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Redistribution and welfare*

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The redistribution of welfare following a redistribution of endowments is arbitrary.

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1. Introduction

If prices are fixed, an individual whose endowment increases gains in utility. At a competitive equilibrium, however, prices depend on the distribution of endowments across individuals.

A redistribution of endowments leaves the aggregate endowment unchanged and, in an economy in which competitive allocations are optimal, the associated redistribution of utilities leaves aggregate welfare unchanged, up to normalization.

We show that the redistribution of endowments and the redistribution of utilities can be independently and arbitrarily specified.

In the framework of a regular economy, the equilibrium allocation is locally a continuously differentiable function of the endowments. Thus, infinitesimal variations avoid utility comparisons across equilibria.

The transfer paradox occurs when donors gain in utility while recipients lose. The phenomenon of advantageous transfers occurs when the individuals involved in the transfer all gain in utility. Both phenomena have been exhibited in the literature, in terms of examples. They follow from our general formulation.

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The argument cautions against the attempt to evaluate the welfare effects of redistributive policies without due attention to the consequent changes in equilibrium prices.

2. The economy

Economic activity occurs under a single period under uncertainty; equivalently, there is a complete market in contingent commodities.

Commodities are

$$\ell \in \boldsymbol{L} = \{0, 1, \ldots, L\},$$

a finite, nonempty set. A commodity bundle is

$$(\lambda_0, \lambda) \in \Lambda_0 \times \Lambda,$$

a column vector of dimension $L+1$.

Commodity prices are

$$(1, p) \in \{1\} \times \boldsymbol{P},$$

a strictly positive row vector of dimension $L+1$. Commodity $\ell = 0$ is numeraire and its price is normalized to equal one.

An individual is characterized by his consumption set, which coincides with the strictly positive orthant of the commodity space, $\boldsymbol{X} = \Lambda_{0+} \times \Lambda_{++}$, his utility function, $u: \boldsymbol{X} \to \boldsymbol{R}$, with domain the consumption set and his initial endowment, $e = (e_0, e) \in \boldsymbol{X}$, an element of the consumption set.

We consider variations in the endowment of the individual which involve only the numeraire commodity, for simplicity and with no loss of generality.

An individual expresses excess demand in order to maximize utility subject to the budget constraint. The behaviour of the individual is regular at (p, e_0) if and only if there exists an open neighborhood where

(i) a solution to the constrained optimization problems exists and is unique,

$$(z_0(p, e_0), z(p, e_0)),$$

and satisfies the budget constraint,

$$z_0 = -pz;$$

(ii) the excess demand function is continuously differentiable and satisfies the Slutzky equation,

$$D_p z = S - vz',$$

where v is the vector of income effects, $v=D_{e_0}z$, and S is the matrix of substitution effects, which is symmetric and negative semi-definite, and
(iii) the indirect utility function,

$$u(p,e_0),$$

is continuously differentiable and satisfies

$$\mathrm{d}u=-z'\mathrm{d}p+\mathrm{d}e_0;$$

without loss of generality we normalize the positive marginal utility of income to equal 1 at (p,e_0). If, in addition, the substitution matrix, S, is negative definite, the behavior of the individual is strongly regular. We argue in terms of the excess demand function for commodities other than the numeraire, no homogeneity or adding up constraints bind the substitution or income effects. The utility function need not be differentiable for individual behavior to be regular, in particular, the fixed coefficient utility functions which have been employed very fruitfully in the literature on the welfare consequences of redistribution do indeed yield regular individual behavior.

The Slutzky decomposition is not only necessary, but also sufficient for strong regularity: given a continuously differentiable function which, at given prices and endowment, satisfies the budget constraint and the Slutzky decomposition with a negative definite substitution matrix for prices other than the numeraire, there exists a twice continuously differentiable, strictly monotone and strictly quasi-concave utility function for which the exces demand function coincides in value and Jacobian with the given function at the given prices and endowment. The strongly regular behavior of an individual at a point is fully described by the triple of characteristics $\{z,v,S\}$, which can be arbitrarily specified up to the negative definiteness of the substitution matrix.

An economy is a collection of individuals,

$$h\in \boldsymbol{H}=\{1,\dots,H\},$$

a finite, nonempty set.

The allocation of initial endowments of the numeraire is a row vector,

$$e_0^H=(\dots,e_0^h,\dots)\in \times_{h\in H}\Lambda_{0,+}^h,$$

an element of the strictly positive orthant of dimension H.

