mathbf{D}_{NA}=\{(p,[\lambda],V)\in\mathbf{D}\colon \lambda b\leq 0,\text{ for }\lambda\in[\lambda],\text{ and }b\in V\Rightarrow b\not>0\}.$$

An open cover for $\mathbf{D}_{NA}$ is $\{\mathbf{D}_{NA,1},\mathbf{D}_{NA,2}\}$, where $\mathbf{D}_{NA,1}=\{(p,[\lambda],V)\in\mathbf{D}_{NA}\colon V\in\Gamma_1^{2,1}\}$, and similarly for $\mathbf{D}_{NA,2}$. A chart on $\mathbf{D}_{NA,1}$ is defined by $\beta_1^{-1}\colon\mathbb{R}^2\to\mathbf{D}_{2,1}$. Let $\mathbf{Q}(\alpha)=\{q\in\mathbb{R}\colon [\lambda]=\{\lambda\colon\lambda[\binom{1}{\alpha}]=q\}$ does not allow for arbitrage for $V=[\binom{1}{\alpha}]\}$ or, equivalently, $\mathbf{Q}(\alpha)=\mathbb{R}$, if $\alpha<0$, while $\mathbf{Q}(\alpha)=\mathbb{R}_{++}$, if $\alpha\geq 0$. It follows that $\{(1,1,1)\}\times\beta_1^{-1}(\{(\alpha,q)\colon q\in\mathbf{Q}(\alpha)\})=\mathbf{D}_{NA,1}$, and hence β_1^{-1} defines a chart on a chart on $\mathbf{D}_{NA,1}$, since $\{(\alpha,q)\colon q\in\mathbf{Q}(\alpha)\}\subset\mathbb{R}^2$ is open. A similar construction for $\mathbf{D}_{NA,2}$ completes the argument that $\mathbf{D}_{NA}$ is a smooth manifold of dimension 2, without boundary.

For any set $\mathbf{F}$, $Bd\mathbf{F}$ and $Cl\mathbf{F}$ are, respectively, its boundary and its closure.

Lemma 3: *For any sequence* $(d_n\colon d_n\in D_{NA}, n=1,\ldots)$, *either there exists a subsequence* $(d_{n(k)}\colon d_{n(k)}\in D_{NA}, k=1,\ldots)$ *with* $d_{n(k)}\to d\in cl\mathbf{D}_{NA}$ or $\|\lambda_n\|\to+\infty$, *for all* $(p_n,\lambda_n,V_n)\in d_n, n=1,\ldots$.

Proof: Since $\mathbf{D}_1$ is bounded and $\mathbf{\Gamma}^{S,A}$ is compact, without any loss of generality, $(p_n,V_n)\to(p,V)$. Let $\sigma\in\mathbf{\Sigma}^A$ be such that $V\in\mathbf{\Gamma}_\sigma^{S,A}$. Then, by the definition of the set $\mathbf{\Gamma}_\sigma^{S,A}$ and of the map β_σ, $\sigma\in\mathbf{\Sigma}^A$, $V_n\in\mathbf{\Gamma}_\sigma^{S,A}$ and $\left((q_\sigma([\lambda]_n,V_n),0)\pi_\sigma^{-1},\left[\pi_\sigma\begin{pmatrix}I_A\\ \psi_\sigma(V_n)\end{pmatrix}\right]\right)\in([\lambda]_n,V_n)$, for $n>N$, large enough. Consider the sequence $q_n=q_\sigma([\lambda]_n,V_n)$, $n>N$. Evidently, without loss of generality, either $q_n\to q$ or $\|q_n\|\to+\infty$. But then, the definition of the map $q_\sigma([\lambda],V)$, implies that if $\|q_n\|\to+\infty$, $\|\lambda_n\|\to+\infty$, for all $(\lambda_n,V_n)\in([\lambda_n],V_n)$, while if $q_n\to q$, $(p_n,\lambda_n,V_n)\to\left(p,[(q,0)\pi_\sigma^{-1}],\left[\pi_\sigma\begin{pmatrix}I_A\\ \psi_\sigma(V)\end{pmatrix}\right]\right)$. □

At $d\in\mathbf{D}_{NA}$ and at $w^h\in\text{Int}\mathbf{X}^h$, a solution to the modified individual optimization problem,

$$z^h(d,w^h),$$

exists and is unique. The excess demand function,

$$z^h\colon\mathbf{D}_{NA}\times\text{Int}\mathbf{X}^h\to\mathbb{R}^{(S+1)L},$$

is thus well defined.

