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Competitive equilibria without free disposal or nonsatiation*

H.M. Polemarchakis

CORE, Catholic University of Louvain, Louvain-la-Neuve, Belgium

P. Siconolfi

Columbia University, New York, NY, USA

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When nonsatiation or free disposal fail, competitive equilibria are either weak, when the individual budget constraint is imposed with equality, or strong, when the budget constraint is imposed as a weak inequality. Weak competitive equilibria exist under standard assumptions since the aggregate demand function satisfies Walras' law. This is the case even though prices cannot be restricted to a contractible domain. Strong competitive equilibria may fail to exist under standard assumptions. A boundary condition on the aggregate demand function guarantees, nevertheless, the existence of strong competitive equilibria.

1. Introduction

In the argument for the existence of competitive equilibria [Arrow and Debreu (1954)], free disposal guarantees that competitive equilibrium prices, whenever they exist, are nonnegative. Nonsatiation guarantees that the value of a consumption bundle chosen by an individual does not fall below the value of his initial endowment, and thus Walras' law is satisfied by the aggregate demand function. In the argument for the optimality of competitive equilibrium allocations [Arrow (1951) and Debreu (1951)] nonsatiation guarantees that an individual is not indifferent between his equilibrium consumption bundle and a bundle which costs less, and this in turn implies that an allocation which Pareto dominates the equilibrium allocation is not feasible.

Correspondence to: Heraklis Polemarchakis, CORE, 37 Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium.

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In an exchange economy, free disposal is captured by the assumption that individual utility functions are weakly monotonic: starting from a feasible consumption bundle, a bundle with at least as much of each commodity remains feasible and yields at least as high a level of utility. If free disposal fails, an individual consumption set may be bounded or a utility function may be decreasing in some directions, even all. In the latter case, the failure of free disposal essentially implies the failure of nonsatiation.

Free disposal as well as nonsatiation may fail in an incomplete asset market.

Consider an elementary exchange economy under uncertainty. Assets are traded in the first period. In the second period, one of finitely many states of nature is realized and assets pay off. For simplicity, only a single consumption good is available at each state of nature and thus commodity spot markets are inactive. Suppose that no consumption occurs in the first, asset trading period. Further, the asset market is incomplete and, in particular, any nonzero portfolio has a negative payoff in some state of nature. This, evidently, cannot be the case if the asset market is complete. That no consumption occurs in the first period captures the idea that in a multiperiod economy assets are traded more frequently than consumption occurs. With the consumption set of an individual in the second period closed and bounded from below, the set of feasible portfolios for the individual in the first period is compact. Thus, in the asset market, free disposal fails. The indirect utility function over portfolios attains a maximum, and thus nonsatiation fails as well.

The budget constraint in the individual optimization problem can be imposed as an equality or as an inequality. When nonsatiation fails, the two are not equivalent. As long as the mechanism of transactions is left unspecified, a conclusive argument for one or the other is not possible. We consider both weak equilibria, in which the budget constraint is imposed as an equality, and strong equilibria, in which the budget constraint is imposed as an inequality.

Weak competitive equilibria exist under standard assumptions. The argument is essentially that of Hart and Kuhn (1975). We simply observe that nonsatiation, which we allow to fail along with free disposal and which they do not, does not interfere with the argument. The proof allows for negative prices and considers the unit sphere as the normalized price domain. It is based on the topology of spheres and the observation that the aggregate demand function, which satisfies Walras' law, is symmetric and coincides at any price vector and its negative. Weak competitive equilibrium allocations need not be Pareto optimal. Nevertheless, under standard assumptions, a Pareto optimal allocation can be supported as a weak competitive equilibrium allocation.

Strong competitive equilibria may fail to exist. Indeed they do not exist whenever it is not feasible for all individuals to attain their satiation points while the cone in the price domain in which the value of all individual satiation points has nonnegative value shrinks to the origin; this is the cone in the price domain in which the aggregate demand satisfies Walras' law even though the individual budget constraint is imposed as a weak inequality. We give a sufficient condition for the existence of strong equilibria. This condition restricts the behavior of aggregate demand at the boundary of the cone in the price domain where Walras' law is satisfied and we refer to it as the 'boundary condition'. In Bergstrom (1976), the existence of strong equilibria without free disposal follows from a restriction that excludes economies in which an individual may attain satiation at a feasible allocation. In Nielsen (1990), this yields strong equilibria in asset markets in which, as we pointed out earlier, free disposal naturally fails. Our boundary condition restricts the behavior of aggregate demand on the boundary only of the price domain and not over the entire set of feasible allocations, and, as we show, it is strictly weaker. Strong competitive equilibrium allocations are optimal; also, an optimal allocation can be supported as a strong competitive equilibrium allocation.

