

## OBSERVABILITY AND OPTIMALITY\*

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Submitted January 1988, accepted December 1988

Observability of an individual's excess demand function for assets and commodities as all prices and revenue vary suffices in order to recover his von Neumann–Morgenstern utility function. This is generically the case, even when the asset market is incomplete and the cardinal utility indices state dependent, as long as there are at least two commodities traded in spot markets at each state of nature. On the contrary, if the response of individuals' excess demand for assets as prices in spot commodity markets vary is not observable, recoverability fails when the asset market is incomplete. In particular, it is not possible to contradict the claim that the competitive allocation is fully optimal in spite of the incompleteness of the asset market. This provides a characterization of the efficacy of intervention in an economy with an incomplete asset market based on the information available to a planner from the observable behavior of individuals.

### 1. Introduction

A criterion of optimality should not employ knowledge of the characteristics of individuals that cannot be recovered from their observable behavior.

When the asset market is complete, competitive equilibria are fully optimal [Arrow (1951, 1953); Debreu (1951)]: no variation in the distribution of assets or commodities can improve on a competitive allocation. Thus, an argument against intervention can be made no matter how much information a central planner possesses. And this is fortunate, since observability of individuals' excess demand functions suffices in order to recover their preferences when the asset market is complete [Mas-Colell (1977)].

When the asset market is incomplete, competitive equilibria are typically constrained suboptimal [Geanakoplos and Polemarchakis (1986)]: there exist improving variations in the distribution of assets; variations, that is, which

\*This work was supported by National Science Foundation grants SES-8812051, SES 88-47005 and SES 87-06670. Also by the Deutsche Forschungsgemeinschaft, Gottfried-Wilhelm-Leibniz-Förderpreis during BoWo 88.

improve on the equilibrium allocation of every individual, after prices and quantities in the commodity spot markets adjust to maintain market clearing.

Constrained suboptimality indicates that the market fails to make optimal use of the available assets.<sup>1</sup> However, this claim ignores the possibly restricted information about individuals' characteristics under which the planner would operate. The preference characteristics of individuals are unobservable. What is observable, at least in principle, is the excess demand behavior of individuals. And when the asset market is incomplete, it is not obvious that excess demand behavior identifies preferences unambiguously.

In this paper we consider whether the information about the characteristics of individuals that can be recovered from their observable excess demand behavior suffices in order to improve on a competitive allocation.

We show first that if a central planner can observe excess demands for assets as well as commodities as all prices and revenue vary, he can recover information sufficient to determine improving interventions. This is so generically, provided there are at least two commodities traded in each community spot market; it is essential that the planner be able to observe the response in individuals' excess demand for assets and commodities as prices in commodity spot markets vary. Our method of proof extends a long tradition in the literature on the recoverability of von Neumann–Morgenstern preferences from demand functions. In this literature, counterexamples were constructed [McLennan (1979)], and restrictive conditions were introduced to guarantee recoverability [Dybvig and Polemarchakis (1981); Green, Lau and Polemarchakis (1979)], in particular, state-dependent cardinal utility indices were excluded. We show here that these earlier results depended crucially on the implicit assumption that only one commodity was available at each date–event and hence spot markets were degenerate.

On the contrary, it is evident that if a central planner can observe only individuals' excess demands for assets and commodities at the market clearing prices, in the absence of additional information about the characteristics of individuals, he will find it impossible to determine interventions that are sure to improve all individuals' welfare; this is so no matter how extensive the planner's power to intervene. We show that the same conclusion holds if the planner can observe the response of individuals' excess demands for assets to variations in asset prices and revenue while future commodity spot prices are held fixed at their equilibrium levels; also, the response of individuals' excess demands for commodities while spot commodity prices and revenue vary while asset prices are held fixed. What is unobservable is the response of individuals' excess demands for assets as spot

<sup>1</sup>Hart (1975) first constructed an example. Newbury and Stiglitz (1984) introduced the definition of constrained optimality which was formalized in Geanakoplos and Polemarchakis (1986).

commodity prices vary. If markets clear sequentially, then indeed a planner cannot have access to more information even if he can observe the adjustment process leading to equilibrium. And then, the information available to him is compatible with the claim that the equilibrium allocation is fully optimal in spite of the incompleteness of the asset market. Thus, unlike the argument against intervention when the asset market is complete, an argument here can be based on the limited information afforded to a central planner by the observable demand behavior of individuals in an incomplete asset market.

