JOURNAL OF ECONOMIC THEORY 46, 414–421 (1988)

Portfolio Choice, Exchange Rates, and Indeterminacy

H. M. POLEMARCHAKIS*

Graduate School of Business, Columbia University, New York, New York 10027 and Churchill College, University of Cambridge, Cambridge, England

Received May 18, 1987; revised October 7, 1987

Consider an exchange economy under uncertainty. The asset market is incomplete: there are fewer assets, A, than states of nature, S. Assets are nominal and are denominated in N distinct "currencies." Competitive equilibria are distinct if the associated allocations are distinct. An economy displays K degrees of indeterminacy if the set of distinct competitive equilibria contains a K-dimensional open set. Generically, the economy displays $NS - A(N-1) - N$ degrees of indeterminacy. *Journal of Economic Literature* Classification Numbers: 021, 431. © 1988 Academic Press, Inc.

INTRODUCTION

When competitive equilibrium allocations are indeterminate, not locally unique, market clearing fails to determine the allocation of resources even for small changes in the economy. It was the contribution of Debreu [7] to show that in economies with a complete asset market (equivalently, with a complete market in contingent commodities), generically, indeterminacy does not arise.

Indeterminacy arises generically when the asset market is incomplete and assets are nominal:[1] denominated in abstract units of account. This was shown by Cass [3]. The intuition is as follows: It is evident that the price level (equivalently, the purchasing power of the units of account) at each state of nature is indeterminate. When the asset market is incomplete, different choices for the price level at each state of nature alter the redistributions of revenue across states of nature which are attainable through the available nominal assets. With sufficiently diverse individuals,

* I thank A. Brandenburger and Y. Balasko for very helpful conversations. This work was supported by National Science Foundation Grant SES 87-06670.

[1] When assets are real, i.e., denominated in commodities, the incompleteness of the asset market does not interfere with the argument for the generic determinacy of equilibria (Geanakoplos and Polemarchakis [11]). Thus the equivalence between nominal and real assets which prevails when the asset market is complete (Arrow [1]; Debreu [6]) breaks down when the asset market is incomplete.

0022-0531/88 $3.00

the nominal indeterminacy of the price level is generically reflected on the allocation of resources at equilibrium and indeterminacy arises.

Competitive equilibria are distinct if the associated allocations are distinct.

An economy displays K degrees of indeterminacy if the set of distinct competitive equilibria contains a K-dimensional open set.[2]

For an exchange economy in which assets are nominal, the asset market is incomplete: there are fewer assets, A, than states of nature, S, and the number of individuals, H, is sufficiently large: it exceeds the number of assets, A, Geanakoplos and Mas-Colell [10] showed that there are generically $S-1$ degrees of indeterminacy. This is possibly surprising: the number of assets, A, does not affect the degrees of indeterminacy (as long as $A<S$). It can be understood by observing that the degrees of freedom are precisely $S-1$: the price level at each state of nature, minus one which amounts to a normlization and has no bearing on the allocation.[3]

Here we allow assets to be dominated in N distinct units of account or "currencies." In addition to the purchasing power of one currency, the rates of exchange across currencies may now vary. We show that, generalically, the economy displays $NS-A(N-1)-N$ degrees of indeterminacy. In particular, neutrality, the separation of real from nominal variables at equilibrium, fails.

THE ECONOMY

Economic activity extends over two periods, under uncertainty. States of nature are $s=1, ..., S$; all uncertainty is resolved in the second period. Commodities are $l=1, ..., L$. Consumption occurs only in the second period. A commodity bundle is thus[4] $x=(..., x(s), ...)'$, where $x(s)=(..., x_l(s), ...)'$.

