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OPTIONS AND EQUILIBRIUM*

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When the asset market is incomplete, options at a priori specified exercise prices may prevent the existence of competitive equilibria. The failure of existence may be robust to perturbations.

When the asset market is incomplete, competitive equilibria may fail to exist. This failure of existence may be robust to perturbations.

Hart (1975) gave an example of an economy with forward and spot markets for which no competitive equilibria exist. The importance of Hart's example was subsequently questioned by Kreps (1979) who argued that the failure of existence in the example is exceptional. In a recent series of papers [Duffie and Shafer (1985, 1986), Magill and Shafer (1984, 1985), McMannus (1984), and Repullo (1984)], it was shown that, indeed, for a class of economies, a perturbation of the endowment allocation and the asset structure is sufficient to restore the existence of competitive equilibria.

The canonical economy for which, under standard assumptions, competitive equilibria can be shown to exist generically is the following: Economic activity extends for two periods. Assets are traded in the first period and pay off in the second. Commodities are traded in spot markets in the second period. When asset returns are denominated in multiple commodities, the space spanned by the matrix of asset payoffs varies with prices in the commodity spot markets. This dependence may be discontinuous, when the payoff matrix drops rank, which in turn may prevent competitive equilibria from existing. Generically, this is not, however, the case. What drives the argument for generic existence is that, at fixed, positive, spot commodity prices, a perturbation of the asset structure suffices to restore the rank of the matrix of asset payoffs.

Not all assets conform to this set up. Options, in particular, have the property that they yield non-degenerate payoffs on a domain defined by inequality constraints with respect to the price of an underlying asset. Thus, a perturbation of the asset structure, parameterized by a priori specified exercise prices, does not suffice to restore the rank of the payoff matrix. We

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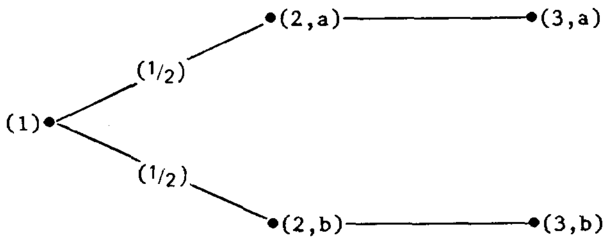

Fig. 1

shall construct an example to show that, as a consequence, the argument for the generic existence of competitive equilibria may fail.

Economic activity occurs over three periods, $t=1,2,3$.

One of two equiprobable states of nature, $s=a,b$, is realized in the second period. The pattern of resolution of uncertainty is thus as depicted in fig. 1.

In the first period only assets are traded. In periods two and three a single, perishable consumption good is available; its quantity is x. A consumption bundle is (x_2, x_3); a consumption plan is $((x(2,s), x(3,s)) : s=a,b)$.

There are two individual consumer–investors, $h=i,j$. Each individual is characterized by his intertemporal utility index, u^h, which maps positive consumption bundles to the real line, and by his initail endowment, e^h, a positive consumption plan.

There are three assets, each in 0 net supply:

- a bond, b, which is traded in the second period and pays off a unit of the good in the third; its price, with the consumption good as numeraire, is q, a positive number;
- options on the bond, a put, p, as well as a call, c, which are traded in the first period and are exercised in the second; the exogenously specified exercise price is k, a positive number; the price of the put is q_p and of the call q_c; option prices are positive, unless the payoff of an option is zero with certainty in which case its price is zero as well.

A price system is $(q_p(1), q_c(1), (q(2,s) : s=a,b))$.

A competitive equilibrium is a price system such that the bond and option markets clear.

Individual h solves the following optimization problem:

$$\max \mathrm{E}u^h \quad \text{s.t.}$$

$$0 = q_p(1)p^h(1) + q_c(1)c^h(1),$$

$$x^h(2,s) = e^h(2,s) + p^h(1)[k - q(2,s)]^+ + c^h(1)[k - q(2,s)]^- - q(2,s)b^h(2,s),$$

$$x^h(3,s) = e^h(3,s) + b^h(2,s),$$

$$s = a,b. \tag{1}$$

The asset structure is parametrized by the exercise price k.

With the utility indices held fixed, for simplicity, the individual consumer–investors are fully described by the endowment allocation $\boldsymbol{e}=(e^1, e^2)$.

An economy is a pair $(\boldsymbol{e}, k)$ of an endowment allocation and an asset structure.

We shall give a robust example of an economy $(\hat{\boldsymbol{e}}, \hat{k})$ for which no competitive equilibria exist.

Let

$$u^h = \ln x_2^h + \delta^h \ln x_3^h, \qquad \delta^h > 0, \quad h = i, j.$$

Denote the aggregate endowment by

$$e(t,s) = \sum_{h=i,j} e^h(t,s), \qquad t = 2, 3, \quad s = a, b.$$

First, suppose that

$$q(2,a) > k > q(2,b), \quad \text{or} \quad q(2,b) > k > q(2,a). \tag{2}$$

It follows from (1) that then, and only then, individuals optimize in a complete asset market. As long as

$$e(2,a) = e(2,b),$$

the bond markets in the second period clear only at

$$q(2,s) = \frac{2l(2)e(3,s')}{m(3,s)m(3,s') - l(3,s)l(3,s')}, \qquad s, s' = a, b, \tag{3}$$

where

$$m(t,s) = \sum_{h=i,j} \frac{2+\delta^h}{1+\delta^h} e^h(t,s),$$

$$t = 2, 3, \quad s = a, b,$$

$$l(t,s) = \sum_{h=i,j} \frac{\delta^h}{1+\delta^h} e^h(t,s),$$

and

$$l(t) = \sum_{s=a,b} l(t,s), \qquad t = 2, 3.$$

Alternatively, suppose that

$$k \leqq q(2,s) \quad \text{or} \quad k \geqq q(2,s), \qquad s=a,b. \tag{4}$$

In the first case the put option has a payoff of zero with certainty and in the second the call option does. In either case, and only then, no trade occurs in the first period asset market, which clears trivially. The bond markets in the second period clear only at