Aggregate excess demand is $z(d,w^H)=\sum_{h\in\mathbf{H}}z^h(d,w^h)$ and the aggregate excess demand function is

$$z\colon\mathbf{D}_{NA}\times\text{Int}\mathbf{X}^H\to\mathbb{R}^{(S+1)L}.$$

Lemma 4: *The excess demand function,* z^h, $h\in\mathbf{H}$, *is bounded from below,* $z\geq -w^h$, *it is continuously differentiable, and it satisfies the budget constraint,* $p_1z_1^h+\lambda(p_2\otimes z_2^h)=0$ *and* $p_2\otimes z_2^h\in V$. *For any sequence* $((d_n,w_n^h)\colon (d_n,w_n^h)\in$

$\mathbf{D}_{NA} \times \mathrm{Int}\mathbf{X}^h, n=1,\ldots)$, *with* $w_n^h \to w^h \in \mathrm{Int}\mathbf{X}^h$, *as long as* $d_n \to d \in \mathrm{Bd}\mathbf{D}_{NA}$ *or* $\|\lambda_n\| \to +\infty$, *for all* $(p_n, \lambda_n, V_n) \in d_n, n=1,\ldots, \|z^h(d_n, w_n^h)\| \to +\infty$. *The aggregate excess demand function z inherits the natural extensions of these properties.*

Proof: We just prove that the boundary conditions hold, since the rest is obvious.

The claim is trivial if $d_n \to d \in \mathrm{Bd}\mathbf{D}_{NA}$.

Therefore, suppose that $(d_n, n=1,\ldots)$ is such that $p_{n,2} \to p_2 \gg 0$ and $\|\lambda_n\| \to +\infty$, for all $(p_n, \lambda_n, V_n) \in d_n, n=1,\ldots$.

Since $\mathbf{\Gamma}^{S,A}$ is compact, without loss of generality, $V_n \to V$. Let σ be such that $V \in \mathbf{\Gamma}_\sigma^{S,A}$. Then, by the definition of the set $\mathbf{\Gamma}_\sigma^{S,A}$ and of the map β_σ, $\sigma \in \mathbf{\Sigma}^A$, $V_n \in \mathbf{\Gamma}_\sigma^{S,A}$ and $\left((q_\sigma([\lambda]_n, V_n), 0)\pi_\sigma^{-1}, \left[\pi_\sigma\begin{pmatrix} I_A \\ \psi_\sigma(V_n)\end{pmatrix}\right]\right) \in ([\lambda]_n, V_n)$, $n>N$, N large enough. Let $\lambda_n = (q_\sigma([\lambda]_n, V_n), 0)\pi_\sigma^{-1}$, $*\lambda_n = \lambda_n / \|(p_{n,1}, \lambda_n)\|$, $*p_{n,1} = p_{n,1}/\|(p_{n,1}, \lambda_n)\|$ and $*p_n = (*p_{n,1}, p_{n,2})$. The homogeneity of degree one in (p_1, λ) of the first period budget constraint and the definition of λ_n imply that $z(p_n, [\lambda_n], V_n, w_n^H) = z(*p_n, [*\lambda_n], V_n, w_n^H)$. But then, without any loss of generality, $*\lambda_n \to *\lambda \neq 0$, and, since the definition of $*\lambda$ implies that $*\lambda\pi_\sigma\begin{pmatrix} I_A \\ \psi_\sigma(V)\end{pmatrix} \neq 0$, the result follows. □

Consider the set

$$\mathbf{W} = \{(d, w^H, R) \in \mathbf{D}_{NA} \times \mathrm{Int}\mathbf{X}^H \times \mathbf{R}\colon z(d, w^H) = 0, \text{ and } R(p) \subseteq V\}.$$

If $(d, w^H, R) \in \mathbf{W}$, each pair $((p,\lambda), V)$, $(\lambda, V) \in ([\lambda], V)$, are pseudo-equilibrium prices and attainable reallocations for the economy $(w^H, R) \in \mathrm{Int}\mathbf{X}^H \times \mathbf{R}$.