2. The economy

The objects of exchange, commodities or assets, are

$$a \in \mathscr{A} = \{1, \ldots, A\}, \quad A < \infty,$$

and a bundle, a commodity bundle or a portfolio, is a column vector

$$y = (\ldots, y_a, \ldots) \in \mathscr{R}^A.$$

Individuals are

$$h \in \mathscr{H} = \{1, \ldots, H\}, \quad H < \infty.$$

An individual is characterized by the feasible set

$$\mathscr{Y}^h \subseteq \mathscr{R}^A, \quad h \in \mathscr{H},$$

and the objective function, the direct or indirect utility function,

$$u^h : \mathscr{Y}^h \to \mathscr{R}, \quad h \in \mathscr{H},$$

with domain the feasible set.

An economy is thus a collection

$$\mathscr{E}=\{(\mathscr{Y}^h,u^h):h\in\mathscr{H}\}.$$

Assumption 1. For $h\in\mathscr{H}$, the feasible set, $\mathscr{Y}^h$, is compact and convex and contains the origin as an interior point, $0\in\operatorname{Int}\mathscr{Y}^h$.

Assumption 2. For $h\in\mathscr{H}$, the objective function, u^h, is continuous and strictly quasi-concave.

Hence, for each individual, there is a unique global satiation bundle,

$$s^h\in\mathscr{Y}^h,\quad h\in\mathscr{H}.$$

An allocation is an array

$$y^{\mathscr{H}}=\{y^h:h\in\mathscr{H}\},$$

such that

$$y^h\in\mathscr{Y}^h,\quad h\in\mathscr{H}.$$

An allocation is feasible if and only if

$$\sum_{h\in\mathscr{H}}y^h=0.$$

That the feasible set of each individual be compact is not important. It suffices to suppose that the feasible set of each individual compatible with a feasible allocation be compact. That 0 be an interior point of the feasible set of each individual serves to avoid discontinuities in the demand behavior of individuals as we show later (Lemma 1), which is well known. That the objective function of each individual be strictly quasi-concave serves to imply that the demand of individuals is single valued, as we argue later, which is well known, and thus simplifies the argument. Nevertheless it is restrictive and should be relaxed.

An allocation, $y'^{\mathscr{H}}$, Pareto dominates another, $y^{\mathscr{H}}$, if and only if

$$u^h(y'^h)\geqq u^h(y^h),\quad h\in\mathscr{H},$$

with some strict inequality.

Definition 1. An allocation is *Pareto optimal* if and only if it is feasible and no feasible allocation Pareto dominates it.

Prices are a row vector,

$$q=(\ldots,q_a,\ldots)\in\mathscr{R}^A.$$

The budget sets of individuals can be defined with the budget constraint imposed as a weak inequality,

$$\mathscr{B}^h(q)=\{y\in\mathscr{Y}^h: qy\leqq 0\},\quad h\in\mathscr{H},$$

or as an equality,

$$\mathscr{B}^h(q)=\{y\in\mathscr{Y}^h: qy=0\},\quad h\in\mathscr{H}.$$

The optimization problem of an individual at asset prices q is to

$$\max u^h(y),$$

$$\text{s.t.}\quad y\in\mathscr{B}^h(q),\quad h\in\mathscr{H}.$$

A solution to this problem,

$$y^h(q),\quad h\in\mathscr{H},$$

exists and is unique. This follows from the compactness and convexity of the feasible set, Assumption 1, the linearity of the budget constraint and the continuity and strict quasi-concavity of the objective function, Assumption 2.

The individual demand function,

$$y^h:\mathscr{R}^A\to\mathscr{R}^A,\quad h\in\mathscr{H},$$

is thus well defined; also the aggregate demand function,

$$y=\sum_{h\in\mathscr{H}} y^h:\mathscr{R}^A\to\mathscr{R}^A.$$

Lemma 1. Under Assumptions 1 and 2, for $h\in\mathscr{H}$, the individual demand function y^h, is homogeneous of degree 0, and it is continuous at $\bar{q}\neq 0$. The aggregate demand function inherits these properties.

