CHAPTER 2

Asset pricing and observability

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1 Introduction

We explore observable restrictions on asset prices. We consider an exchange economy with diverse preferences and endowments. The asset structure may be incomplete, with fewer assets than states of nature.

Asset prices do not allow for arbitrage: a portfolio of assets with positive payoff does not have a zero price. This nonarbitrage condition is equivalent to the existence of a positive price associated with each state of nature (Ross, 1978; Harrison and Kreps, 1979; also Geanakoplos and Polemarchakis, 1986). State prices are the implicit prices for elementary securities, often referred to as Arrow prices (Arrow, 1953).

In fact, the nonarbitrage condition is exhaustive. Neither equilibrium nor optimality have any observable implications for asset prices other than those derived from the nonarbitrage condition. If the asset market is in equilibrium, Arrow prices exist, irrespective of the asset structure, and are positive; and, conversely, if the Arrow prices implied by the asset payoffs and prices are positive, it is possible to construct an economy that yields these prices at a competitive equilibrium (Harrison and Kreps, 1979). Perhaps more strongly, it is possible to construct an economy for which these prices characterize an optimum, even if the number of assets is smaller than the number of states of nature; this construction is valid locally to an equilibrium (Geanakoplos and Polemarchakis, 1990).

When the asset market is complete, the Arrow prices that support any nonarbitrage asset prices are unique, and they are fully recoverable from the asset payoffs and prices. All possible assets are attainable, since their payoffs are in the span of the payoffs of the marketed assets, so that it is possible to speak of fully determinate nonarbitrage prices for all

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assets. When the asset market is incomplete, there are many Arrow prices compatible with every vector of asset prices; generically, assets that are not attainable cannot be unambiguously valued by nonarbitrage.

Asset pricing in the theory of finance is concerned with empirical statements about the prices of assets. We use the term empirical in a broad sense. This includes predictions that relate the prices and returns of assets to the means and variances of their payoffs, expectations, and variances taken with respect to an empirical probability measure. In addition, it includes restrictions on asset prices, such as subspace restrictions, stated in terms of observables.

What does the nonarbitrage condition imply for the empirical properties of asset prices? To derive this, we first use the normalized Arrow prices as a probability measure. The nonarbitrage restriction is then equivalent to the condition that asset prices equal expected payoffs – the martingale property – when expectations are taken with respect to the Arrow measure. Generically, this measure is different from the empirical measure, even if the empirical measure is the commonly held prior distribution of all individuals (Drèze, 1971; also Lucas, 1978; Bewley, 1980). Then, we show that, empirically, the price of all attainable assets must equal their expected payoffs, with a correction factor. This factor represents the covariance of asset payoffs with a random variable that is the likelihood ratio that transforms the Arrow measure to the empirical one.

Evidently, there are as many distinct likelihood ratios as there are distinct Arrow prices that support the asset prices, so that this pricing rule is not unambiguous if the asset market is incomplete. However, only one such likelihood ratio is observable: it is the return to a portfolio of marketed assets. We call this the "benchmark" portfolio. Any attainable asset is unambiguously priced by its expected return and its covariance with the return of the benchmark portfolio. In addition, the benchmark portfolio weights can be derived, in closed form. The weights depend only on the empirical expectation and variance–covariance matrices of returns on the marketed assets, that is, the portfolio is of the familiar mean–variance form. Clearly, if the asset market is complete, the benchmark portfolio recovers the unique Arrow prices. Otherwise, it selects one of the many state prices compatible with equilibrium, the selected one being linear in the payoffs of marketed assets. Since all assets are priced relative to it, this portfolio provides a summary measure of the attitudes toward risk in a competitive market.

Once the benchmark portfolio is available, we can use it to derive asset pricing rules. For example, the Capital Asset Pricing Model (Lintner, 1965; Mossin, 1966; Sharpe, 1964; Treynor, 1961) posits a relation between the return of any asset and its covariance with the return of the market, the aggregate consumption portfolio; and the relation is valid only if all individuals choose portfolios that trade off expectation for variance in returns, which is only possible with restrictive assumptions on preferences or on the structure of payoffs. We show that the same relation is valid with general preferences and asset payoffs as long as we define the "market" portfolio appropriately and restrict attention to attainable assets. Once again, this representation is equivalent to the nonarbitrage condition; the market portfolio is stated entirely in terms of observables.

Even with fewer assets than states of nature, the asset market may be effectively complete: it may be possible to price assets, not necessarily attainable, so that, if marketed, they would leave the solutions to the optimization problems of all individuals essentially unaffected. Under restrictive assumptions on the utility functions and initial endowments of individuals and on the asset structure, 2-fund separation obtains and every asset, not necessarily attainable, can be priced by its expected payoff and its covariance with the return of the benchmark portfolio. Also the Capital Asset Pricing Model holds with respect to the return of the aggregate consumption portfolio.