Individuals are $h=0, ..., H$. An individual, h, is characterized by his objective function U^h over commodity bundles, and by his endowment w^h, a commodity bundle. Assets are $a=1, ..., A$. Distinct monetary authorities, "countries," are $n=1, ..., N$. Country issues A_n assets, $a=\underline{a}_n, ..., \bar{a}_n$, where $\underline{a}_n=(\sum_{n'=1}^{n-1} A_{n'-1})+1$ and $\bar{a}_n=\sum_{n'=1}^{n} A_{n'}$. The payoffs of an asset are denominated in the currency of the country which issues the asset. The vector of payoffs of asset a is $r_a=(..., r_a(s), ...)'$. The asset structure in currency n is $R_n=(r_{\underline{a}_n}, ..., r_{\bar{a}_n})$.

[2] A K-dimensional open set is the image of a continuously differentiable, one-to-one function with domain an open neighborhood in K-dimensional euclidean space.

[3] In this argument the matrix of nominal assets payoffs is held fixed. Alternatively, Balasko and Cass [2] and Cass [4] fixed the rates of return of nominal assets; they showed that there are generically $S-A$ degrees of indeterminacy.

[4] All vectors are column vectors; $'$ denotes the transpose.

We make the following assumptions:

— The objective function U^h is a twice continuously differentiable function defined over strictly positive commodity bundles. Everywhere on this domain, DU^h is strictly positive, while D^2U^h is negative definite. The closures of the indifference hypersurfaces of U^h are contained in its domain of definition. The endowment w^h is strictly positive.

— The asset structure in currency n, R_n, is in general position: any submatrix of order A_n has full rank.

– There exists a portfolio with positive returns in all states. Without loss of generality, this portfolio coincides with the asset $a=1$; $r_1>0$.

An allocation is $\underset{\sim}{x}=(..., x^h, ...)$ such that $\sum_{h=0}^{H}(x^h-w^h)\leqslant 0$. We hold the preference characteristics of individuals and the asset structure fixed. An economy is thus described by $\underset{\sim}{w}=(..., w^h, ...)$, the distribution of endowments. The space of economies is W; it is an $(H+1)\,LS$-dimensional open set. A property holds generically if and only if it holds for an open set of economies of full lebesgue measure.

We may ignore asset prices and exchange rates in the first period by normalization, and consider only commodity prices and exchange rates in the second period. Exchange rates are $e=(..., e_n, ...)$, where $e_n=(..., e_n(s), ...)'$. The rate of exchange of currency n of currency n' at state s is $e_{n'}(s)/e_n(s)$. At exchange rates e, the matrix of asset returns is

$$R(e)=(..., E_nR_n, ...),$$

where $E_n=\mathrm{diag}(e_n)$. Commodity prices are $p=(..., p(s), ...)'$, where $p(s)=(..., p_l(s), ...)'$.

At exchange rates and commodity prices (e, p) individual h solves the following optimization problem:[5]

$$\begin{aligned} &\text{Max } U^h(x)\\ &\text{s.t. } p'(x-w^h)=0\\ &\qquad p'{*}(x^h-w^h)\in[R(e)], \end{aligned} \tag{1}$$

where

$$p'{*}(x-w^h)=(..., p(s)'(x(s)-w^h(s)), ...)'.$$

Note that from the strict concavity of the objective function and the convexity of the constraint set, a solution to (1) is unique. Considering the individual optimization problem (1), it is natural to say that exchange rates

[5] For matrices or collections of vectors, [ ] denotes the (column) span.

e and e' are equivalent, $e \sim e'$, if $[R(e)] = [R(e')]$: if $e \sim e'$, for any p, the constraint sets at (e, p) and (e', p) coincide. The asset market is complete if and only if $\dim[\mathrm{R}(e)] = S$. When $A < S$, necessarily $\dim[\mathrm{R}(e)] < S$ and the asset market is incomplete. A competitive equilibrium is a pair $(\bar{e}, \bar{p})$ of exchange rates and commodity prices such that $\bar{x}$ is an allocation, where $\bar{x}^h$ solves (1) at $(\bar{e}, \bar{p})$.

Remark. The choice between assets denominated in different currencies is a pure portfolio choice. Individuals have no liquidity preference for assets denominated in the currency of their country which, as a consequence, we do not specify. Similarly, commodities are costlessly tradeable across countries. At each state relative commodity prices are equal across countries and purchasing power parity prevails; thus we specify commodity prices without reference to a particular country.