**2. The economy**

Exchange occurs over two periods. The resolution of uncertainty in the second period is described by states of nature  $s=0, 1, \dots, S$ .

Commodities  $l=0, 1, \dots, L$  are traded in spot markets in the second period after the resolution of uncertainty. For simplicity we suppose that there is no consumption in the first period when assets are traded. A commodity bundle in state  $s$  is  $x_s = (\dots, x_{s,l}, \dots)$ ; a commodity bundle is  $x = (\dots, x_s, \dots)$ .

Assets  $a=0, 1, \dots, A$  are traded in the first period and pay off in the second. Asset payoffs are denominated in commodity 0. The payoff of asset  $a$  in state  $s$  is  $r_{s,a}$ . The vector of payoffs of asset  $a$  is  $r_a = (\dots, r_{s,a}, \dots)$  a column vector; the vector of asset payoffs in state  $s$  is  $r_s = (\dots, r_{s,a}, \dots)$ , a row vector. The matrix of asset payoffs or asset structure is  $R = (\dots, r_a, \dots)$ . A portfolio is  $y = (\dots, y_a, \dots)$ .

An individual is characterized by his initial endowment  $e = (\dots, e_s, \dots)$ , a commodity bundle; and by his von Neumann–Morgenstern utility function<sup>2</sup>

$$W = \sum_{s=0}^S u_s,$$

defined on the consumption set of non-negative commodity bundles:  $x \geq 0$ ; the domain of the cardinal utility index  $u_s$  is the consumption set of non-negative commodity bundles in state  $s$ :  $x_s \geq 0$ .

We make the following assumptions concerning the asset structure:

A.1. The matrix of asset payoffs,  $R$ , has full column rank.

A.2. There are at least two assets:  $(A + 1) \geq 2$ , and two commodities:  $(L + 1) \geq 2$ .

<sup>2</sup>It does not affect the argument to restrict the utility function to the form

$$W = \sum_{s=0}^S \pi_s u_s;$$

that is, with objective probabilities and a state-independent cardinal utility index. However, our argument makes essential use of the additive separability of the objective function.

A.3. There exists a portfolio  $\bar{y}$  such that  $R\bar{y} > 0$ .

A.4. At each state  $s$ , some asset has non-zero payoff:  $r_s \neq 0$ .

Concerning the individual's characteristics we make the following assumptions:

A.5. The initial endowment is strictly positive:  $e \gg 0$ .

A.6. At each state  $s$ , the cardinal utility index  $u_s$  is a continuous, strictly monotonically increasing and strictly concave function which takes values on the extended real line. Everywhere on the interior of its domain,  $u_s$  is twice continuously differentiable,  $Du_s \gg 0$  and  $D^2u_s$  is negative definite. Along any sequence of strictly positive commodity bundles,  $(x_s^n; n=1, \dots)$  converging to a bundle  $\bar{x}_s$  on the boundary of the consumption set,

$$((x_s^n)' Du_s(x_s^n) / \|Du_s(x_s^n)\|) \rightarrow 0, \quad \text{while} \quad \|Du_s(x_s^n)\| \rightarrow \infty.$$

Assumption A.1 eliminates redundant assets that do not affect the argument; A.2 allows for trade in the asset and commodity spot markets and is essential for our argument; A.3 is a non-trivial restriction on the asset structure; evidently, it guarantees a direction of preference over portfolios for all objective functions that are monotonically increasing in consumption; A.4 guarantees that all states are accessible through the asset market; with individual objective functions separable across states; inaccessible states can be handled separately without affecting the argument. Note that assumptions A.3 and A.4 are together weaker than the alternative assumption that there exists a riskless portfolio: a portfolio  $\bar{y}$  such that  $R\bar{y} \gg 0$  or, after appropriate normalization of the price level at each state,  $R\bar{y} = (1, \dots, 1)'$ . Assumptions A.5 and A.6 are strong but standard.