Since the benchmark portfolio does select one out of the many possible Arrow prices, it can be used to approximately price nonattainable assets even when the asset market fails to be effectively complete. Formally, the pricing rule operates as follows. Consider an asset whose payoff is not spanned by those of marketed assets. Find the portfolio of marketed assets that yields the return closest to the return of this asset – closest in empirically weighted Euclidean distance. The market price of this portfolio is the "projection" price of the asset. We give an example of the projection pricing rule to determine the price of a nonattainable asset. Of course, this is only one of many possible approximate prices. More importantly, it abstracts from the fact that, generically, the introduction of a previously nonattainable asset in an incomplete asset market affects nontrivially the solution to the optimization problems of individuals. Interestingly, it is precisely this caveat that turns projection pricing into an empirical implication.

2 The asset market

Economic activity extends over two periods, 1 and 2, under uncertainty. States of nature are $s=1,\dots,S$; all uncertainty is resolved in the second period. Date–event pairs are 1 and $(2,1),\dots,(2,S)$.

Commodities $l=1,\dots,L$ are marketed and consumed in both periods. A commodity bundle is

$$x=(x(1),x(2))=(x(1),\ldots,x(2,s),\ldots),$$

a vector of dimension $L(S+1)$.

Individuals are $h=1,\ldots,H$. An individual is described by his utility function u^h defined on the consumption set X^h, a set of commodity bundles; and by his initial endowment w^h, a consumption bundle.

For $h=1,\ldots,H$, the consumption set is

$$X^h=\{x: x\gg 0\},$$

the set of strictly positive commodity bundles.

For $h=1,\ldots,H$, the utility function u^h is twice continuously differentiable; $Du^h\gg 0$ and D^2u^h is negative definite on $[Du^h]^\perp$; for $(x_n: n=1,\ldots)$, a sequence of consumption bundles, and for $x\neq 0$, a commodity bundle on the boundary of the consumption set,

$$\lim_{n\to\infty} x_n=x \Rightarrow \lim_{n\to\infty}\frac{x_n' Du^h(x_n)}{\|Du^h(x_n)\|}=0.$$

For $h=1,\ldots,H$, the initial endowment is

$$w^h\gg 0,$$

a consumption bundle.

These are strong but standard.

Individuals' utility functions may, but need not, satisfy the expected utility hypothesis. The probability of occurrence of the states of nature may not be agreed upon, even defined.

Assets $a=1,\ldots,A$ are marketed in the first period and pay off in the second. A portfolio is

$$y=(\ldots,y_a,\ldots).$$

A marketed asset is described by its payoff, b_a.

For $a=1,\ldots,A$, the payoff is

$$b_a=(\ldots,b_a(2,s),\ldots),$$

a second-period commodity bundle of dimension $LS\times 1$. The asset structure is a matrix of payoffs of marketed assets,

$$B=(\ldots,b_a,\ldots)=(\ldots,B(2,s),\ldots),$$

of dimension $LS\times A$.

The matrix of payoffs of marketed assets has full column rank:

$$\dim[B]=A.$$

This eliminates redundant assets.

There exists a portfolio $\bar{y}$ such that

$$B(2,s)\bar{y}>0, \quad s=1,\ldots,S.$$

This allows commodity prices to be normalized so that the portfolio $\bar{y}$ has a state-independent payoff in terms of revenue.

An economy is parametrized by the array of initial endowments, and possibly by the array of utility functions and the asset structure. The family of economies can be identified with a finite dimensional open set.

A property holds generically if and only if it holds for a generic set of economies: an open set of full Lebesgue measure. A property is robust if and only if it holds for an open set of economies.

Commodity prices are

$$p=(p(1),p(2))=(p(1),\ldots,p(2,s),\ldots),$$

where

$$p(1)=(\ldots,p_l(1),\ldots)$$

and

$$p(2,s)=(\ldots,p_l(2,s),\ldots), \quad s=1,\ldots,S.$$

At commodity prices p, for $a=1,\ldots,A$, the payoff in terms of revenue is

$$d_a(p)=P(2)'b_a,$$

where

$$P(2)=\mathrm{diag}(\ldots,p(2,s),\ldots).$$

The matrix of payoffs of marketed assets in terms of revenue is

$$D(p)=P(2)'B,$$

a matrix of dimension $S\times A$.

We restrict attention to strictly positive commodity prices such that

$$\dim[D(p)]=A,$$

which we normalize so that

$$p(1)'1^L=1 \quad \text{and} \quad D(p)\bar{y}=1^S.$$

The payoff in terms of revenue of the portfolio $\bar{y}$ is state-independent. This is simply a normalization; the portfolio $\bar{y}$ is not risk-free, which would require a payoff at each state of nature inversely proportional to the marginal utility of income of each individual.

Prices of marketed assets are

$$q=q(1)=(\ldots,q_a,\ldots).$$

At commodity prices p, prices q for marketed assets do not allow for arbitrage if and only if

$$D(p)y > 0 \Rightarrow q'y > 0.$$

At commodity prices p, the attainable reallocations of revenue across states of nature coincide with

$$[D(p)],$$

the column span of the matrix of asset payoffs in terms of revenue. The asset market is complete if and only if

$$\dim[D(p)] = S;$$

all reallocations of revenue across states of nature are then attainable. Evidently, the asset market is complete whenever

$$A = S.$$

Otherwise it is incomplete.