Proof. Homogeneity of degree 0 of the individual demand function follows from the homogeneity of degree zero of the budget constraint.

In order to show that the individual demand function, y^h, is continuous at $q\neq 0$, we consider first the case in which the budget contraint is imposed as an equality. Let $(q_n: n=1,\ldots)$ be a sequence of prices with $\lim_{n\to\infty} q_n=q$, and let $(y_n : y_n = y^h(q_n), n=1,\ldots)$, with $\lim_{n\to\infty} y_n = y$ be the associated bundles demanded by the individual; evidently, $qy=0$. Suppose $q\neq 0$. We argue by contradiction. Suppose $y \neq y^h(q)$. It follows then, from the strict quasi-concavity of the utility function, that $u^h(y^h(q)) > u^h(y)$. Since $0 \in \operatorname{Int}\mathscr{Y}^h$ and $y^h(q) \in \mathscr{Y}^h$, $y_\lambda = \lambda y^h(q) \in \operatorname{Int}\mathscr{Y}^h$, for $0<\lambda<1$. By the continuity of the utility function, there exists $\bar{\lambda}<1$ such that $u^h(y_\lambda) > u^h(y)$, for $\lambda \geqq \bar{\lambda}$. Thus, we may choose $y' \in \operatorname{Int}\mathscr{Y}^h$ such that $qy'=0$ and $u^h(y') > u^h(y)$. Since $q \neq 0$, there exists $\bar{n}$ such that $q_n \neq 0$ for $n=\bar{n},\ldots$. Consider the sequence $(y_n : y'_n = y' - (q_n y'/\|q_n\|)q_n, n=\bar{n},\ldots)$. Evidently, $q_n y'_n = 0$, for $n=\bar{n},\ldots$. Since $\lim_{n\to\infty} q_n = q$, $\lim_{n\to\infty} y'_n = y'$, and since $y' \in \operatorname{Int}\mathscr{Y}^h$, there exists $\bar{\bar{n}} \geqq \bar{n}$ such that $u^h(y'_n) > u^h(y_n)$, for $n=\bar{\bar{n}},\ldots$. But this contradicts the optimality of y_n for the individual at prices q_n, for $n=\bar{\bar{n}},\ldots$.

In order to complete the argument, we must consider the case in which the budget constraint is imposed as a weak inequality. But the argument is essentially the same as the argument above.

The aggregate demand function evidently inherits these properties. □

Definition 2. A competitive equilibrium is a pair,

$$(q^*, y^{*\mathscr{H}}),$$

of prices and an allocation, such that

$$y^{*h} = y^h(q^*), \quad h \in \mathscr{H}.$$

A competitive equilibrium is weak if the budget constraint is imposed as an equality and *strong* if the budget constraint is imposed as a weak inequality.

Definition 3. A feasible allocation, $\bar{y}^{\mathscr{H}}$, can be supported as a competition equilibrium allocation if and only if $\hat{y}^{\mathscr{H}}=0$ is a competitive equilibrium allocation for the economy

$$\mathscr{E}_{\bar{y}^{\mathscr{H}}} = \{(\mathscr{Y}^h_{\bar{y}^h}, u^h_{\bar{y}^h}) : h \in \mathscr{H}\}$$

obtained from the economy by defining $\mathscr{Y}^h_{\bar{y}^h} = \{y \in \mathscr{R}^A : (y+\bar{y}^h) \in \mathscr{Y}^h\}$ and $u^h_{\bar{y}^h} = u^h(y+\bar{y}^h)$, for $h \in \mathscr{H}$.

In the following sections we consider whether competitive equilibria exist, whether competitive equilibrium allocations are optimal, and whether optimal allocations can be supported as competitive equilibrium allocations.

3. Weak equilibria

In this section we consider weak competitive equilibria, in which the individual budget constraint is imposed as an equality.

Proposition 1. Under Assumptions 1 and 2, weak competitive equilibria exist.

Proof. Observe that the aggregate demand function satisfies Walras' law, $qy(q)=0$, since individual budget constraints are satisfied with equality.

Let $\mathcal{S}^{A-1}=\{q:\|q\|=1\}\subset\mathcal{R}^A$ be the unit sphere in the price domain.

Note that prices cannot he restricted to a contractible domain.

In order to show that competitive equilibrium asset prices exist, we distinguish between two cases: The case in which $(A-1)$ is even and the case in which $(A-1)$ is odd [Milnor (1965)].