*Remark 1.* Our construction allows for consumption in the first period as a special case. It suffices to interpret consumption in state  $s=0$  as consumption in the first period and to suppose that some asset, say  $a=0$ , pays off 1 at  $s=0$  and 0 at  $s=1, \dots, S$ . Note that assumption A.3 is then immediately satisfied.

The asset structure is complete if the matrix of asset payoffs has full row rank as well; equivalently, if and only if  $(A+1) = (S+1)$ . If  $(A+1) < (S+1)$ , the asset structure is incomplete.

An economy is a finite collection of heterogeneous individuals, together with an asset structure.

An allocation of commodities is an array of commodity bundles, one for

each individual, such that the commodity bundle of each individual lies in his consumption set while their sum, the aggregate commodity bundle, does not exceed the aggregate endowment.

An allocation of assets is an array of portfolios, one for each individual, whose sum is equal to 0.

The asset structure and the utility functions of individuals are held fixed. An economy can thus be characterized as an array of initial endowments, one for each individual. The space of economies is a finite dimensional manifold. We say that a property holds *generically* if it holds for a generic set of economies: an open set of full Lebesgue measure.

An allocation fails to be optimal if and only if there exists another allocation that dominates it: it yields at least as high a value for the objective function of each individual and strictly higher for some. We use at times the term fully optimal to draw the distinction between this standard notion of optimality and the notion of constrained optimality which we define below.

Commodity prices in state  $s$ , that is, spot commodity prices, are

$$p_s = (p_{s,0}, \hat{p}_s) = (p_{s,0}, p_{s,1}, \dots, p_{s,l}, \dots, p_{s,L}) \gg 0.$$

Commodity prices are  $p = (\dots, p_s, \dots)$ .

Asset prices are

$$q = (q_0, \hat{q}) = (q_0, q_1, \dots, q_a, \dots, q_A)$$

such that  $q' = \pi'R$  for some  $\pi = (\dots, \pi_s, \dots) \gg 0$ .

*Remark 2.* Asset prices  $q$  do not allow for arbitrage if and only if  $q'y > 0$  whenever  $Ry > 0$ . The domain of asset prices that do not allow for arbitrage consists of asset prices of the form  $q' = \pi'R$  for some  $\pi > 0$ ; its interior consists of asset prices of the form  $q' = \pi'R$  for some  $\pi \gg 0$ . Note that  $\pi$  is, up to normalization, the measure with respect to which asset prices satisfy the martingale property. Also, from assumption A.3 the requirement that  $q' = \pi'R$  for some  $\pi > 0$  does rule out some  $q$ : for example, for  $\pi > 0$ ,  $-\bar{y}' \neq \pi'R$ , while for  $\pi \gg 0$ ,  $0 \neq \pi'R$ .

Let prices be  $(q, p)$  and suppose that, in addition to his initial endowment at each state  $s$  in the second period, an individual receives in the first period exogenous revenue  $t$ . The individual expresses excess demand  $y$  for assets and  $z = (\dots, z_s, \dots)$  for commodities by solving the following constrained optimization problem:

$$\max W(e + z) = \sum_{s=0}^S u_s(e_s + z_s)$$

s.t.  $p'_s z_s = p_{s,0} r_s y$ , for each  $s$ ,

$$q'y = t. \quad (1)$$

Suppose that for some portfolio  $y$ ,  $p'_s e_s + p_{s,0} r_s y > 0$  for each  $s$ , while  $q'y = t$ . On a neighborhood of  $(q, p, t)$ , a solution to the optimization problem (1) exists and is unique. The excess demand function  $(y, z) = (y, \dots, z_s, \dots)$  is continuously differentiable;  $z \gg -e$ .

For simplicity, when  $t = 0$ , we write  $(y, z)(q, p, 0) = (y, z)(q, p)$ .

Competitive equilibrium prices are such that the aggregate excess demand is equal to 0.

Associated with competitive equilibrium prices, there is a unique allocation of commodities; also of assets.

A competitive equilibrium exists [Geanakoplos and Polemarchakis (1986)].