At prices (p, q), an individual solves the following optimization problem: for $h = 1, \ldots, H$,

$$\operatorname{Max} u^h(x)$$

subject to the budget constraints

$$P(2)'(x(2) - w^h(2)) \leq D(p)y,$$

in the second-period commodity markets, and

$$p(1)'(x(1) - w^h(1)) + q'y \leq 0,$$

in the first-period commodity and asset market.

A solution to the individual optimization problem exists, is unique, and is characterized by the following first-order necessary and sufficient conditions: for $h = 1, \ldots, H$,

$$Du^h(2) = P(2)\gamma^h(2),$$

in the second-period commodity markets,

$$Du^h(1) = p(1)\gamma^h(1),$$

in the first-period commodity market, and

$$D(p)'\gamma^h(2) = q\gamma^h(1),$$

in the first-period asset market, for some

$$\gamma^h(1) > 0 \quad \text{and} \quad \gamma^h(2) = (\ldots, \gamma^h(2, s), \ldots) \gg 0.$$

Evidently, $\gamma^h(1)$ is the individual's marginal utility of revenue in the first-period commodity and asset market, and $\gamma^h(2)$ the state-dependent marginal utility of revenue in the second-period commodity markets.

At prices (p, q), the asset market is effectively complete if and only if, for $h = 1, \ldots, H$,

$$\frac{1}{\gamma^h(1)} \gamma^h(2) = \gamma$$

for some

$$\gamma = (\ldots, \gamma(2, s), \ldots) \gg 0.$$

Evidently, when $A = S$ and the asset market is complete, it is effectively complete.

When $A < S$ and the asset market is incomplete, generically, the asset market is not effectively complete.

An asset, i, marketed or not, is described by its payoff in terms of revenue

$$d_i = (\ldots, d_i(s), \ldots).$$

At commodity prices, p, an asset is attainable if and only if

$$d_i \in [D(p)];$$

equivalently, an attainable asset is identified by the unique portfolio of marketed assets such that

$$d_i = D(p) y_i.$$

At prices q for marketed assets, the price of an attainable asset is

$$q_i = q' y_i.$$

An asset is nonattainable if and only if

$$d_i \notin [D].$$

Attainable assets are priced at the unique prices at which if marketed they would leave the solutions to the optimization problems of all individuals essentially unaffected. For nonattainable assets, generically, such prices do not exist.

The return of an asset is

$$r_i = \frac{1}{q_i} d_i - 1^S;$$

evidently, this is an abuse of language for a nonattainable asset.

The matrix of returns of the marketed assets is

$$R(p,q)=(\ldots,r_a(p,q),\ldots).$$

The return on the portfolio $\bar{y}$ is state-independent and equal to

$$\rho(q)=\frac{1}{q'\bar{y}}-1;$$

this is its expected return for any probability measure on the states of nature.

We chose units of measurement for the marketed assets such that

$$q=1^A,$$

and restrict attention to portfolios with

$$y'1^A=1;$$

it follows that the return to any portfolio is

$$r(y)=Ry.$$

A return is attainable if and only if it is the return to a portfolio.

3 Nonarbitrage asset prices

All properties of asset prices or returns follow from the nonarbitrage condition alone. The following result characterizes the set of prices for marketed assets which do not allow for arbitrage.

Proposition 1 (Geanakoplos and Polemarchakis, 1986). *At commodity prices p, prices q for marketed assets do not allow for arbitrage if and only if they satisfy the nonarbitrage condition:*

$$q=D(p)'\lambda$$

for some

$$\lambda=(\ldots,\lambda(2,s),\ldots)\gg 0.$$

Proof: Let

$$\bar{Q}(p)=\{q: q=D(p)'\lambda \text{ for some } \lambda\geq 0\}.$$

By construction, $\bar{Q}(p)$ is a finite cone. All finite cones are convex and closed. Since $D(p)$ has linearly independent columns, the interior of this cone is a non-empty open set and coincides with

$$Q(p)=\{q: q=D(p)'\lambda \text{ for some } \lambda\gg 0\}.$$

Let $\tilde{q} \in Q(p)$; then $\tilde{q} = D(p)'\lambda$ for some $\lambda \gg 0$. Now suppose $D(p)y > 0$ for some y, while $\tilde{q}'y \leq 0$. Since $\lambda \gg 0$, this is a contradiction. Thus, if $\tilde{q} \in Q(p)$, $D(p)y > 0 \Rightarrow \tilde{q}'y > 0$. The open cone $Q(p)$ consists only of prices that do not allow for arbitrage.

Note that $D(p)\bar{y} = 1^S > 0$. It follows that $0 \notin Q(p)$ and hence that $\bar{Q}(p) \neq \mathcal{E}^A$.