INDETERMINACY

Two competitive equilibria, $(\bar{e}, \bar{p})$ and $(\bar{\bar{e}}, \bar{\bar{p}})$, are distinct if the associated allocations, $\bar{x}$ and $\bar{\bar{x}}$, respectively, are distinct: $\bar{x} \neq \bar{\bar{x}}$. An economy displays K degrees of indeterminacy if the set of distinct competitive equilibria contains a K-dimensional open set.

PROPOSITION. *Suppose $A < S$: the asset market is incomplete; also the number of individuals is large: $A < H$. Generically, the economy displays $NS - A(N-1) - N$ degrees of indeterminacy.*

Proof. We break down the argument into four steps.

Step 1. First, we establish a normalization for exchange rates and then for commodity prices. Write $e_n = (e_{1,n}, ..., e_{n',n}, ..., e_{n,n}, ..., e_{N,n}, e_{N+1,n})'$, where $e_{n',n}$ is of length $A_{n'}$ and $e_{N+1,n}$ of length $S - A$. Similarly, write $R_n = (R_{1,n}, ..., R_{n',n}, ..., R_{n,n}, ..., R_{N,n}, R_{N+1,n})'$. Let[6] $\mathbf{G} = \{e: e_{n',n} = 0, n' = 1, ..., n-1, n+1, ..., N, e_{n,n} \gg 0, e_{N+1,n} \gg 0\}$; that is, for $e \in \mathbf{G}$, the matrix of returns is of the form

$$R(e) = \begin{bmatrix} E_{11} R_{11} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & E_{n,n} R_{n,n} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & E_{N,N} R_{N,N} \\ E_{N+1,1} R_{N+1,1} & \cdots & E_{N+1,n} R_{N+1,n} & \cdots & E_{N+1,N} R_{N+1,N} \end{bmatrix},$$

where $E_{n,n} = \mathrm{diag}(e_{n,n})$ and $E_{N+1,n} = \mathrm{diag}(e_{N+1,n})$. Evidently, $\mathbf{G}$ is a

$NS-A(N-1)$-dimensional open set. So is the set $\mathbf{F}=\{e\in\mathbf{G}$: the matrices $(E_{n,n}R_{n,n}, E_{N+1,n}R_{N+1,N})'$ are in general position$\}$; this follows from the assumption that the matrices R_n, and hence also the submatrices $(R_{n,n}, R_{N+1,n})'$ are in general position.

Observe now that for $e, e'\in\mathbf{F}$, $e\sim e'$ if and only if $e'_n=\delta_n e_n$, for some scalar $\delta_n>0$. To see this suppose that $e\sim e'$. Then there exists a $(A\times A)$ matrix of full rank, T, such that $R(e)\,T=R(e')$. Restricting our attention to the first A rows, we obtain that

$$T=\begin{bmatrix} T_{1,1} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & T_{n,n} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & T_{N,N} \end{bmatrix},$$

where $T_{n,n}=(E_{n,n}R_{n,n})^{-1}(E'_{n,n}R_{n,n})$. Let $\delta_{n,n}=e'_{n,n}/e_{n,n}$ and $\delta_{N+1,n}=e'_{N+1,n}/e_{N+1,n}$, where vectors are divided component-by-component, and let $\Delta_{n,n}=\text{diag}(\delta_{n,n})$ and $\Delta_{N+1,n}=\text{diag}(\delta_{N+1,n})$. It follows that $T_{n,n}=(E_{n,n}R_{n,n})^{-1}\Delta_{n,n}(E_{n,n}R_{n,n})$. From the assumption of general position, we may write $(E_{N+1,n}R_{N+1,n})=\Lambda'_n(E_{n,n}R_{n,n})$, where Λ_n is a matrix with no zero elements. But then, $(E_{N+1,n}R_{N+1,n})\,T_{n,n}=E'_{N+1,n}R_{N+1,n}$; or $\Lambda_n(E_{n,n}R_{n,n})(E_{n,n}R_{n,n})^{-1}\Delta_{n,n}(E_{n,n}R_{n,n})=\Delta_{N+1,n}\Lambda'_n(E_{n,n}R_{n,n})$. It follows that $\delta_{n,n}=\delta_n(1, ...,)$ and $\delta_{N+1,n}=\delta_n(1, ..., 1)$, for some positive scalar δ_n, as claimed.