For fixed revenue  $t_s$  in state  $s$ , the individual expresses excess demand for commodities  $\zeta_s$  by solving the following constrained optimization problem:

$$\begin{aligned} \max u_s(e_s + \zeta_s) \\ \text{s.t. } p'_s \zeta_s = t_s. \end{aligned} \quad (2-s)$$

Suppose  $p'_s e_s + t_s > 0$ . On a neighborhood of  $(p_s, t_s)$  a solution to the optimization problem (2-s) exists and is unique. The excess demand function  $\zeta_s$  is continuously differentiable;  $\zeta \gg -e_s$ .

We refer to the excess demand function  $\zeta_s$  as the excess demand function in the spot commodity market in state  $s$ .

*Remark 3.* If the excess demand function for assets and commodities  $(y, z)$  is observable, so is the demand function  $\zeta_s$  in the spot commodity market in each state  $s$ . The argument is as follows: Given  $(p_s, t_s)$ , choose  $y$  such that  $t_s = p_{s,0} r_s y$ ; this is possible, since by assumption A.4  $r_s \neq 0$ . Then choose  $p_{s'}$ , for  $s' \neq s$ ,  $q$ , and  $t$  such that  $y = y(q, p, t)$ ; this is possible simply from the concavity of the utility function  $W$ . It follows that  $\zeta_s(p_s, t_s) = z_s(q, p, t)$ .

An allocation of revenue in state  $s$  is an array of revenues, one for each individual, whose sum is equal to 0.

An allocation of assets determines an allocation of revenue in each state.

Competitive equilibrium commodity prices relative to a fixed allocation of assets are commodity prices such that in each state the aggregate excess demand for commodities at the fixed allocation of revenue is equal to 0.

Associated with competitive equilibrium prices relative to a fixed allocation of assets there is a unique allocation of commodities.

A competitive equilibrium is fully optimal if and only if the associated allocation of commodities is fully optimal.

A competitive equilibrium is constrained suboptimal if and only if there exists an allocation of assets and a competitive equilibrium relative to this allocation such that the associated allocation of commodities dominates the competitive equilibrium allocation.

When the asset market is complete, all competitive equilibria are fully optimal.

When the asset market is incomplete, under regularity assumptions, typically, all competitive equilibria of an economy are constrained suboptimal. Furthermore, a dominating commodity allocation can be found in any neighborhood of the competitive allocation [Geanakoplos and Polemarchakis (1986)].

### 3. Observability and indeterminacy

We first suppose that the excess demand function of the individual is observable; in particular, that it is possible to observe the response in the individual's excess demand for assets and commodities as commodity prices vary.

*Proposition 1. Suppose that at fixed prices and revenue  $(q, p, t)$  it is possible to observe the following:*

- (i) *the excess demand for assets and commodities  $(y, z) = (y, z)(q, p, t)$ ;*
- (ii) *the first derivatives of the excess demand function with respect to asset and commodity prices and income:*

$$D_{(q, p, t)}(y, z)(q, p, t) = (D_q(y, z)(q, p, t), D_p(y, z)(q, p, t), D_t(y, z)(q, p, t)).$$

*Suppose further that in each state  $s$ , the excess demand function in the spot commodity market satisfies the following conditions at  $(p_s, t_s) = (p_s, p_s, 0, r_s, y)$ :*

- (iii) *the vectors  $\zeta_s(p_s, t_s)$  and  $D_{t_s}\zeta_s(p_s, t_s)$  are linearly independent;*
- (iv)  *$(D_q y(q, p, t) + D_t y(q, p, t))y' r_s \neq 0$ .*

It is then possible to recover the first and second derivatives of the objective function at  $x = z + e$ ,  $DW(x)$  and  $D^2W(x)$ , up to a positive scalar multiple.

*Proof.* First note that, following Remark 3, since the excess demand for assets and commodities  $(y, z)(q, p, t)$  and its first derivatives  $D_{(q, p, t)}(y, z)(q, p, t)$

are observable, so are the excess demand  $\zeta_s(p_s, t_s)$  and its derivatives  $D_{(p_s, t_s)} \zeta_s(p_s, t_s)$  in the commodity spot market in each state  $s$ .