Let $\tilde{q} \notin Q(p)$. Since $Q(p)$ is convex, there exists a nontrivial hyperplane, $\mathcal{H}_{\hat{y}} = \{q : q'\hat{y} = \tilde{q}'\hat{y}\}$, with $\hat{y} \neq 0$, through $\tilde{q}$, such that $q'\hat{y} \geq \tilde{q}'\hat{y}$ for all $q \in Q(p)$. Since $0 \in Q(p)$, $\tilde{q}'\hat{y} \leq 0$, and $q'\hat{y} \geq 0$ for all $q \in \bar{Q}(p)$; equivalently, $\lambda' D(p)\hat{y} \geq 0$ for all $\lambda \geq 0$. Since $\hat{y} \neq 0$ and $D(p)$ has full column rank, $D(p)\hat{y} > 0$. Thus, if $\tilde{q} \notin Q(p)$, there exists a portfolio $\hat{y}$ such that $D(p)\hat{y} > 0$ while $\tilde{q}'\hat{y} \leq 0$. Asset prices not in $Q(p)$ allow for arbitrage.

Note that in the previous argument, if $\tilde{q}$ is on the boundary of $Q(p)$, there is an arbitrage portfolio, $\hat{y}$, with $\tilde{q}'\hat{y} = 0$. □

We refer to λ as the Arrow prices.

We argue at given prices (p, q). We omit the reference to prices when no confusion arises; also the distinction between asset payoffs and asset payoffs in terms of revenue.

Let the set of Arrow prices compatible with asset payoffs D and prices q for marketed assets be

$$\Lambda = \{\lambda \gg 0 : D'\lambda = q\}.$$

When the asset market is incomplete, Arrow prices are not unique:

$$\dim(\Lambda) = S - A,$$

with

$$\lambda_1, \lambda_2 \in \Lambda \Rightarrow (\lambda_1 - \lambda_2) \in [D]^{\perp}.$$

Note that, for $h = 1, \ldots, H$,

$$\frac{1}{\gamma^h(1)} \gamma^h(2) \in \Lambda.$$

Evidently, the prices of attainable assets are independent of the choice of Arrow prices compatible with the payoffs and prices of the marketed assets.

3.1 *The martingale property*

Asset prices satisfy the martingale property with respect to a probability measure ν on the states of nature if and only if, for some $\rho > 0$, and for every attainable asset,

$$E_\nu r_i = \rho;$$

equivalently, if and only if

$$\nu' R = 1^A \rho.$$

Asset prices satisfy the martingale property if and only if they satisfy the martingale property with respect to some measure.

Asset prices do not allow for arbitrage if and only if they satisfy the martingale property for some strictly positive probability measure; the two conditions are equivalent.

Note that, for all $\lambda \in \Lambda$,

$$\frac{1}{\lambda' 1^S} = \rho + 1.$$

For $\lambda \in \Lambda$, define

$$\nu(\lambda) = \lambda(\rho + 1);$$

note that $\nu(\lambda)$ is a strictly positive probability measure on the states of nature. From the nonarbitrage condition, it follows that

$$E_{\nu(\lambda)} r_i = \nu(\lambda)' r_i = (\rho+1)\frac{\lambda' d_i}{q' y_i} - 1 = (\rho+1)\frac{\lambda' D(p) y_i}{\lambda' D(p) y_i} - 1 = \rho.$$

Let

$$\mathfrak{N} = \{\nu : \nu' R = 1^A \rho\}$$

be the set of probability measures on the states of nature compatible with the martingale property.

For $\nu \in \mathfrak{N}$, define

$$\lambda(\nu) = \nu \frac{1}{\rho + 1};$$

note that $\lambda(\nu) \gg 0$. From the martingale property, it follows that

$$\lambda(\nu)' d_i = \lambda(\nu)' q_i (r_i + 1) = q_i \left(\frac{\rho}{\rho+1} + \frac{1}{\rho+1} \right) = q_i.$$

4 Nonarbitrage and empirical properties

The testing of asset pricing theories is based on the actual realization of payoffs and prices. We use this as the point of departure and assume that the empirical probability measure is directly observable. This is just the frequency function

$$\pi = (\ldots, \pi(s), \ldots),$$

which we assume to be strictly positive.

Any vector $x=(\ldots, x(s), \ldots)$ can be viewed as a random variable, whose probability distribution is fully described by any probability measure on the set of states of nature, in particular π. We consider empirical properties – that is, properties of random variables endowed with the empirical measure. Clearly, the return of an asset is such a random variable. Define expectations with respect to the empirical measure to write

$$\mu_i = Er_i,$$
$$\sigma_i^2 = \operatorname{var}(r_i) = E(r_i - \mu_i)^2,$$
$$\sigma_{i,j} = \operatorname{cov}(r_i, r_j) = E(r_i - \mu_i)(r_j - \mu_j),$$

as expectations, variances, and covariances.

The nonarbitrage condition imposes a restriction on the prices of attainable assets: these prices equal the expectation of payoffs, with respect to the measure ν; this measure need not equal the empirical one. In addition, when the asset market is incomplete, ν is not even unique. It is then not clear that the nonarbitrage restriction has any empirical counterpart. We derive here the empirical implications of the nonarbitrage restrictions. Clearly, this is a useful exercise only when the ν is distinct from π.