For $\sigma \in \mathbf{\Sigma}^A$, consider the function

$$G_\sigma\colon \mathbf{D}_{NA,\sigma} \times \mathrm{Int}\mathbf{X}^H \times \mathbf{R} \to \mathbb{R}^{(S+1)L-(1+S-A)+A(S-A)},$$

defined by

$$G_\sigma(d, w^H, R) = (z_\sigma(d, w^H), \phi_\sigma(V, R(p)) = 0,$$

where

$$\mathbf{D}_{NA,\sigma} = \mathbf{D}_{NA} \cap (\mathbf{D}_1 \times \mathbf{D}_{2,\sigma}),$$

and z_σ is the aggregate excess demand function for commodities other than $(1,1)$ and $(1,s)$, for $s \notin \sigma$, and $\phi_\sigma(V, M) = (-\psi_\sigma(V), I_{S-A})\pi_\sigma^{-1}R(p) = 0$.

Let

$$\mathbf{W}_\sigma = \{(d, w^H, R) \in \mathbf{D}_{NA,\sigma} \times \mathrm{Int}\mathbf{X}^H \times \mathbf{R}\colon G_\sigma(d, w^H, R) = 0\}.$$

Observe that the collection $\{\mathbf{D}_{NA,\sigma} \times \mathrm{Int}\mathbf{X}^H \times \mathbf{R}, \sigma \in \mathbf{\Sigma}^A\}$ is an open cover of $\{\mathbf{D}_{NA} \times \mathrm{Int}\mathbf{X}^H \times \mathbf{R}\}$ and, moreover, the sets $\{\mathbf{D}_{NA,\sigma} \times \mathrm{Int}\mathbf{X}^H \times \mathbf{R}\}$, $\sigma \in \mathbf{\Sigma}^A$, are connected. By Walras' law and the definition of the function ϕ_σ, $z(p, [\lambda], V, w^H, R) = 0$ and $R(p) \subseteq V$ if and only if $G_\sigma(p, [\lambda], V, w^H, R) = 0$ whenever $V \in \mathbf{\Gamma}_\sigma^{S,A}$. Moreover, if $(p, [\lambda], V) \in \mathbf{D}_{NA,\sigma_1} \cap \mathbf{D}_{NA,\sigma_2}$ for some $\sigma_1 \neq \sigma_2, \sigma_1, \sigma_2 \in \mathbf{\Sigma}^A$, and if $G_{\sigma_1}(p, [\lambda], V, e^H, R) = 0$ for some $(e^H, R) \in \mathrm{Int}\mathbf{X}^H \times R$, $G_{\sigma_2}(p, [\lambda], V, e^H, R) = 0$. Therefore, $\mathbf{W} = \bigcup_{\sigma \in \mathbf{\Sigma}^A} \mathbf{W}_\sigma$, and if G_σ is transversal to 0, $\sigma \in \mathbf{\Sigma}^A$, $\mathbf{W}$ is a smooth manifold.

Lemma 5: *For all* $(d, w^H) \in \mathbf{D}_{NA,\sigma} \times \mathrm{Int}\mathbf{X}^H$,

$$\mathrm{Rank}(D_{w^H} z_\sigma(d, w^{\mathbf{H}})) = L(S+1) - (S+1-A) \tag{i}$$

For $w^{*H} \in \mathrm{Int}\mathbf{X}^H$ *a Pareto optimal allocation, and* $d^* \in \mathbf{D}_{NA,\sigma}$ *such that* $z^h(d^*, w^{*h}) = 0$, *for* $h \in \mathbf{H}$,

$$\mathrm{Rank}(D_{p,[\lambda]} z_\sigma(d^*, w^{*H})) = L(S+1) - (S+1-A), \quad \text{(ii)}$$

and

$$D_V z^h(d^*, w^{*H}) = 0. \quad \text{(iii)}$$

Proof: For $([\lambda], V) \in \mathbf{D}_2$, there exists a matrix M of dimension $S \times A$ and a vector $q \in \mathbb{R}^A$, such that $[M] = V$ and $q = \lambda M$, for $\lambda \in [\lambda]$. Let the excess demand $\zeta^h(p, q, M, w^h)$ be the solution, when defined, to the optimization problem

$$\begin{aligned} &\max u^h(z + w^h) \\ &\text{s.t. } p_1 z_1 + qy = 0 \\ &\qquad p_2 \otimes z_2 = My. \end{aligned} \quad (4.1)$$