To develop the argument for recoverability, we introduce two auxiliary steps:

*Step 1.* The solution of the individual optimization problem (2-s) at commodity prices and revenue  $(p_s, t_s)$  in the spot commodity market in state  $s$  is characterized by the following first-order conditions:

$$\begin{aligned} Du_s(e_s + \zeta_s) &= \lambda_s p_s, \\ p'_s \zeta_s &= t_s \end{aligned} \tag{3-s}$$

where  $\lambda_s > 0$  is the Lagrange multiplier associated with the budget constraints in the spot commodity market in state  $s$ .

For

$$\begin{bmatrix} D^2 u_s & -p_s \\ -p'_s & 0 \end{bmatrix}^{-1} = \begin{bmatrix} K_s & -v_s \\ -v'_s & \alpha_s \end{bmatrix}, \tag{4-s}$$

it follows from applying the implicit function theorem to eqs. (3-s) that

$$\begin{aligned} D_{p_s} \zeta_s(p_s, t_s) &= \lambda_s K_s - v_s \zeta'_s, \\ D_{t_s} \zeta_s(p_s, t_s) &= v_s, \\ D_{p_s} \lambda_s(p_s, t_s) &= -\lambda_s v'_s + \alpha_s \zeta'_s, \\ D_{t_s} \lambda_s(p_s, t_s) &= -\alpha_s. \end{aligned} \tag{5-s}$$

*Step 2.* The solution to the optimization problem (1) of the individual at prices and revenue  $(q, p, t)$  is characterized by the following first-order conditions:

$$\begin{aligned} \sum_{s=0}^s \lambda_s(p_s, t_s) r_s &= \lambda q, \\ q' y &= t, \end{aligned} \tag{6}$$

where  $\lambda > 0$  is the Lagrange multiplier associated with the budget constraint in the asset market. Evidently, for each state  $s$ ,  $t_s = p_{s,0} r_s y(q, p, t)$ ,  $z_s(q, p, t) = \zeta_s(p_s, t_s)$ , and  $\lambda_s(p_s, t_s)$  is the Lagrange multiplier obtained from the first-order condition (3-s). Note that  $\partial t_s / \partial y = r_{sa}$ , hence  $\partial y_s / \partial y_a = (\partial \lambda_s / \partial t_s) r_{sa} = -\alpha_s r_{sa}$ .

For



$$\begin{bmatrix} -\sum_{s=0}^S \alpha_s r'_s & -q \\ -q' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} K & -v \\ -v' & \alpha \end{bmatrix}, \tag{7}$$

it follows that

$$D_{p_s} y(q, p, t) = K r_s (\lambda_s v'_s - \alpha_s \zeta'_s), \tag{8-s}$$

while

$$D_q y(q, p, t) = \lambda K - v y', \tag{9}$$

$$D_t y(q, p, t) = v;$$

evidently the vector  $v_s$  and the scalar  $\alpha_s$  are obtained from (4-s) for each state  $s$ .

We can now complete the argument for recoverability:

Let  $\lambda$  be an arbitrary positive scalar; without loss of generality, let  $\lambda = 1$ .

From the observability of  $y$  and the first derivatives  $D_q y$  and  $D_t y$  in (9), we can recover the matrix of substitution effects  $K$  and the vector of income effects  $v$  in the asset market.

From the observability of the first derivatives  $D_{t_s} \zeta_s$  in (5-s), we can recover  $v_s$ .

From the observability of  $\zeta_s$  and the first derivatives  $D_{p_s} y$  in (8-s), we can recover the marginal utility of revenue,  $\lambda_s$  and its first derivative  $\alpha_s$  for each state  $s$ : This is possible because of conditions (iii) and (iv):

$$K r_s = (D_q y(q, p, t) + D_t y(q, p, t) y') r_s \neq 0,$$

while the vectors  $v_s = D_{t_s} \zeta_s(p_s, t_s)$  and  $\zeta_s(p_s, t_s)$  are linearly independent.

Substituting  $\lambda_s$  into the first-order condition (3-s) we obtain the first derivatives of the cardinal utility function at  $x_s = z_s + e_s$ ,  $Du_s(x_s)$ , for each state  $s$ .