Generically,

$$\pi \notin \mathfrak{N},$$

even when individuals agree that π defines the probability of occurrence of the states of nature and maximize expected utility.

The following lemma restates the nonarbitrage condition in terms of the empirical measure.

Lemma 1. *Asset prices satisfy the nonarbitrage condition if and only if there exists a bounded random variable v with*

$$v < Ev + 1,$$

such that, for every attainable asset i,

$$\mu_i - \rho = \operatorname{cov}(v, r_i).$$

Proof: Suppose asset prices satisfy the nonarbitrage condition. This implies that they satisfy the martingale property with respect to any probability measure $\nu \in \mathfrak{N}$. For any constant k, define

$$v(k, \nu) = 1^S k + \Pi^{-1}\nu,$$

where

$$\Pi = \operatorname{diag}(\ldots, \pi(s), \ldots).$$

Note that $v < k = Ev + 1$; it is bounded since π is strictly positive. It follows that

$$\mathrm{cov}(v, r_i) = Evr_i - (k-1)\mu_i = \mu_i - \rho.$$

Suppose such a random variable v exists. Define

$$\nu(v) = \Pi(1^S(1 + Ev) - v).$$

Evidently, ν is a strictly positive probability measure on the states of nature. Since

$$E_\nu r_i = \mu_i - \mathrm{cov}(v, r_i) = \rho,$$

asset prices satisfy the martingale property with respect to ν and hence do not allow for arbitrage. □

Clearly, the random variable $v = v(k, \nu)$ is not uniquely defined; we need to specify k and, more importantly, $\nu \in \mathfrak{N}$. As we show in what follows, there is a unique v^* that is the payoff of an attainable asset; as a result, there is a unique portfolio of that asset that generates returns colinear with v^*. This portfolio is fully specified by the empirical characteristics of returns.

4.1 *Pricing and the benchmark portfolio*

We use some simplifying notation in the derivations; this should also make the correspondence with existing literature on asset pricing transparent.

The portfolio with state-independent payoff coincides with asset $a = 1$:

$$\bar{y} = (1, 0, \ldots, 0).$$

Portfolios are

$$y = (y_1, y_2) = (y_1, y_2, \ldots, y_A).$$

The matrix of returns of marketed assets is

$$R = (r_1, R_2) = (r_1, r_2, \ldots, r_A).$$

The return on a portfolio is

$$r(y) = Ry = r_1 y_1 + R_2 y_2.$$

For $a = 2, \ldots, A$, the returns r_a are nondegenerate random variables. Let

$$\mu = E(r_2, \ldots, r_A) = R_2' \pi$$

be the vector of expectations and

$$\Sigma = E(r_2 - \mu_2, \ldots, r_A - \mu_A)(r_2 - \mu_2, \ldots, r_A - \mu_A)' = R_2' \Pi R_2 - \mu\mu'$$

be the variance–covariance matrix that is square, symmetric, of dimension $A-1$, and strictly positive definite.

Proposition 2. *The benchmark portfolio*

$$y^*=(1-(1^{A-1})'\Sigma^{-1}(\mu-\rho^{1^{A-1}}),\Sigma^{-1}(\mu-\rho^{1^{A-1}}))$$

with return

$$r^*=r_1+(R_2-r_1(1^{A-1})')\Sigma^{-1}(\mu-\rho^{1^{A-1}})$$

is the unique portfolio with the property that, for every attainable asset i,

$$\mu_i-\rho=\operatorname{cov}(r^*,r_i).$$

Proof: From Lemma 1, the benchmark property holds for

$$v(k,\nu)=1^S k-\Pi^{-1}\nu$$

for any constant k, and any strictly positive measure such that

$$R_2'\nu=1^{A-1}\rho$$

and

$$1^{S'}\nu=1.$$

Let

$$\theta=\Sigma^{-1}(\mu-\rho^{1^{A-1}}).$$

Define

$$k^*=1+\rho(1-\theta'1^{A-1})+\mu'\theta;$$

and

$$\nu^*=\Pi(1^S k^*-r^*)=\Pi(1^S+(1^S\mu'-R_2)\theta).$$

By construction,

$$r^*=1^S k^*=\Pi^{-1}\nu^*.$$

Since

$$R_2'\pi=R_2'\Pi 1^S=\mu,$$

by direct substitution,

$$R_2'\nu^*=1^{A-1}\rho,$$

and

$$1^{S'}\nu^*=1.$$

Hence, y^* has the benchmark property.