$$q(2,s)=\frac{l(2,s)}{n(3,s)}, \qquad s=a,b, \tag{5}$$

where

$$n(3,s)=\sum_{h=i,j}\frac{1}{1+\delta^h}e^h(3,s), \qquad s=a,b.$$

Let

$$\delta^i=1, \quad \delta^j=\tfrac{2}{3}.$$

No competitive equilibria exist[1] in an open neighborhood of the economy $(\hat{e},\hat{k})$ with

$$\hat{e}^i=(\hat{e}^i(2,a)=2, \quad \hat{e}^i(3,a)=1, \quad \hat{e}^i(2,b)=1, \quad \hat{e}^i(3,b)=2),$$

$$\hat{e}^j=(\hat{e}^j(2,a)=1, \quad \hat{e}^j(3,a)=2, \quad \hat{e}^j(2,b)=2, \quad \hat{e}^j(3,b)=3),$$

and

$$\hat{k}=0.81.$$

With a complete asset market, the competitive equilibrium allocation is

$$x^i=(x^i(2,a)\cong 1.1893, x^i(3,a)\cong 1.4884, x^i(2,b)\cong 1.1893, x^i(3,b)\cong 2.4828),$$

$$x^j=(x^j(2,a)\cong 1.8107, x^j(3,a)\cong 1.5116, x^j(2,b)\cong 1.8107, x^j(3,b)\cong 2.5172),$$

From (3), the supporting prices yield that

$$q(2,a)\cong 0.798, \qquad q(2,b)\cong 0.479.$$

[1]Competitive equilibria do exist for other values of the exercise price, away from $\hat{k}$, say $0.479<k<0.798$.

But then (2), which is necessary for the put and call options to provide a complete market, fails.

Alternatively, with the bond traded in the second period as the only asset, no revenue is transferred across states of nature; the competitive equilibrium allocation is

$$x^i=(x^i(2,a)\cong 1.4116, x^i(3,a)\cong 1.7150, x^i(2,b)\cong 1.9643, x^i(3,b)\cong 2.0776),$$

$$x^j=(x^j(2,a)\cong 1.5884, x^j(3,a)\cong 1.2850, x^j(2,b)\cong 2.0357, x^j(3,b)\cong 2.0123),$$

From (5), the prices of the bond at equilibrium are

$$q(2,a)\cong 0.823, \qquad q(2,b)\cong 0.464.$$

But then (4), which is necessary for the call or put option to be out of the money and thus for the option market to be inactive, fails.

That equilibria fail to exist in an open neighborhood of the economy $(\hat{e},\hat{k})$ follows from the uniqueness and continuity of equilibrium prices with respect to the allocation of initial endowments[2] and the exercise price of the option, which in turn follows from the gross substitutability which is displayed by the excess demand functions.

When competitive equilibria fail to exist, the options are not redundant assets.[3,4]

In our example, the bond is not traded in the first period; equivalently no one-period risk-free asset is available then. This is only for computational simplicity.

When the number and exercise prices of the options traded are not specified a priori, competitive equilibria exist generically, and the asset market is complete; the argument is standard.[5]

The example illustrates explicitly the difference between asset structures for which competitive equilibria may fail to exist but do exist generically, and asset structures for which the failure of existence may be robust to perturbations. Let

[2]We do not restrict perturbations to satisfy the condition $e(2,a)=e(2,b)$, which we imposed only in order to simplify the expression for equilibrium prices when the asset market is complete. Similarly, a perturbation of the bond payoffs does not alter the argument.

[3]The options are redundant assets when the economy aggregates. This is indeed the case when $\delta^i=\delta^j$. The equilibrium price of the bond is then independent of the distribution of revenue in the second period and hence of any transactions in the option market. An equilibrium exists and is unique, and the options can be 'priced' as is done in finance – Rubinstein (1976); in continuous time models, options are again priced as redundant assets – Black and Scholes (1973).

[4]Detemple and Seldon (1986) characterized the interaction between stock and option prices when competitive equilibria exist, yet exaggeration fails and options are not redundant.

[5]Ross (1976) developed the argument that options can 'complete the asset market' but did not solve for general equilibrium; McMannus (1986) exploited this construction to prove the generic existence of competitive equilibria when sufficiently many options complete the asset market.

$$e(2,a)=e(2,b) \quad \text{and} \quad e(3,a)=e(3,b),$$

yet

$$(l(2,a)/n(3,a)) \neq (l(2,b)/n(3,b)).$$

Consider a variant of the economy. The bond is traded in the first as well as in the second period; also, a risk-free asset is traded in the first period which pays off a unit of the good in the second. If, at equilibrium, $q(2,a) \neq q(2,b)$, the equilibrium is one with a complete asset market; but, from our earlier computation, at the complete market equilibrium, $q(2,a)=q(2,b)$ – a contradiction. If alternatively, at equilibrium, $q(2,a)=q(2,b)$, the risk-free asset is redundant; but with no trade in the first period asset market, $q(2,a) \neq q(2,b)$ – again a contradiction. Hence, with the bond traded in both periods and a risk-free asset, no competitive equilibia exist. However, a perturbation of the asset structure and the endowment allocation is sufficient to restore existence: at the complete market equilibrium the two-period asset and the one-period asset need not yield collinear payoffs in the second period and the equilibrium can be sustained. On the contrary, with the put and call options traded in the first period and the bond in the second, the failure of existence is, as we showed above, robust to perturbations.

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