If $(A-1)$ is even, the Euler characteristic of the sphere $\mathcal{S}^{A-1}$ does not vanish and hence the sphere does not allow for a nonvanishing vector field. From Walras' law it follows that the aggregate asset demand function defines a vector field and hence $y(q^*)=0$, for some $q^*\in\mathcal{S}^{A-1}$. Thus q^* are competitive equilibrium prices.

If $(A-1)$ is odd, we argue by contradiction. Suppose $y(q)\neq 0$ everywhere on $\mathcal{S}^{A-1}$. It follows that the function

$$f=\frac{y}{\|y\|}:\mathcal{S}^{A-1}\to\mathcal{S}^{A-1}$$

is well defined. Since $(A-1)$ is odd, there exists $\bar{q}\in\mathcal{S}^{A-1}$ such that $f(\bar{q})=-f(\bar{q})$. But that is incompatible with the homogeneity of degree zero that the function f inherits from the aggregate demand function. Thus, competitive equilibrium prices exist. □

Consider an economy with a single, representative individual, $\mathcal{H}=\{1\}$. Suppose further that, in addition to Assumptions 1 and 2 which are satisfied, the individual's utility function is differentiable and $du^1(0)=0$. This does not imply that $q=0$ are the only weak equilibrium prices. Indeed, any prices q are weak competitive equilibrium prices in this case, in particular $q\in\mathcal{S}^{A-1}$ as in the argument in Proposition 1.

The allocation at a weak competitive equilibrium, $(q^*, y^{*\mathcal{H}})$, need not be optimal. This is evident since, with the budget constraint imposed as an equality, it may well be that a Pareto dominating allocation, $y'^{\mathcal{H}}$, is such that $q^*y'^{h_1}<0$, for some $h_1\in\mathcal{H}$, while $q^*y'^{h_2}>0$, for some $h_2\in\mathcal{H}$. For the first inequality to be satisfied, it is necessary that $q^*s^{h_1}<0$, while for the second that $q^*s^{h_2}>0$, which may indeed occur at weak competitive equilibrium prices. The standard argument for the optimality of competitive equilibrium allocations breaks down. This is also the case in economies in which the consumption sets of individuals are not compact and thus do not contain satiation points. Examples of weak competitive equilibrium allocations which fail to be optimal are straightforward.

The failure of free disposal and nonsatiation do not interfer with the possibility of supporting Pareto optimal allocations as weak competitive equilibrium allocations.

Corollary 1. Let $\bar{y}^{\mathscr{H}}$ be a Pareto optimal allocation. Under Assumptions 1 and 2, if $\bar{y}^h \in \operatorname{Int} \mathscr{Y}^h$, for $h \in \mathscr{H}$, $\bar{y}^{\mathscr{H}}$ can be supported as a weak competitive equilibrium allocation.

Proof. Since $\bar{y}^h \in \operatorname{Int} \mathscr{Y}^h$, for $h \in \mathscr{H}$, the economy $\mathscr{E}_{\bar{y}^{\mathscr{H}}}$ satisfies Assumptions 1 and 2. Let $(\hat{q}, \hat{y}^{\mathscr{H}})$ be a weak competitive equilibrium for the economy $\mathscr{E}_{\bar{y}^{\mathscr{H}}}$ which exists from Proposition 1. Since $u^h_{\bar{y}^h}(\hat{y}^h) \geqq u^h_{\bar{y}^h}(0)$, $u^h(\bar{y}^h + \hat{y}^h) \geqq u^h(\bar{y}^h)$ for $h \in \mathscr{H}$. Since the allocation $\bar{y}^{\mathscr{H}}$ is Pareto optimal, $u^h(\bar{y}^h + \hat{y}^h) = u^h(\bar{y}^h)$ for $h \in \mathscr{H}$. If for some $\bar{h} \in \mathscr{H}$, $\hat{y}^{\bar{h}} \neq 0$, this contradicts the optimization of individual $\bar{h}$ in the equilibrium $(\hat{q}, \hat{y}^{\mathscr{H}})$ for the economy $\mathscr{E}_{\bar{y}^{\mathscr{H}}}$, since, from the strict quasi-concavity of the objective function of individual $\bar{h}$, $u^{\bar{h}}_{\bar{y}^{\mathscr{H}}}(\lambda \hat{y}^{\bar{h}}) > u^{\bar{h}}_{\bar{y}^{\mathscr{H}}}(\hat{y}^{\bar{h}})$ for $0 < \lambda < 1$, while $\lambda \hat{y}^{\bar{h}}$ is feasible and satisfies the budget constraint. Thus $\hat{y}^{\mathscr{H}} = 0$ or, equivalently, $\bar{y}^{\mathscr{H}}$ can be supported as a competitive equilibrium allocation. □

Optimal allocations that fail to satisfy the interiority assumption may fail to be supportable as weak competitive equilibrium allocations. This is the case even in economics with free disposal and nonsatiation.