Finally, substituting  $\lambda_s$  and  $\alpha_s$  into the first derivatives of the demand function in the commodity spot market (5-s) and (4-s) we obtain the second derivatives of the cardinal utility function at  $x_s$ ,  $D^2 u_s(x_s)$ , for each state  $s$ .

Since  $DW(x) = (\dots, Du_s(x_s), \dots)$ , while  $D^2 W(x) = \text{diag}(\dots, D^2 u_s(x_s), \dots)$ , this completes the argument for recoverability.

That recoverability is obtained up to a positive scalar multiple, i.e., up to a positive linear transformation follows from  $\lambda = 1$ .  $\square$

*Remark 4.* It is easy to show that conditions (iii) and (iv) hold generically as long as at least two commodities are traded in spot markets:  $(L + 1) \geq 2$ .

Furthermore, by continuity, if they are satisfied at a point, they are satisfied at a neighborhood of the point. Thus, the infinitesimal recoverability result of proposition 1 extends immediately to local recoverability.

*Remark 5.* Condition (iii) fails when a single commodity is traded in spot markets:  $(L+1)=1$ . This is the case that was treated in the earlier literature on recoverability

It follows from proposition 1 that knowledge of the excess demand functions of individuals for assets and commodities and of their first derivatives with respect to asset as well as commodity prices and revenue at the competitive equilibrium prices suffices in order to determine improving variations in the distribution of assets.

Next, we suppose that what is observable at prices and revenue  $(q, p, t)$  is the excess demand of the individual for assets and commodities and contemporaneous first derivatives: the first derivatives of the excess demand for assets with respect to asset prices and revenue while commodity prices are held fixed; also the first derivatives of spot commodity demands with respect to spot commodity prices. What is unobservable is the first derivatives of asset demands with respect to future spot commodity prices.

The following proposition 2 makes clear that the observability of contemporaneous derivatives does not allow for the recoverability of marginal utilities even up to a scalar multiple.

*Proposition 2.* Consider the individual with initial endowment  $e$  and objective function  $W = \sum_{s=0}^S u_s$ ; his excess demand function for assets and commodities is  $(y, z)$ .

At fixed asset and commodity prices and revenue  $(q, p, t)$ , let  $\tilde{\lambda} = (\dots, \tilde{\lambda}_s, \dots) \gg 0$  be a strictly positive vector and  $\tilde{\lambda} > 0$  a positive scalar such that

$$\sum_{s=0}^S \tilde{\lambda}_s r_s = \tilde{\lambda} q.$$

It is possible to attribute to the individual an objective function  $\tilde{W} = \sum_{s=0}^S \tilde{u}_s$ , such that his excess demand function for asset and commodities  $(\tilde{y}, \tilde{z})$  satisfies the following conditions at  $(q, p, t)$ :

- (i)  $(\tilde{y}, \tilde{z})(q, p, t) = (y, z)(q, p, t)$ ;
- (ii)  $D_{(q, t)} \tilde{y}(q, p, t) = D_{(q, t)} y(q, p, t)$ ;

- (iii)  $D_{(p_s, t_s)} \tilde{\zeta}_s(p_s, t_s) = D_{(p_s, t_s)} \zeta_s(p_s, t_s)$ , where  $t_s = p_s \cdot 0 r_s y$ , for all  $s$ ;
- (iv)  $D\tilde{u}_s(e_s + \tilde{z}_s) = \tilde{\lambda}_s p_s$ , for all  $s$ .

*Proof.* First observe that as long as  $Du_s(e_s + z_s) = \tilde{\lambda}_s p_s$  for each  $s$  while  $\sum_{s=0}^S \tilde{\lambda}_s r_s = \tilde{\lambda} q$ ,  $(\tilde{y}, \tilde{z}) = (y, z)$ ; this follows from the first-order necessary and sufficient condition for individual optimization in the commodity spot market in each  $s$ , (3- $s$ ), as well as in the asset market, (6).

Hence, it suffices to construct the cardinal utility index  $\tilde{u}_s$  for each  $s$  such that (ii) though (iv) are satisfied.