Suppose any other portfolio $y=(1-(1^{A-1})'y_2,y_2)$ has the benchmark property. Suppose that $y_2=\theta+e$. Then

$$\operatorname{cov}(r(y), r_i) = \operatorname{cov}(r^*(b), r_i) + \operatorname{cov}(R_2 e, r_i)$$
$$= \mu_i - \rho + \operatorname{cov}(R_2 e, r_i),$$

whenever r_i is attainable. The benchmark property requires that

$$\operatorname{cov}(R_2 e, r_i) = 0$$

for each attainable asset. Now, the portfolio $y_e = (1 - e'1^{A-1}, e)$ is attainable and has return $r(y_e) = R_2' e$, so that

$$\operatorname{cov}(r(y_e), r(y_e)) = e'\Sigma e = 0,$$

whenever y is a benchmark portfolio. Since Σ is positive definite, this can hold if and only if $e = 0$. □

By construction, the benchmark portfolio selects the unique Arrow prices compatible with the payoffs of the marketed assets such that

$$\lambda^* \in [\Pi D] = [\Pi R].$$

4.2 *The Capital Asset Pricing Model*

The Capital Asset Pricing Model specifies a relationship between the excess returns of attainable assets and their covariances with the return to a portfolio, the market portfolio.

Corollary 1. *Let a constant, $m \neq 0$, index the market portfolio:*

$$r_m = (1-m) r_1 + m r^*.$$

The Capital Asset Pricing Model holds with respect to r_m as the return to the market portfolio: for every attainable asset,

$$\mu_i - \rho = \beta_i (\mu_m - \rho),$$

where

$$\beta_i = \frac{\operatorname{cov}(r_m, r_i)}{\operatorname{var}(r_m)}.$$

Proof: For $m \neq 0$,

$$\operatorname{cov}(r_m, r_i) = m \operatorname{cov}(r^*, r_i) = m(\mu_i - \rho).$$

Further,

$$\operatorname{var}(r_m) = m^2 \operatorname{var}(r^*) = m^2 (Er^* - \rho) = m(Er_m - \rho).$$

The result follows from Proposition 2 by substitution. □

A one-parameter family of portfolios serves as market portfolios for the pricing of assets. Each of these is a portfolio of asset $a=1$ and the benchmark asset. The indexing parameter m characterizes the trade-off between mean excess return and variance in the return of the market:

$$m=\frac{\operatorname{var}(r_m)}{Er_m-\rho}.$$

Evidently, $Er_m-\rho$ is positive if and only if $m>0$. Since

$$m^2=\frac{\operatorname{var}(r_m)}{\theta'\Sigma\theta},$$

the family of market returns r_m have the property that

$$\frac{(Er_m-\rho)^2}{\operatorname{var}(r_m)}=\theta'\Sigma\theta.$$

For $m>0$, this defines a monotonically increasing and linear relation between the excess returns of market portfolios and their standard deviations; the locus of expected excess returns and standard deviations of market portfolios coincides with the efficient frontier.

Of course, generically, the market portfolio so specified is different from the aggregate market portfolio or the aggregate consumption of individuals.

5 Pricing nonattainable assets

A nonattainable asset i can be priced if and only if, for $h=1,\ldots,H$,

$$d_i'\gamma^h(2)=\gamma^h(1)q_i;$$

otherwise it cannot be priced.

Evidently, every asset, not necessarily attainable, can be priced if and only if the asset market is effectively complete; then

$$q_i=d_i'\gamma.$$

For simplicity, we suppose that there is only one commodity: $L=1$; also, that commodity prices are

$$p=(p(1),p(2))=(p(1),\ldots,p(2,s),\ldots)=(1,\ldots,1,\ldots)$$

and

$$D=B.$$

This is not essential, but simplifies the argument.

Consider the following restrictive assumptions on the utility functions and initial endowments of individuals and on the asset structure.

For $h=1,\dots,H$, the utility function is intertemporally separable:

$$u^h = u^h(1) + u^h(2).$$

Over second-period consumption, the utility function satisfies the expected utility hypothesis with respect to the empirical measure over states of nature; locally, at the solution to the individual optimization problem, the cardinal utility index is state-independent and quadratic:

$$u^h(2) = E\alpha^h x(2) - (1/2)x(2)^2.$$

For $h=1,\dots,H$, the initial endowment of second-period commodities lies in the span of the matrix of asset payoffs:

$$w^h(2) \in [B].$$

There exists a riskless asset: for some portfolio of marketed assets,

$$B\bar{y} = 1^S.$$

Lemma 2. *Under the restrictive assumptions on the utility functions and the initial endowments of individuals and on the asset structure, the asset market is effectively complete and 2-fund separation obtains: for* $h=1,\dots,H$,

$$w^h(2) + By^h \in [1^S, B(B'\Pi B)^{-1}q]$$

(Cass and Stiglitz, 1970).

Proof: For $h=1,\dots,H$, from the first-order conditions for individual optimization in the second-period commodity markets,

$$\Pi(1^S\alpha^h - w^h(2) - By^h) = \gamma^h(2);$$

thus,

$$\gamma^h(2) \in [\Pi B].$$

Then from the first-order conditions for optimization in the first-period asset market,

$$\gamma^h(2) = \Pi B(B'\Pi B)^{-1}q\gamma^h(1);$$

hence, the asset market is effectively complete. Finally, by substitution,

$$w^h(2) + By^h = 1^S\alpha^h - B(B'\Pi B)^{-1}q\gamma^h(1). \qquad \square$$

Evidently, for $h=1,\dots,H$, the return to the consumption portfolio of individual h is