4. Strong equilibria

In this section we consider strong competitive equilibria, in which the individual budget constraint is imposed as an inequality.

Consider the subset of the price domain

$$\mathscr{Q} = \{q : q's^h \geqq 0,\ h \in \mathscr{H}\}.$$

Note that for $q \in \mathscr{Q}, qy^h(q) = 0$, for $h \in \mathscr{H}$, and hence

$$qy(q) = 0, \quad \text{for} \quad q \in \mathscr{Q}.$$

Thus $\mathscr{Q}$ is the cone in the price domain where the aggregate demand function satisfies Walras' law. Also, $-s^h \notin \mathscr{Q}$, for $h \in \mathscr{H}$, and hence $\mathscr{Q} \neq \mathscr{R}^A$ whenever $s^h \neq 0$, for some $h \in \mathscr{H}$.

Proposition 2. Under Assumptions 1 and 2, no strong competitive equilibria exist when

(*i*) $\sum_{h\in\mathscr{H}} s^h \neq 0$,

(*ii*) $\mathscr{Q}=\{0\}$.

Proof. First, observe that it follows from (i) that $q=0$ are not equilibrium prices.

In order to show that no strong equilibria exist, we argue by contradiction. Suppose $\bar{q}\neq 0$ are equilibrium prices. Let $\mathscr{H}(\bar{q})=\{h:\bar{q}'s^h<0\}$ and note that it follows from (ii) that $q\notin\mathscr{Q}$ and hence $\mathscr{H}(\bar{q})\neq\phi$. Evidently, $y^h(\bar{q})=s^h$, for $h\in\mathscr{H}(\bar{q})$, while $\bar{q}'y^h(\bar{q})=0$ for $h\in\mathscr{H}/\mathscr{H}(\bar{q})$. Hence

$$\bar{q}y(\bar{q})=\bar{q}\left(\sum_{h\in\mathscr{H}(\bar{q})} y^h(\bar{q})+\sum_{h\in\mathscr{H}/\mathscr{H}(\bar{q})} y^h(\bar{q})\right)$$

$$=\bar{q}\left(\sum_{h\in\mathscr{H}(\bar{q})} s^h+\sum_{h\in\mathscr{H}/\mathscr{H}(\bar{q})} y^h(\bar{q})\right)<0.$$

But this contradicts that $y(\bar{q})=0$ since $\mathscr{H}(\bar{q})\neq\phi$. □

Evidently, a strong equilibrium is a weak equilibrium. It suffices to observe that $\hat{q}\notin\mathscr{Q}$ can not be strong equilibrium prices since $\hat{q}y(\hat{q})<0$, which is incompatible with market clearing, $y(\hat{q})=0$. But for $q\in\mathscr{Q}, qy^h(q)=0$ for $h\in\mathscr{H}$ and hence if $\bar{q}$ are strong equilibrium prices they are weak equilibrium prices as well.

Even if the cone $\mathscr{Q}$ in the price domain does not vanish, $\mathscr{Q}\neq\{0\}$, strong competitive equilibria may fail to exist.

Example 1. Consider an elementary exchange economy under uncertainty. Assets, $a\in\mathscr{A}=\{1,2\}$, are traded in the first period. In the second period, one of four states of nature is realized, assets pay off and consumption occurs; only a single consumption good is available at each state of nature and thus commodity spot markets are inactive.