It follows that

$$z^h(p, [\lambda], V, w^h) = \zeta^h(p, q, M, w^h),$$

and

$$z(p, [\lambda], V, w^H) = \zeta(p, q, M, w^H) = \sum_{h \in H} \zeta^h(p, q, M, w^h),$$

whenever $V = [M]$ and $q = \lambda M$, for $\lambda \in [\lambda]$.

Therefore, for $(d, w^H, R) \in \mathbf{W}$ and for $V = [M]$ and $q = \lambda M$,

$$D_{w^H} \zeta(p, q, M, w^H) = D_{w^H} z(p, [\lambda], V, w^H), \quad (4.2)$$

and, since M is a matrix of dimension $S \times A$ of full column rank,

$$\mathrm{rank}(D_{(p,q)} \zeta_\sigma(p, q, M, w^H)) = \mathrm{rank}(D_{(p,[\lambda])} z_\sigma(p, [\lambda], V, w^H)). \quad (4.3)$$

The individual maximization problem (4.1) define a standard economy with incomplete financial markets. Therefore, claims (i) and (ii) of the Lemma follows directly from Geanakoplos and Polemarchakis (1986), (4.2) and (4.3).

Claim (iii) follows from Duffie and Shafer (1985). □

Lemma 6: *The set of pseudo-equilibria* W *is a smooth manifold of dimension* $H(S+1)L + ASL$ *without boundary.*

Proof: It suffices to show that 0 is a regular value of the map G_σ, for $\sigma \in \Sigma^A$. By the definition of the map G_σ it follows that

$$D_{(w^H,R)} G_\sigma(d, w^H, R) = \begin{pmatrix} D_{w^H} Z_\sigma(d, w^H), & 0 \\ 0, & D_R(\phi_\sigma(V, R(p))) \end{pmatrix}.$$

By lemma 4, the matrix $D_{w^H} z_\sigma(d, w^H)$ has rank $(S+1)L - (S+1-A)$ and, as shown in Duffie and Shafer (1985, fact 7), the matrix $D_R(\phi_\sigma(V, R(p)))$ has rank $A(S-A)$. □

Lemma 7: *The natural projection map*

$$proj_{\mathrm{Int}X^H \times R}\colon \mathbf{W} \to \mathrm{Int}\mathbf{X}^H \times \mathbf{R}$$

is proper.

Proof: Let $\mathbf{F} \subset \mathrm{Int}\mathbf{X}^H \times \mathbf{R}$ be a compact set and let $((d_n, w_n^H, R_n)$: $(d_n, w_n^H, R_n) \in (\mathrm{proj}_{\mathrm{Int}X^H \times R})^{-1}(\mathbf{F})$, $n = 1, \ldots)$ be an arbitrary sequence. We have to show that, without any loss of generality, $(d_n, w_n^H, R_n) \to (d, w^H, R) \in (\mathrm{proj}_{\mathrm{Int}X^H x R})^{-1}(\mathbf{F})$.

Since $p_n \in \mathbf{D}_1$, $V_n \in \mathbf{\Gamma}^{S,A}$ and $(w_n^H, R_n) \in \mathbf{F}$, and since $\mathbf{F}, \mathbf{\Gamma}^{S,A}$ and the closure of $\mathbf{D}_1$ are compact, without loss of generality, $(p_n, w_n^H, R_n) \to (p, w^H, R)$. Pick, $\sigma \in \mathbf{\Sigma}^A$ such that $V \in \mathbf{\Gamma}_\sigma^{S,A}$. Then, by the continuity of ϕ_σ and by the definition of the matrix $R(p)$, $\Phi_\sigma(V, R(p)) = 0$. Moreover, $z_\sigma(d_n, w_n^H) = 0$, for $n = 1, \ldots,$ and Lemma 4 imply that $d \in \mathbf{D}_{NA}$. Evidently, the continuity of the aggregate excess demand function, implies that $z(d, w^H) = 0$ or, equivalently, $(d, w^H, R) \in \pi^{-1}(\mathbf{F})$. □