Let $\mathbf{H}=\{e: e\sim e'$ for some $e'\in\mathbf{F}\}$. We take the domain of exchange rates to be the quotient space $\mathbf{E}=\mathbf{H}/\sim$. It can be identified with an open set of dimension $NS-A(N-1)-N$. For simplicity we do not distinguish between exchange ates and their equivalence class in $\mathbf{E}$. Note that for $e\neq e'$, $[R(e)]\neq[R(e')]$.

Remark. If $e_n(s)=0$, currency n has no purchasing power in state s. This can be interpreted as hyperinflation. Note, however, that exchange rates which display hyperinflation in some states may well be equivalent to exchange rates e' which do not.[6]

We take the domain of commodity price systems to be $\mathbf{P}=\{p: p\gg 0$ and $\|p\|=1\}$; equivalently, $\mathbf{Q}=\{q: (1, q)/\|(1, q)\|\in\mathbf{P}\}$. Either can be identified with an open set of dimension $LS-1$.

Consider the alternative individual optimization problem

$$\begin{aligned} &\text{Max } U^h(x) \\ &\text{s.t. } p'(x-w^h)=0. \end{aligned} \tag{2}$$

[6] We restrict our attention to non-negative exchange rates only for simplicity.

Suppose individual $h=0$ expresses excess demand $f^0 = x^0 - w^0$ by solving (2), while individuals $h=1, ..., H$ express excess demand $f^h = x^h - w^h$ by solving (1). Let $f = f^0 + \sum_{h=1}^{H} f^h$ be the aggregate excess demand thus defined. The projection of the aggregate excess demand function to components other than the first is g.

This completes the normalization of exchange rates and commodity prices: If $f(\bar{e}, \bar{p}; \underset{\sim}{w}) = 0$, $(\bar{e}, \bar{p})$ is competitive emquilibrium for the economy $\underset{\sim}{w}$. Conversely, if $(\bar{e}, \bar{p})$ is a competitive equilibrium for the economy $\underset{\sim}{w}$, there exist exchange rates $\bar{\bar{e}}$ and commodity prices $\bar{\bar{p}}$ such that for all individuals the solution to (1) at $(\bar{\bar{e}}, \bar{\bar{p}})$ and $(\bar{e}, \bar{p})$ coincide, while $f(\bar{\bar{e}}, \bar{\bar{p}}; \underset{\sim}{w}) = 0$.

Since $f=0$ if and only if $g=0$, while the price domain **P** and **Q** are isomorphic we may identify competitive equilibria with pairs $(\bar{e}, \bar{q})$ such that $g(\bar{e}, \bar{q}) = 0$. We do so later.

Step 2. In this step we simply oberve that for every economy $\underset{\sim}{w}$ and any exchange rates $\bar{e}$, there exist commodity prices $\bar{p}$ such that $f(\bar{p}, \bar{e}; \underset{\sim}{w}) = 0$; $(\bar{e}, \bar{p})$ are competitive equilibrium exchange rates and prices. This follows from the continuity of the excess demand function $f(p, \bar{e}; \underset{\sim}{w})$ in commodity prices, Walras' law: $p'f(p, \bar{e}; \underset{\sim}{w}) = 0$, and the boundary behavior: $\lim_{p \to \partial \mathbf{P}} f(p, \bar{e}; \underset{\sim}{w}) = \infty$; the boundary behavior follows from the boundedness from below of the excess demand function $f^h(p, \bar{e}; \underset{\sim}{w}^h)$, $h = 1, ..., H$, and the strict positivity of the endowment and the strict monotonicity of the objective function of individual $h=0$.[7]

Step 3. In this step we characterize the conditions under which $(\bar{e}, \bar{q})$ and $(\bar{\bar{e}}, \bar{\bar{q}})$, both elements of $g^{-1}(0; \underset{\sim}{w})$, yield distinct competitive equilibria. Let

$$y^h(e, q; w^h) = (R'(e)\, R(e))^{-1}\, R(e)\, p'^*(e, q; w^h);$$

for $h=1, ..., H$, $y^h(e, q; \underset{\sim}{w})$ is the portfolio of assets held by the individual.