The matrix of asset payoffs is

$$R=\begin{pmatrix}1 & 0\\ -1 & 0\\ 0 & -1\\ 0 & 1\end{pmatrix}.$$

Individuals, $h \in \mathscr{H} = \{1,2\}$, are characterized by their utility functions over nonnegative state contingent consumption and by their endowments,

$$u^1 = 3\ln x_1 + 4\ln x_2 + 6\ln x_3 + 7\ln x_4,$$

$$w^1 = (1,1,1,1),$$

$$u^2 = 3\ln x_1 + 2\ln x_2 + 3\ln x_3 + 2\ln x_4,$$

$$w^2 = (1,1,1,1).$$

The feasible sets and objective functions, indirect utility functions, of individuals over portfolios are

$$\mathscr{Y}^1 = \{y = (y_1, y_2) : |y_1| \leqq 1, |y_2| \leqq 1\},$$

$$u^1 = 3\ln(1+y_1) + 4\ln(1-y_1) + 6\ln(1-y_2) + 7\ln(1+y_2),$$

$$\mathscr{Y}^2 = \{y = (y_1, y_2) : |y_1| \leqq 1, |y_2| \leqq 1\},$$

$$u^2 = 3\ln(1+y_1) + 2\ln(1-y_1) + 3\ln(1-y_2) + 2\ln(1+y_2).$$

The satiation points of individuals are

$$s^1 = \left(-\frac{1}{7}, \frac{1}{13}\right),$$

$$s^2 = \left(\frac{1}{5}, -\frac{1}{5}\right).$$

The individual budget constraint in the asset market, imposed as a weak inequality, is

$$q_1 y_1 + q_2 y_2 \leqq 0.$$

Note that

$$\mathscr{Q} = \{q : q s^h \geqq 0,\ h \in \mathscr{H}\} = \left\{q : q_1 \geqq \frac{7}{13} q_2,\ q_2 \leqq 0\right\} \neq \{0\}.$$

Even stronger, Int $\mathscr{Q} \neq \phi$.

Observe first that $q=(q_1,q_2)$ with $q_1=0$ or $q_2=0$ cannot be competitive equilibrium prices. This follows from the separability of the objective function of each individual across portfolios and the fact that $\sum_{h\in\mathscr{H}} s_1^h \neq 0$ and also $\sum_{h\in\mathscr{H}} s_2^h \neq 0$.

Note further that a competitive equilibrium, $(q^*, y^{*\mathscr{H}})$, since from the individual budget constraint $q^* y^{*h} \leqq 0$, while from the feasibility constraint $\sum_{h\in\mathscr{H}} y^{*h}=0, q^* y^{*h}=0$, for $h\in\mathscr{H}$.

In order to show that competitive equilibria do not exist, we may restrict our attention to asset prices $q^*=(q_1^*,q_2^*)$ with $q_1^*\neq 0$ and $q_2^*\neq 0$. For $\rho^*=q_1^*/q_2^*, y_1^{*1}=-y_1^{*2}$ and $y_2^{*1}=-y_2^{*2}=-\rho^* y_1^*$, also $|y_1^{*1}|\leqq 1$ and $|\rho^* y_1^{*1}|\leqq 1$.

We argue by contradiction. From the first-order conditions for individual optimization, it follows that

$$\frac{3}{1+y_1^{*1}}-\frac{4}{1-y_1^{*1}}+\frac{6\rho^*}{1+\rho^* y_1^{*1}}-\frac{7\rho^*}{1-\rho^* y_1^{*1}}=0,$$

$$\frac{3}{1-y_1^{*1}}-\frac{2}{1+y_1^{*1}}+\frac{3\rho^*}{1-\rho^* y_1^{*1}}-\frac{2\rho^*}{1+\rho^* y_1^{*1}}=0.$$

Summing across individuals, we obtain as a necessary condition for equilibrium that

$$y_1^{*1}\left(\frac{1}{1-(y_1^{*1})^2}+\frac{4(\rho^*)^2}{1-(\rho^* y_1^{*1})^2}\right)=0.$$

The only solution to this equation is $y_1^{*1}=0$ and thus $\rho^*=-1$. However, $\rho^*=-1$ implies that either $q_1^*=-q_2^*<0$ or $q_1^*=-q_2^*>0$. Subject to the weak inequality budget constraint, in the first case individual $h=2$ chooses his satiation portfolio and in the second individual $h=1$ does. Since $s^1\neq 0$ and $s^2\neq 0$, this is a contradiction.

That the nonexistence of competitive equilibria is robust to perturbations in the structure of the economy, the endowments of individuals, for example, is evident.

Further restrictions on the aggregate demand function are required to guarantee the existence of competitive equilibria.

For $q\in\mathscr{Q}$, let

$$\mathcal{N}(q)=\left\{y: y=\sum_{h\in\mathcal{H}} \lambda^h s^h, \quad \text{with } \lambda^h \leqq 0 \text{ and } \lambda^h q s^h=0, \quad h\in\mathcal{H}\right\}.$$

This is the nonpositive cone generated by the satiation points whose value vanishes at q.