Without loss of generality we may suppose that  $\tilde{\lambda} = \lambda$ . It then follows from (7) and (9) that  $D_{(q, t)} y = D_{(q, t)} \tilde{y}$  if and only if  $\tilde{\alpha}_s = \alpha_s$  for each  $s$ ; evidently,  $\tilde{\alpha}_s$  is derived from  $D^2 \tilde{u}_s(e_s + z_s)$  as in (4- $s$ ). From (4- $s$ ) it also follows that  $\alpha_s = -v'_s D^2 u_s(e_s + z_s) v_s$  while  $\tilde{\alpha}_s = -v'_s D^2 \tilde{u}_s(e_s + z_s) v_s$ .

From the individual's optimization problem in state  $s$ , we can obtain the expenditure function  $m_s$ . For commodity bundles in the interior of the consumption set in state  $s$ , consider the function  $f_s$  defined by

$$f_s(x_s) = m_s(u_s(x_s); p_s).$$

Recall that the expenditure function  $m_s(u_s; p_s)$  is convex in  $u_s$ .

The function  $f_s$  is twice continuously differentiable and strictly monotonically increasing. Let  $x_s = e_s + z_s$ . From the definition of the expenditure function  $m_s$  it also follows as an identity that  $v'_s Df_s(x_s) = 1$ , while  $v'_s D^2 f_s(x_s) v_s = 0$ . Also, at  $x_s$  the function  $f_s$  is locally concave: Let  $kx'_s + (1-k)x''_s = x_s$ ,  $0 \leq k \leq 1$ ; then

$$\begin{aligned} f_s(kx'_s + (1-k)x''_s) &= f_s(x_s) = p_s x_s = k p_s x'_s + (1-k) p_s x''_s \geq k f_s(x'_s) \\ &\quad + (1-k) f_s(x''_s). \end{aligned}$$

It follows that  $D^2 f_s$  is negative semi-definite at  $x_s$ .

Consider the function  $\phi_s$  defined by

$$\phi_s(\hat{x}_s) = u_s(\hat{x}_s) + k_s f_s(\hat{x}_s)$$

on the interior of the consumption set in state  $s$ ;  $k_s$  is a scalar that we shall choose. Clearly

$$D\phi_s(\hat{x}_s) = Du_s(\hat{x}_s) + k_s Du_s m_s(u_s; p_s) Du_s(\hat{x}_s),$$

which is a scalar multiple of  $Du_s(\hat{x}_s)$ . Thus, in any region in which  $\phi_s$  is monotonically increasing it is a monotonically increasing transformation of  $u_s$ .

At  $x_s = e_s + z_s$ ,  $Du_s m_s = 1/\lambda_s$ ; thus

$$D\phi_s(x_s) = Du_s(x_s) + \frac{k_s}{\lambda_s} Du_s(x_s).$$

It follows that

$$D\phi_s(x_s) = \tilde{\lambda}_s Du_s(x_s)$$

for

$$k_s = (\tilde{\lambda}_s - 1)\lambda_s.$$

It is also clear that  $v'_s D^2\phi_s(x_s)v_s = v'_s D^2u_s(x_s)v_s$ . Letting  $W = \sum_{s=0}^S \phi_s$ , we have thus verified (ii) through (iv); it only remains to check that  $\phi_s$  is globally concave and monotonic.

By a straightforward computation,

$$\begin{aligned} D^2\phi_s(x_s) &= D^2u_s(x_s) + k_s D^2f_s(x_s) \\ &= D^2u_s(x_s) + \frac{k_s}{\lambda_s} D^2u_s(x_s) + k_s D^2_{u_s^2} m_s(u_s, p_s) Du_s(x_s) Du_s(x_s)'. \end{aligned}$$

If  $k_s < 0$ ,  $D^2\phi_s(x_s)$  is negative definite since  $D^2_{u_s^2} m_s > 0$  from the convexity of  $m_s$  in  $u_s$ , while  $(1 + k_s/\lambda_s) = \tilde{\lambda}_s > 0$ . If  $k_s > 0$ , the negative definiteness of  $D^2\phi_s(x_s)$  follows from the concavity of  $f_s$  at  $x_s$ .