Proof of the proposition:

It suffices to show that there is a regular value, (w^{*H}, R^*), of the map $\mathrm{proj}_{\mathrm{Int}XxR}$ such that

$$\#(\mathrm{proj}_{\mathrm{Int}X^H x R})^{-1}(w^{*H}, R^*) = 1.$$

Let w^{*H} be a Pareto optimal allocation and let (p^*, λ^*) be such that $(p_1^*, \lambda_1^* p_{(2,1)}^*, \ldots, \lambda_s^* p_{(2,s)}^*, \ldots)$ is the unique price, with $p^* \in \mathbf{D}_1$, supporting w^{*H} as an equilibrium of the complete market economy.

Let $R^* \in \mathbf{R}$, be such that $R^*(p^*)$ has the first A rows linearly independent and let V^* be the unique plane in $\mathbf{\Gamma}_{id}^{S,A}$ satisfying $[R^*(p^*)] = V$, where $\sigma = id$ is the identity permutation, $\pi_{id} = \mathbf{I}_S$. Evidently, given the Pareto optimality of w^{*H} and the fact that w^{*H} $h \in \mathbf{H}$, is affordable by individual h at all $d \in \mathbf{D}$, d^* is the unique element of $\mathbf{D}_{NA}$ supporting $w^{*\mathbf{H}}$ as a pseudo-equilibrium allocation.

To show that (w^{*H}, R^*) is regular value of the natural projection, consider the map G_{id}, then

$$D_{(p,[\lambda],V)} G_{id}(d^*, w^{*H}, R^*) = \begin{pmatrix} D_{(p,[\lambda])} z_{id}(d^*, w^{*H}), & D_V z_{id}(d^*, w^{*H}) \\ D_{(p,[\lambda])} \Phi_{id}(V^*, R^*(p^*)), & D_V x_{id}(V^*, R^*(p^*)) \end{pmatrix}.$$

By lemma 5, $D_{(p,[\lambda])} z_{id}(d^*, w^{*H})$ is invertible and $D_V z_{id}(d^*, w^{*\mathbf{H}}) = 0$, moreover Duffie and Shafer (1985, theorem 1), prove that the matrix $D_V(\phi_{id}(V^*) R^*(p^*))$ is non-singular. Therefore, the matrix $D_{(p,[\lambda],V)} G_{id}(d^*, w^{*H}, R^*)$ is invertible or, equivalently, (w^{*H}, R^*) is a regular value of the natural projection, $\mathrm{proj}_{\mathrm{Int}X^H x R}$. □

5. A large economy

We outline the argument for the existence of pseudo equilibria in a large economy. In particular, we restrict ourselves to perturbations which are finite dimensional and which can be used, subsequently, to establish the generic existence of competitive equilibria in a large economy, which we omit. Our formulation of excess demand and the derivation of its properties are a natural extension of the argument in Mas–Colell (1985) for the case of a complete market economy.

The set of individuals is the closed interval $[0, 1]$. An individual, $h \in [0, 1]$, is described by the characteristics (u^h, w^h), which satisfy Assumption 2.ii and 2.iii, while the consumption set coincides with the nonnegative orthant of the commodity space, as in 2.i. Let $\mathbf{U}$ be the space of utility functions that satisfy assumption 2.ii. We endow $\mathbf{U}$ with the C^2-uniform convergence topology, which makes $\mathbf{U}$ a complete metric space.

A utility-endowment assignment is a map $E:[0,1] \rightarrow \mathbf{U} \times \mathbb{R}^{L(S+1)}_{++}$ which associates to each individual, $h \in [0,1]$, the pair of a utility function and an endowment, $E^h = (u^h, w^h)$.

Assumption 3: The utility-endowment assignment map E is Borel-measurable and satisfies $E^h \in K$, for all $h \in [0,1]$, for some compact set $K \subset \mathbf{U} \times \mathbb{R}^{L(S+1)}_{++}$.