The aggregate excess demand for commodities other than the numeraire is

$$z^a(p,e_0^H)=\sum_{h\in H} z^h(p,e_0^h).$$

A competitive equilibrium price vector, $p(e_0^H)$, is such that aggregate excess demand vanishes,

$$z(p(e_0^H), pe_0^H) = 0.$$

A pair, $(p(e_0^H), (e_0^H)$, of an allocation of initial endowments of the numeraire and of competitive equilibrium prices is a regular competitive equilibrium if and only if

(i) the behavior of each individual, $h \in \boldsymbol{H}$, is regular at $(p(e_0^H), e_0^h)$, and
(ii) the Jacobian of the aggregate excess demand function with respect to prices is of full rank,

$$|D_p z^a| \neq 0.$$

If the behavior of each individual is strongly regular at the regular competitive equilibrium $(p(e_0^H), e_0^H)$, the equilibrium is strongly regular. It follows that, at a strongly regular equilibrium, the matrix

$$S^a = \sum_{h \in \boldsymbol{H}} S^h$$

is of full rank,

$$|S^a| \neq 0.$$

A redistribution is a variation in the allocation in the initial endowment of the numeraire, $\mathrm{d}e_0^H \neq 0$, which leaves the aggregate endowment unchanged,

$$\sum_{h \in \boldsymbol{H}} \mathrm{d}e_0^h = 0.$$

Let $(p(e_0^H), e_0^H)$ be a regular competitive equilibrium. Totally differentiating and substituting, we obtain that

$$\mathrm{d}p = -\left(S^a - \sum_{h \in \boldsymbol{H}} v^h (z^h)^{\mathrm{T}}\right)^{-1} \left(\sum_{h \in \boldsymbol{H}} v^h \,\mathrm{d}e_0^h\right), \tag{1}$$

and hence

$$\mathrm{d}u^h = (z^h)' \left(S^a - \sum_{h' \in \boldsymbol{H}} v^{h'} (z^{h'})^{\mathrm{T}}\right)^{-1} \left(\sum_{h' \in \boldsymbol{H}} v^{h'} \,\mathrm{d}e_0^{h'}\right) + \mathrm{d}e_0^h, \quad h \in \boldsymbol{H}. \tag{2}$$

Eqs. (1) and (2) express explicitly in terms of the characteristics of individuals the variations in relative prices at equilibrium, $\mathrm{d}p$, and the variations in utilities,

$$\mathrm{d}u^H=(\mathrm{d}u^0,\mathrm{d}u^1,\dots \mathrm{d}u^H),$$

following the variations in endowments,

$$\mathrm{d}e^H=(\mathrm{d}e_0^0,\mathrm{d}e_0^1,\dots,\mathrm{d}e_0^H).$$

Evidently, the variations in utilities satisfy the adding up constraint

$$\sum_{h\in H}\mathrm{d}u^h=\sum_{h\in H}\mathrm{d}e_0^h=0,$$

which follows from the Pareto optimality of competitive equilibria and the normalization of the marginal utility of revenue of each individual to 1. The variation in utlity for each individual decomposes into a relative price effect and a revenue effect.

If, for example, all individual preferences display constant marginal utility for the numeraire commodity, income effects for other commodities vanish, $v^h=0$, for $h\in \boldsymbol{H}$. It follows that equilibrium prices are independent of variations in endowments, $\mathrm{d}p=0$, and the variations in utility coincides with the variations in endowments, $\mathrm{d}u^h=\mathrm{d}e_0^h$, for $h\in \boldsymbol{H}$. There is no reason, of course, to restrict attention to individual characteristics in this class.

Proposition. Let $a=(\dots,a^h,\dots)$ *and* $b=(\dots,b^h,\dots)$ *be column vectors of dimension* H *such that* $\sum_{h\in H}a^h=\sum_{h\in H}b^h=0$.

As long as

$$a^hb^h\neq 0 \quad \textit{for some} \quad h\in \boldsymbol{H},$$

there exists an economy with a strongly regular competitive equilibrium where

$$\mathrm{d}e^H=a \quad \textit{and} \quad \mathrm{d}u^H=b.$$

Proof. We consider first the case of an economy with two goods, $L=1$.

It suffices to show that there exist $\{z^h:h\in \boldsymbol{H}'\}$, $\{w^h:h\in \boldsymbol{H}'\}$ and $S^a<0$, where $\boldsymbol{H}'=\boldsymbol{H}/\{0\}$, such that

$$b^h=\frac{z^h(\sum_{h\in H'}w^{h'}a^{h'})}{S^a-\sum_{h'\in H'}w^{h'}z^{h'}}+a^h,\quad h\in \boldsymbol{H}'. \tag{3}$$

Eq. (3) is obtained from (1) by substituting

$$z^0=-\sum_{h'\in H'}z^{h'} \quad \text{and} \quad w^{h'}=v^{h'}-v^0, \qquad h'\in \boldsymbol{H}'.$$

By rearranging terms, we obtain a system of H linear equations in the variables $\{z^h : h \in \boldsymbol{H}'\}$. To guarantee that a solution exists, it suffices to show that the $H \times H$ matrix of coefficients is of full rank. This we guarantee by appropriate choice of the variables $\{w^h : h \in \boldsymbol{H}'\}$.