It is then straightforward to modify the function  $\phi_s$  outside a neighborhood of  $(x_s)$  into a cardinal utility index  $\tilde{u}_s$  that is globally monotonic and concave. This does not alter (i)–(iv).  $\square$

Let  $(q^*, p^*)$  be competitive equilibrium prices. Suppose that it is only possible to observe each individual's excess demands for assets and commodities at the equilibrium prices and only contemporaneous first derivatives: the first derivatives of the excess demand for assets with respect to asset prices and revenue, while commodity prices are held fixed at their equilibrium levels; also, the first derivatives of the excess demand for commodities with respect to spot commodity prices and revenue, while asset prices are held fixed at their equilibrium levels. It follows from proposition 2 that it is possible to attribute characteristics to individuals such that their observable behavior is unchanged while the gradients of their objective functions at equilibrium are colinear; equivalently, it cannot be ruled out that the competitive equilibrium allocation is fully optimal.

#### 4. Conclusion

The criterion of optimality appropriate to a particular market structure

should take into account the constraints under which the market operates. When the asset market is incomplete, constrained optimality does indeed restrict attention to the available assets and does not allow for instruments that the market does not have at its disposal. It ignores, however, informational constraints: With a restricted set of assets, the observable demand behavior of individuals need not suffice for a central planner to recover their unobservable characteristics – their preferences and endowments.

In this paper we considered whether the observable demand behavior of individuals suffices in order to determine improving interventions in the asset market.

We distinguished two cases. In the first, a central planner could observe the response in individuals' excess demands for assets and commodities as all prices varied; in particular, as relative prices in spot commodity markets varied. We showed that as long as at least two commodities were traded in each spot commodity market, it was possible for the planner to recover the von Neumann–Morgenstern objective functions of individuals and thus determine improving interventions. In the second case, the planner could not observe the response of individuals' excess demands for assets as commodity prices varied. We showed that based on this restricted information, a central planner could not contradict the claim that, in spite of the market incompleteness, a competitive allocation was fully optimal; in particular, he could not determine improving interventions.

## References

- Arrow, K.J., 1951, An extension of the basic theorems of classical welfare economics, in: J. Neyman, ed., *Proceedings of the second Berkeley symposium on mathematical statistics and probability* (University of California Press, Berkeley, CA) 507–532.
- Arrow, K.J., 1953, Le rôle des valeurs boursières pour la repartition la meilleure des risques, *Econometrie, Colloques Internationaux du CNRS*, 11, 41–47.
- Debreu, G., 1951, The coefficient of resource utilization, *Econometrica* 19, no. 2, 273–292.
- Dybvig, P.L. and H.M. Polemarchakis, 1981, Recovering cardinal utility, *Review of Economic Studies* 48, no. 1, 159–166.
- Geanakoplos, J.D. and H.M. Polemarchakis, 1986, Existence, regularity and constrained suboptimality of competitive allocations when the asset structure is incomplete, in: W.P. Heller, R.M. Starr and D.A. Starrett, eds., *Uncertainty, information and communication: Essays in honor of K.J. Arrow*, Vol. 3 (Cambridge University Press, New York) 65–95.
- Green, J.R., L.J. Lau and H.M. Polemarchakis, 1979, On the identifiability of the von Neumann–Morgenstern utility function from asset demands, in: J.R. Green and J.A. Scheinkman, eds., *Equilibrium, growth and trade: Essays in honor of L. McKenzie* (Academic Press, New York) 151–161.
- Hart, O.D., 1975, On the optimality of equilibrium when the market structure is incomplete, *Journal of Economic Theory* 11, no. 3, 418–443.
- McLennan, A., 1979, Examples of von Neumann–Morgenstern utility functions not recoverable from asset demands, *Economics Letters* 3, no. 2, 149–152.
- Mas-Colell, A., 1977, The recoverability of consumer preferences from market demand behavior, *Econometrica* 45, no. 6, 1409–1430.
- Newbury, D.M.G. and J.E. Stiglitz, 1984, Pareto inferior trade and optimal trade policy, *Review of Economic Studies* 51, no. 1, 1–12.