$$r^h = \frac{1}{(w^h(2)+By^h)'\gamma} w^h(2) + By^h - 1^S.$$

Proposition 3. *Under the restrictive assumptions on the utility functions and initial endowments of individuals and on the asset structure, for every asset, not necessarily attainable,*

$$\mu_i - \rho = \text{cov}(r^*, r_i).$$

Proof: It suffices to show that, for $h=1,\dots,H$,

$$\frac{1}{\gamma^h(1)}\gamma^h(2) = \lambda^*.$$

But this is evident, since the Arrow prices selected by the benchmark return are the unique Arrow prices such that

$$\lambda^* \in [\Pi B],$$

and, from the proof of Lemma 2, for $h=1,\dots,H$,

$$\frac{1}{\gamma^h(1)}\gamma^h(2) \in [\Pi B].$$

□

The return to the aggregate consumption portfolio is

$$r^A = \frac{1}{(w^A + By^A)'\gamma}(w^A + By^A) - 1^S,$$

where

$$w^A = \sum_h w^h$$

and

$$y^A = \sum_h y^h.$$

Corollary 2 (Geanakoplos and Shubik, 1989). *The Capital Asset Pricing Model holds with respect to r^A, the return to the aggregate consumption portfolio: for every asset, not necessarily attainable,*

$$\mu_i - \rho = \beta_i(\mu^A - \rho)$$

where

$$\beta_i = \frac{\text{cov}(r^A, r_i)}{\text{var}(r^A)}.$$

Proof: Evidently,

$$\lambda^* = \frac{1}{\sum_h \gamma^h(1)} \sum_h \gamma^h(2).$$

From the first-order conditions for individual optimization in the second-period commodity markets and in the first-period asset market, it follows then that

$$\lambda^* = \frac{1+\rho}{\pi'(1^S\alpha-(w+By))}\Pi(1^S\alpha-(w+By)),$$

where

$$\alpha = \sum_h \alpha^h.$$

Since

$$r^* = 1^S k^* - \Pi^{-1}\nu^* = 1^S k^* - \frac{1}{1+\rho}\Pi^{-1}\lambda^*,$$

it follows that

$$r^* \in [r_1, r^A].$$

The result then follows from Proposition 3. □

Evidently, the Capital Asset Pricing Model holds for any nondegenerate return spanned by the return to the risk-free asset and the return to the aggregate consumption portfolio.

Generically, the restrictive assumptions on the utility functions and initial endowments of individuals and on the asset structure do not hold; more importantly, the asset market is not effectively complete. We analyze the properties of "approximate pricing" of nonattainable assets in such a situation.

Let r_i, r_j be the returns of any two assets, not necessarily attainable. We define the distance between the returns as

$$\delta(r_i, r_j) = (r_i - r_j)'\Pi(r_i - r_j) = E(r_i - r_j)^2.$$

The distance δ is the weighted Euclidean metric with weights corresponding to the empirical probabilities; alternatively, it is the empirical mean-squared error.

For any asset i, with return r_i, not necessarily attainable, we define the portfolio

$$\hat{y}_i = \arg\min \delta(r_i, r(y))$$

and the return

$$\hat{r}_i = r(\hat{y}_i) = r_1(1-(1^{A-1})'y_{i,2}) + R_2 y_{i,2}.$$

Evidently, $\hat{y}_i$ is the portfolio of marketed assets that approximates the return of the asset i most closely: we call this the projection portfolio of the asset i and the return of the projection portfolio the projection return. It is straightforward that $r_i = \hat{r}_i$ if and only if the asset i is attainable.

Lemma 3. *Let m be a market portfolio. For any asset, not necessarily attainable,*

$$\beta_i = \frac{\text{cov}(r_m, r_i)}{\text{var}(r_m)} = \frac{E\hat{r}_i - \rho}{Er_m - \rho}.$$

Proof: Since $\hat{r}_i$ is attainable, it follows from Corollary 1 that

$$\hat{\beta}_i = \frac{\text{cov}(r_m, \hat{r}_i)}{\text{var}(r_m)} = \frac{E\hat{r}_i - \rho}{Er_m - \rho};$$

we need to show that

$$\text{cov}(r_m, r_i - \hat{r}_i) = 0.$$

But this is evident, since by definition $\text{cov}(r_m, r_i - \hat{r}_i) = \delta(r_i - \hat{r}_i, r_m)$, and by construction, the quantity $e_i = r_i - \hat{r}_i$ is orthogonal to r_a, $a = 1, \ldots, A$ and r_m is in the span $[r_1, R]$. □

This result establishes that approximately pricing a nonattainable asset at its expected payoff with a correction for covariance with the benchmark return is equivalent to pricing the projection portfolios of the asset. We refer to this price as the projection of an asset.

We provide an example of projection pricing in an economy with two assets: a risk-free bond and a risky asset. We compute the projection portfolio in closed form and use it to find the projection price of an option. An option on the risky asset, which has a truncated payoff, is not in the span of marketed assets. In order to derive restrictions on prices, we specify the example in terms of asset payoffs and prices.