All allocation is an assignment $x:[0,1] \rightarrow \mathbb{R}^{L(S+1)}_{++}$, each coordinate of which is Lebesgue integrable over $[0,1]$, for which $\int_{[0,1]} x^h dh = \int_{[0,1]} w^h dh$.

The individual optimization problems and the modified individual optimization problems are as before, as are the definitions of a competitive equilibrium and a pseudo-equilibrium.

We focus on the existence of a pseudo-equilibrium for each matrix of asset payoffs, $\mathbf{R} \in \mathbf{R}$, which satisfies Assumption 1.

We fix the utility assignment and use finite-dimensional perturbations of the endowment assignment. In particular, we restrict ourselves to perturbations for which the endowments of all individuals are perturbed in exactly the same way. This is the essential difference between the argument here, for a large economy, and the preceding argument, where endowment perturbations are a fortiori finite-dimensional. Given the map E and vector $\delta \in \mathbb{R}^{L(S+1)}$, define E_δ to be the utility-endowment assignment defined by

$$E^h_\delta = (u^h, w^h + \delta), \; h \in [0,1].$$

Evidently, since $w^h \gg 0$, for all $h \in [0,1]$, there exists an open set $\mathbf{T} \subset \mathbb{R}^{L(S+1)}$, such that $w^h + \delta \gg 0$ for all $(h, \delta) \in [0,1] \times \mathbf{T}$. Moreover, for each $\delta \in \mathbf{T}$, the map E_δ satisfies assumption 3 for some compact set $K_\delta \subset \mathbf{U} \times \mathbb{R}^{L(S+1)}_{++}$.

As already discussed, the individual demand function $z^h(d, E^h_\delta)$ is a continuously differentiable map, $d \in \mathbf{D}_{NA}$ and $\delta \in \mathbf{T}$. Since Assumption 3 implies that $z^h(d, E^h_\delta)$ is bounded, the aggregate excess demand $z(d, E_\delta) = \int_{[0,1]} z^h(d, E^h_\delta) dh$ exists. Moreover, the aggregate excess demand function $z: \mathbf{D}_{NA} \times \mathbf{T} \rightarrow \mathbb{R}^{L(S+1)}$ is continuously differentiable, since Assumption 3 implies that $Dz(d, E^h_\delta) = \int_{[0,1]} Dz^h(d, E_\delta) dh$.

Since individual excess demand functions satisfy the boundary conditions, Lemma 4 holds also for each E that satisfies Assumption 3.

Let x^*: $[0,1] \rightarrow \mathbb{R}^{(S+1)L}_{++}$ be one of the complete market competitive equilibrium allocations associated with E. For $\alpha > 0$, for $\mu \in (-\alpha, 1+\alpha)$, and for $\delta \in \mathbb{R}^{L(S+1)}$, let $E_{\mu,\delta}$ be the utility-endowment assignment defined by $E^h_{\mu,\delta} = (u^h, \mu e^h + (1-\mu) x^{*h} + \delta)$, $h \in [0,1]$. The definition of x^* implies that there exists a scalar $\alpha > 0$, close enough to 0, an open set $\mathbf{N} \subset \mathbb{R}^{L(S+1)}$, and a compact set $\mathbf{K}^* \subset \mathbf{U} \times \mathbb{R}^{L(S+1)}_{++}$, such that $E^h_{\mu,\delta} \in \mathbf{K}^*$, for $\mu \in (-\alpha, 1+\alpha)$, $\delta \in \mathbf{N}$ and for $h \in [0,1]$. In other words, $E_{\mu,\delta}$ satisfies Assumption 3, for all μ and δ.