Proposition 3. Under Assumptions 1 and 2, strong competitive equilibria exist if $\mathcal{Q}\neq\{0\}$ and

$$y(q)\in\mathcal{N}(q)\Rightarrow y(q)=0, \quad \text{for} \quad q\in\mathcal{Q}.$$

If $Q=\{0\}$, strong competitive equilibria exist only if $\sum_{h\in\mathcal{H}} s^h=0$. This is Proposition 2.

If $q\in \operatorname{Int}\mathcal{Q}, qs^h>0$ for $h\in\mathcal{H}$ and hence $\mathcal{N}(q)=\{0\}$. That $y(q)=0$ whenever $y(q)\in\mathcal{N}(q)$ thus only restricts the aggregate demand function on the boundary of the price domain. It requires that for $q\in \operatorname{Bd}\mathcal{Q}$, the aggregate demand, if it does not vanish, it is not a nonpositive linear combination of the satiation points whose value vanishes at q; we thus refer to it as the boundary condition.

Suppose the boundary condition fails. There exists, thus, $\bar{q}\in \operatorname{Bd}\mathcal{Q}$ such that $y(\bar{q})\neq 0$, while $y(\bar{q})=\sum_{h\in\mathcal{H}}\bar{\lambda}^h s^h$, with $\bar{\lambda}^h\leqq 0$ and $\bar{\lambda}^h\bar{q}s^h=0$, for $g\in\mathcal{H}$. Since $y(\bar{q})\neq 0$, we may suppose without loss of generality that $\bar{\lambda}^1<0$ and $-\bar{\lambda}^1\geqq -\bar{\lambda}^h$ for $h\in\mathcal{H}$. Consider the allocation $\bar{y}^{\mathcal{H}}$ defined by

$$\bar{y}^h=\frac{1-\bar{\lambda}^h}{1-\bar{\lambda}^1}\left(\frac{y^h(\bar{q})-\bar{\lambda}^h s^h}{1-\bar{\lambda}^h}\right), \quad h\in\mathcal{H}.$$

Since $(1-\bar{\lambda}^1)\geqq(1-\bar{\lambda}^h)$ and the feasible set $\mathcal{Y}^h$ is convex and contains 0, $\bar{y}^h\in\mathcal{Y}^h$, while by construction $\sum_{h\in\mathcal{H}}\bar{y}^h=(1/(1-\bar{\lambda}^1))\ (y(\bar{q})-\sum_{h\in\mathcal{H}}\bar{\lambda}^h s^h)=0$. Thus $\bar{y}^{\mathcal{H}}$ is a feasible allocation at which some individual, $h=1$, is satiated, while $u^h(\bar{y}^h)\geqq u^h(0)$, for $h\in\mathcal{H}$.

In Bergstom (1976), the sufficient condition used to prove the existence of strong competitive equilibria is that no feasible allocation, $y^{\mathcal{H}}$, exists at which some individual is satiated, $y^{h'}=s^{h'}$, for some $h'\in\mathcal{H}$, while $u^h(y^h)\geqq u^h(0)$ for $h\in\mathcal{H}$. The same condition is used to prove the existence of strong equilibria for an asset market in Nielsen (1990). The argument above shows that our boundary condition is weaker. Furthermore, it is evidently strictly weaker; it restricts aggregate behavior only at the boundary of the price domain $\mathcal{Q}$ and not over the entire set of feasible allocations.

Proof of Proposition 3. If $\sum_{h\in\mathscr{H}} s^h=0, q^*=0$ are competitive equilibrium prices. For the remainder of the proof we suppose that $\sum_{h\in\mathscr{H}} s^h\neq 0$.

Since the cone $\mathscr{Q}$ is closed, convex and does not coincide with $\mathscr{R}^A$, while $\mathscr{Q}\neq\{0\}$, the normalized price domain

$$\mathscr{D}=\{q\in\mathscr{Q}:\|q\|=1\}$$

is closed, nonempty, and convex up to a homeomorphism.