The coefficients are

$$c^{h,h'} = \begin{cases} (b^h - a^h)w^{h'}, & h' \neq h, \\ (b^h - a^h)w^h + \sum_{h'' \in \boldsymbol{H}'} w^{h''} a^{h''} & h, h' \in \boldsymbol{H}'. \end{cases}$$

Without loss of generality,

$$a^H b^H \neq 0.$$

Let

$$w^{h'} = \begin{cases} 0, & h' \neq H, \\ x, & h' = H, \end{cases} \quad h' \in \boldsymbol{H}'.$$

The determinant of the matrix of coefficients does not vanish as long as $(b^H - a^H)(a^H)^{H-1} + a^H = b^H(a^H)^H \neq 0$. But this is implied by $a^H b^H \neq 0$, and hence the matrix of coefficients is invertible, as desired.

In order to complete the argument, we have to consider commodities with more than two commodities. But this is straightforward, it suffices to set $z_\ell^h = v_\ell^h = 0$, for $\ell \in \boldsymbol{L}/\{0,1\}$ and $h \in \boldsymbol{H}$. □

Remark 1. The requirement that, for some $h \in \boldsymbol{H}$, $b^h = \mathrm{d}u^h \neq 0$ as well as $a^h = \mathrm{d}e_0^h \neq 0$ is rather weak; we do not know whether it is necessary. In particular, it allows for the paradox of redistribution, in which, for $h' \neq h'' \in \boldsymbol{H}'$, $\mathrm{d}u^{h'}\, \mathrm{d}e^{h'} < 0$ and $\mathrm{d}u^{h''} \mathrm{d}e^{h''} < 0$, as well as for advantageous reallocations, in which, for $\boldsymbol{H}'' \subseteq \boldsymbol{H}'$, $\mathrm{d}u^{h'} > 0$ as well as $\mathrm{d}e^{h'} > 0$, for all $h' \in \boldsymbol{H}''$.[1]

[1]The possibility of the transfer paradox was first suggested by Leontief (1936). Subsequently, Samuelson (1947) claimed that the paradox cannot obtain a a stable equilibrium. That Samuelson's claim is valid only in an economy with only two individuals is due to Chichilnisky (1980): she gave an example of an economy with three individuals in which the paradox obtains at a unique and stable equilibrium. The argument of Chichilnisky was further elaborated on in Chichilnisky (1983), Geanakoplos and Heal (1983) and Polemarchakis (1983). See also Brecher and Bhagwati (1981), Bhagwati et al. (1983a), Dixit (1983), Jones (1982) and Yano (1983). The distribution between the local and global versions of the paradox for economies with possibly multiple equilibria was first drawn by Balasko (1978) who developed the argument in the framework of a regular economy. Safre (1983) analyzed the relation between the occurrence of the paradox and the multiplicity of equilibria. Bhagwati et al. (1983b) showed that if the Walrasian tâtonnement is relied upon as a global adjustment process, with two individuals, the global paradox fails even at stable equilibria. The possibility of advantageous redistribution in an economy with three individuals was first shown in an example by Gale (1974). Guesnerie and Laffont (1978) developed a general argument.

Remark 2. In the literature on the paradox of redistribution it was further required that the equilibrium be locally asymptotically stable, that the dynamical system

$$\dot{p} = z^a$$

be locally asymptotically stable, in particular that the Jacobian $D_p z^a$, at the equilibrium, have no eigenvalue with positive real part. The proposition can yield a locally asymptotically stable equilibrium as long as the variation in utility of at least one individual is not arbitrary, as long as $b^H = \mathrm{d}u^H$ is not a priori specified. The result can then be strengthened further by requiring that the equilibrium be globally asymptotically stable and unique.[2]

Remark 3. Variations of endowments which are not redistributions do not preserve the aggregate endowment, $\sum_{h\in \boldsymbol{H}} \mathrm{d}e^h \neq 0$. The Pareto optimality of competitive allocations together with the normalization of the marginal utility of income to 1 imply that $\sum_{h\in \boldsymbol{H}} \mathrm{d}u^h = \sum_{h\in \boldsymbol{H}} \mathrm{d}e^H$. The argument extends immediately to this more general case. For example disadvantageous growth occurs when $\mathrm{d}e^h \geqq 0$, for all $h \in \boldsymbol{H}$, and $\mathrm{d}e^{\hat{h}} < 0$, for some $\hat{h} \in \boldsymbol{H}$.[3]

3. Conclusion

The informational required to determine the redistribution of endowments which yields a particular redistribution of utilities is possibly prohibitive. It would have been desirable to establish soime general relation between redistributions of endowments and welfare. This is not possible.

[2]The argument changes in economies with incomplete markets [Forsythe and McCubbins (1983)], in the presence of distortions [Bhagwati et al. (1983b)] or economies of overlapping generations [Donsimoni and Polemarchakis (1989) and Galor and Polemarchakis (1987)].

[3]The possibility of disadvantageous growth was first noted for economies with production by Bhagwati (1958) and Johnson (1955). Aumann and Peleg (1974) simplified the argument for an exchange economy with two individuals. Bhagwati et al. (1982) provided a synthesis.

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