Lemma 4': For $(d, \mu, \delta) \in \mathbf{D}_{NA,\sigma} \times (-\alpha, 1+\alpha) \times \mathbf{N}$,

$$\text{Rank}(D_\delta z_\sigma(d, E_{\mu,\delta})) = L(S+1) - (S+1-A). \qquad \text{(i)}$$

For $\mu = 1$, $\delta = 0$ *and* $d^* \in \mathbf{D}_{NA,\sigma}$ *such that* $z^h(d^*, E^h_{1,0}) = 0$, *for* $h \in [0,1]$

$$\text{Rank}(D_{p,[\lambda]} z_\sigma(d^*, E_{1,0}) = L(S+1) - (S+1-A), \tag{ii}$$

and

$$D_V z(d^*, E_{1,0}) = 0. \tag{iii}$$

Proof: (ii) and (iii) are trivial, since, when $\mu = 1$ and $\delta = 0$, the map $E_{1,0}$ assigns a Pareto optimal utility-endowment profile. Hence, we only prove (i). Consider the perturbation of endowment assignments, $\Delta\delta$, defined as follows:

1) for $(1,\ell)$, $\ell > 1$, $\Delta_{(1,\ell)}\delta \in^{\mathbb{R}L(S+1)}$ is defined by $\Delta\delta_{(1,\ell)} \in \mathbb{R}$, $\Delta\delta_{(1,\ell')} = 0$, for $\ell' \neq \ell$, $\Delta\delta_{(1,1)} = -(p_{(1,\ell)}/p_{(1,1)})\Delta\delta_{(1,\ell)}$ and $\Delta\delta_{((2,s),\ell)} = 0$, for all ℓ and s;
2) for $((2,s),\ell)$, $\ell > 1$, and all s, $\Delta_{((2,s),\ell)}\delta \in \mathbb{R}^{L(S+1)}$ is defined by $\Delta\delta_{((2,s),\ell)} \in \mathbb{R}$, $\Delta\delta_{((2,s),\ell')} = 0$, for $\ell' \neq \ell$, $\Delta\delta_{((2,s),1)} = -(p_{((2,s),\ell)}/p_{((2,s),1)})\Delta\delta_{((2,s),\ell)}$ and $\Delta\delta_{((2,s'),\ell)} = \Delta\delta_{(1,\ell)} = 0$, for all ℓ and $s' \neq s$;
3) for $((2,s),1)$, all $s \in \sigma$, $\Delta_{((2,s),1)}\delta \in \mathbb{R}^{L(S+1)}$ is defined by $\Delta\delta_{((2,s),1)} \in \mathbb{R}$, $\Delta\delta_{(1,\ell)} = 0$, for all $\ell > 1$, $\Delta\delta_{((2,s'),\ell)} = 0$, for all $\ell > 1$ and all s', $\Delta\delta_{((2,s'),1)} = 0$, for $s' \in \sigma \backslash \{s\}$, $\Delta\delta_{((2,s),1)}$, $s \notin \sigma$, such that the vector $p_2 \otimes ((\Delta\delta_{((2,s),1)})_{s \in S}) \in V$, and, finally, $\Delta\delta_{(1,1)} = -\lambda(p_2 \otimes ((\Delta\delta_{((2,s),1)})_{s \in S})/p_{(1,1)})$, for $\lambda \in [\lambda]$.

It is easy to check that the perturbations defined by (1) and (2) not change, at $d \in D$, the wealth of any individual at any market. Moreover, the perturbations defined by (3) do not change, at $d \in D$, the present value of the endowment and produce second period wealth perturbations contained in V. Hence, $z^h(d, E^h_{\mu,\delta+\Delta}) = z^h(d, E^h_{\mu,\delta}) + \Delta\delta$. Let $D_{\delta\rightarrow}$ be the directional derivative taken in all possible directions described by the perturbations (1), (2) and (3). Then, denoting by $I_{L(S+1)-(S+1-A)}$ the identity matrix of dimension $L(S+1)-(S+1-A)$, $D_\delta \rightarrow z^h_\sigma(d, E^h_{\mu,\delta}) = I_{L(S+1)-(S+1-A)}$. Since, $D_\delta \rightarrow z(d, E_{\mu,\sigma}) = \int_{[0,1]} D_\delta \rightarrow z^h(d, E^h_{\mu,\delta})dh$, $\text{Rank}(D_\delta z_\sigma(d, E_{\mu,\delta})) = L(S+1) - (S+1-A)$. □

For $\sigma \in \mathbf{\Sigma}^A$, consider the function

$$G_\sigma\colon \mathbf{D}_{NA,\sigma} \times \mathbf{N} \times (-\alpha, 1+\alpha) \times \mathbf{R} \rightarrow \mathbb{R}^{(S+1)L-(1+S-A)+A(S-A)},$$

defined by

$$G_\sigma(d, \mu, \delta, R) = (z_\sigma(d, E_{\mu,\delta}), \phi_\sigma(V, R(p))) = 0.$$