Consider the correpondence

$$\xi=(\xi_1,\xi_2):\ \underset{h\in\mathscr{H}}{\times}\ \mathscr{Y}^h\times\mathscr{D}\rightarrow\ \underset{h\in\mathscr{H}}{\times}\ \mathscr{Y}^h\times\mathscr{D},$$

defined componentwise by

$$\xi_1^h=y^h(q),\quad h\in\mathscr{H},$$

$$\xi_2=\arg\max\{\hat{q}y(q):\hat{q}\in\mathscr{D}\}.$$

The correspondence ξ is nonempty, compact, convex valued up to a homeomorphism and upper hemi-continuous, and its domain is compact and convex. By Kakutani's fixed point theorem, it has a fixed point, $(q^*, y^{*\mathscr{H}})$.

In order to complete the argument, it remains to show that a fixed point of the correspondence ξ is indeed a competitive equilibrium; in particular, that $y^*=\sum_{h\in\mathscr{H}} y^{*h}=0$.

By the definition of the normalized price domain, Δ, and the correspondence, ξ, q^* is a solution to the maximization problem

$$\max qy^*$$

$$\text{s.t.}\quad qs^h\geqq 0,\quad h\in\mathscr{H},$$

$$\|q\|=1.$$

It follows that there exist $\lambda_h^*\leqq 0$, for $h\in\mathscr{H}$, and μ^* such that $y^*=\sum_{h\in\mathscr{H}}\lambda^{*h}s^h+\mu^*q$ and the complementary slackness conditions are satisfied, $\lambda^h q^*s^h=0$, for $h\in\mathscr{H}$. Since, with $q^*s^h>0, q^*y^h(q^*)=0$, for $h\in\mathscr{H}, q^*y^*=0$ and hence $\mu^*=0$. Thus $y^*=\sum_{h\in\mathscr{H}}\lambda^{*h}s^h$.

With $y^* = \sum_{h \in \mathscr{H}} \lambda^{*h} s^h$, with $\lambda^{*h} \leqq 0$ and $\lambda^{*h} q^* s^h = 0$, for $h \in \mathscr{H}$, it follows from the boundary condition that $y^* = 0$, and thus $(q^*, y^{*\mathscr{H}})$ is a strong competitive equilibrium. □

Under the Assumptions 1 and 2, strong competitive equilibrium allocations are optimal. Suppose a feasible allocation, $y'^{\mathscr{H}}$, Pareto dominates the strong competitive equilibrium allocation, $y^{*\mathscr{H}}$. For $h \in \mathscr{H}$, if $u^h(y'^h) > u^h(y^{*h}), q^* y'^h > 0$; this is evident. If $u^h(y'^h) = u^h(y^{*h}), q^* y'^h \geqq 0$; this follows by observing that either $y^{*h} = s^h$, in which case also $y'^h = s^h$, or $y^{*h} \neq s^h$, in which case also $y'^h \neq s^h$, the individual is nonsatiated at y'^h and hence from the quasi-concavity of the utility function and the linearity of the budget constraint and the convexity of the feasible set, $q^* y'^h < 0$ leads to a contradiction. It follows that $q^* \sum_{h \in h} y'^h > 0$, which contradicts $\sum_{h \in \mathscr{H}} y'^h = 0$, the feasibility of the allocation y'^h. This argument is nevertheless of limited interest, as it depends essentially on the uniqueness of the satiation point or the strict quasi-concavity of the utility function for each individual.

Alternatively, suppose that $\bar{y}^{\mathscr{H}} = 0$ is a Pareto optimal allocation and that Assumptions 1 and 2 are satisfied. If $\sum_{h \in \mathscr{H}} s^h \neq 0$ and $\mathscr{Q} = \{0\}$, which the Pareto optimality of no trade does not preclude, then according to Proposition 2 no strong competitive equilibria exist. Thus a Pareto optimal allocation, even if interior, may fail to be supported as a strong competitive equilibrium allocation.

5. Conclusion

The existence of competitive equilibria may fail when the asset market is incomplete. This is due to the failure of upper hemi-continuity of the attainable reallocations of revenue as prices vary when asset payoffs are denominated in multiple commodities [Hart (1975)]. Evidently, this is not an issue when asset payoffs are denominated in a numeraire [Geanakoplos and Polemarchakis (1986)] and thus a fortiori in economics with a single, composite good. Nevertheless, nonsatiation and free disposal may still fail in the asset market. Here we have characterized the existence of competitive equilibria when nonsatiation and free disposal indeed fail.

Appendix: Notation

$\geqq, >, \gg$:	Vector inequalities.
$'$:	The transpose of a vector.
Int:	The interior of a set.
Bd:	The boundary of a set.
$\mathscr{R}^A$:	The Euclidean space of dimension A.

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