Consider an economy with two marketed assets, $A = 2$. Asset $a = 1$ is a riskless bond with payoff $d_1 = 1^S$. Asset $a = 2$ is a risky asset, with payoff d.

Prices for the two marketed assets are q_1 and q_2, respectively. Asset prices satisfy the nonarbitrage restriction whenever Arrow prices $\lambda \gg 0$ satisfy $\lambda' 1^S = q_1$ and $\lambda' d = q_2$.

The expectation and variance of the payoff of the risky asset in terms of the empirical measure are $\bar{d}$ and σ^2, respectively.

Consider any asset, not necessarily attainable, with payoff d_i. The projection of d_i on the span of $[1^S, d]$ in the weighted Euclidean norm is

$$\hat{d}_i = (Ed_i - \beta_i) + \beta_i d,$$

where

$$\beta_i = \frac{\text{cov}(d, d_i)}{\sigma^2}.$$

It follows that the projection price is

$$q_i = q_1(Ed_i - \beta_i \bar{d}) + \beta_i q_2.$$

Clearly, if d_i is uncorrelated with d, its projection price is equal to the present value of the expected payoff.

A call option with exercise price c is a secondary asset with payoff $d_i = (d-c)^+$. In order to price this asset by projection, we need to compute explicitly Ed_i, and $\text{cov}(d_i, d)$, which are functions of the truncated means and variances of d. Let

$$S_c = \{s \in \mathcal{S} : d(s) > c\}$$

and

$$\pi_c = \sum_{s \in S_c} \pi(s) = \text{Prob}(d > c).$$

Define

$$\bar{d}_c = E_{S_c}(d) = \frac{\sum_{s \in S_c} \pi(s) d(s)}{\pi_c}$$

and

$$\sigma_c^2 = \text{var}_{S_c}(d) = \frac{\sum_{s \in S_c} \pi(s) d^2(s)}{\pi_c} - (\bar{d}_c)^2$$

as the truncated mean and variance of d. By substitution, we obtain

$$Ed_i = \pi_c \bar{d}_c$$

as the expected payoff of the option, while its covariance with d is

$$\text{cov}(d_i, d) = \pi_c(\sigma_c^2 + (\bar{d}_c - \bar{d})(\bar{d}_c - c)).$$

It follows that the projection price of the option is equal to

$$q_i = \pi_c(\bar{d}_c q_1 + \beta_i(q_2 - \bar{d} q_1)),$$

where

$$\beta_i = \frac{\sigma_c^2 + (\bar{d}_c - \bar{d})(\bar{d}_c - c)}{\sigma^2}.$$

It remains to characterize the properties of the projection price as an approximate price when the asset market is not effectively complete.

6 Conclusions

In this chapter, we investigated the properties of asset prices when the asset market is possibly incomplete. Of particular interest are observable properties, which we interpret as the properties of asset returns with respect to their empirical distribution.

We showed that, irrespective of the possible incompleteness of the asset market, all attainable assets are priced by their covariance with a unique portfolio of marketed assets. This portfolio, called the benchmark portfolio, can be explicitly characterized as a function of the empirical means and variances of the returns. The empirical interest of this formulation comes because this construct is based on observations of asset returns alone, without reference to the utility functions and initial endowments of individuals, which are not observable.

We also demonstrated that the benchmark pricing rule can be used to price nonattainable assets. This amounts to finding the value of the projection portfolio. This is the portfolio of marketed assets whose return best approximates that of the nonattainable asset, distance being measured by the mean-squared error, or, equivalently, by the weighted Euclidean metric. This pricing rule is, generically, only approximate. It is exact under the restrictive conditions on the utility functions and the initial endowments of individuals and on the asset structure that imply that the asset market is effectively complete and 2-fund separation.

Projection pricing may be understood as the pricing of a large number of assets by a smaller number of factors; the two factors that are used to price all assets are the risk-free return and the benchmark return. This is precisely the theme of Arbitrage Pricing Theory (Ross, 1976). Our results suggest that this class of pricing rules are valid under restrictive conditions that entail, among others, that the asset market is effectively complete.

The valuation of investment projects by firms is a bothersome problem in the theory of production with incomplete markets. Since projection pricing yields a determinate valuation rule for all such projects, it is worth examining the implications of the implied decision criterion for firms.

Notation

- $\gg, >, \geq$: vector inequalities. For $a=(\dots,a_k,\dots)$ and $b=(\dots,b_k,\dots)$, $a\gg b$ if and only if $a_k > b_k$ for all k; $a>b$ if and only if $a_k \geq b_k$ for all k, with some strict inequality; $a \geq b$ if and only if $a_k \geq b_k$ for all k. The terms strictly positive, positive, and nonnegative refer to $\gg 0$, >0, and ≥ 0, respectively; for scalars, the terms strictly positive and positive coincide.
- $'$: the transpose. All vectors are column vectors; in the text, vectors are written as row vectors.
- [ ]: the space spanned by a collection of vectors; also by the columns of a matrix.

- $\perp$: the orthogonal complement.
- $\mathcal{E}^K$: the Euclidean space of dimension K.

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