Let

$$\mathbf{W}_\sigma = \{(d, \mu, \delta, R) \in \mathbf{D}_{NA,\sigma} \times \mathbf{N} \times (-\alpha, 1+\alpha) \times \mathbf{R}\colon G_\sigma(d, \mu, \delta, R) = 0\},$$

and

$$\mathbf{W} = \{(d, \mu, \delta, R) \in \mathbf{D}_{NA,\sigma} \times \mathbf{N} \times (-\alpha, 1+\alpha) \times \mathbf{R}\colon z(d, E_{\mu,\delta}) = 0, \text{ and } R(p) \subseteq V\}.$$

Observe that Lemma 4′ implies that 0 is a regular value of the map G_σ, for $\sigma \in \mathbf{\Sigma}^A$. This together with what was already explained in Section 4, imply that $\mathbf{W}_\sigma$, for $\sigma \in \mathbf{\Sigma}^A$, and $\mathbf{W}$ are smooth manifolds of dimension $(S+1)L + 1 + ASL$.

Let $\text{proj}_{(-\alpha,1+\alpha)\times N\times R}\colon \mathbf{W} \rightarrow (-\alpha, 1+\alpha) \times \mathbf{N} \times \mathbf{R}$ be the natural projection. The argument of Lemma 7 applies to the present context, and hence $\text{proj}_{(-\alpha,1+\alpha)\times N\times R}\colon \mathbf{W} \rightarrow (-\alpha, 1+\alpha) \times \mathbf{N} \times \mathbf{R}$ is proper.

Proof of the Proposition:

It suffices to show that $\mu = 1$, $\delta = 0$ and R^* is a regular value, of the map.

Let $E_{1,0}$ is a Pareto optimal profile. Let (p^*, λ^*) be such that $(p_1^*, \lambda_1^* p_{(2,1)}^*, \ldots, \lambda_s^* p_{(2,s)}^*, \ldots)$ is the unique price, with $p^* \in \mathbf{D}_1$, supporting the allocation x^* as an equilibrium of the complete market economy.

Let $R^* \in \mathbf{R}$, be such that $R^*(p^*)$ has the first A rows linearly independent, and let V^* be the unique plane in $\mathbf{\Gamma}_{id}^{S;A}$ satisfying $[R^*(p^*)] = V$, where $\sigma = id$ is the identity permutation, $\pi_{id} = I_S$.

Lemma 4′ and the same argument in the proof of the proposition conclude the argument. □

References

Cass, D.: Competitive equilibria with incomplete financial markets. Mimeo, 1984.

Duffie, D., Shafer W.: Equilibrium with incomplete markets, I: A basic model of generic existence. J. Math. Econ. **14**, 285–299 (1985)

Duffie, D., Shafer W.: Equilibrium with incomplete markets, II: A basic model of generic existence. J. Math. Econ. **15**, 199–216 (1986)

Geanakoplos, J., Polemarchakis, H. M.: Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete, pp. 65–90. In: Heller W., Starr, R., Starret D. (eds.) Uncertainty, information and Communication: Essays in honor of K.J. Arrow, Vol. III. Cambridge: Cambridge University Press 1986

Geanakoplos, J. D., Shafer, W.: Solving systems of simultaneous equations in economics. J. Math. Econ. **19**, 69–93 (1990)

Hirsch, M., Magill, M., Mas Colell, A.: A geometric approach to a class of equilibrium existence theorems. J. Math. Econ. **19**, 95–106 (1990)

Husseini, S. Y., Lasry, J.-M., Magill, M.: Existence of equilibrium with incomplete markets. J. Math. Econ. **19**, 39–67 (1990)

Mas-Colell, A.: Private communication, 1982

Mas-Colell, A.: The theory of general equilibrium: A differentiable approach. Cambridge: Cambridge University Press, 1985

Radner, R.: Existence of equilibrium plans, price and price expectations. Econometrica **40**, 289–303 (1972)

Shafer, W.: Work in progress, 1993