Suppose $(\bar{e}, \bar{q})$ and $(\bar{\bar{e}}, \bar{\bar{q}})$ are competitive equilibria such that the associated portfolio holdings $(y^{-1}, ..., \bar{y}^H)$ and $(\bar{\bar{y}}^1, ..., \bar{\bar{y}}^H)$, respectively, satisfy the condition $\dim[\bar{y}^1, ..., \bar{y}^H] = \dim[\bar{\bar{y}}^1, ..., \bar{\bar{y}}] = A$. Then, $(\bar{e}, \bar{q})$ and $(\bar{\bar{e}}, \bar{\bar{q}})$ are distinct competitive equilibria (if and) only if $(\bar{e}, \bar{q}) \neq (\bar{\bar{e}}, \bar{\bar{q}})$. For suppose that the associated allocations coincide: $\underset{\sim}{\bar{x}} = \underset{\sim}{\bar{\bar{x}}}$. From (2), the

[7] The existence argument is a simple variant of existence arguments for economies in which the asset market is possibly incomplete yet the matrix of asset payoffs has rank independent of the spot commodity prices in **P**. This is the case when assets are nominal (Chae [5]; Werner [13]), also when assets are real but are denominated in the same numeraire commodity (Geanakoplos and Polemarchakis [11]). When the rank of the matrix of asset payoffs varies with spot commodity prices, only generic existence can be obtained (Duffie and Shafer [9]). In alternative frameworks, even generic existence may fail (Polemarchakis and Ku [12]).

optimization problem solved by individual $h=0$, and by the smoothness of his objective function U^0, it follows that $\bar{q}=\bar{\bar{q}}$. Since $R(\bar{e})\,\bar{y}^h = R(\bar{\bar{e}})\,\bar{\bar{y}}^h$ for $h=1, ..., H$ while $\dim[\bar{y}^1, ..., \bar{y}^H] = \dim[\bar{\bar{y}}^1, ..., \bar{\bar{y}}] = A$, $[R(\bar{e})] = [R(\bar{\bar{e}})]$; it follows that $\bar{e}=\bar{\bar{e}}$.

Step 4. We complete the argument as follows: We show that for each economy $w \in W^*$, an open set of full lebesque measure, there exists a non-empty, open set of exchange rates $V \subset \mathbf{E}$ and a continuously differentiable parametrization of competitive equilibria $(p(e), e)$ such that the associated portfolio holdings satisfy the rank condition $\dim[y^1(e), ..., y^H(e)] = A$.

Consider the excess function g with domain $\mathbf{Q} \times \mathbf{E} \times \mathbf{W}$ and range $(LS-1)$-dimensional euclidan space. Consider also the augmented function $\gamma = (g, \det(y^1, ..., y^A))$ with the same domain and range LS-dimensional euclidean space, where $\det(y^1, ..., y^A)$ denotes the determinant of the matrix whose columns are the portfolio holdings of individuals $h=1, ..., A$. Both functions are $k-2$ times continuously differentiable: this follows as in Debreu [8].

Both functions are transverse to the origin. That $g \pitchfork 0$ follows simply by perturbing the endowment of individual $h=0$. To show that $\gamma \pitchfork 0$ we must show in addition that there exists a perturbation of endowments which changes $\det(y^1, ..., y^A)$ while leaving g unaffected. But this is straightforward: perturb$(y^1, ..., y^A)$; compensate by adjustng the endowments of individuals $h=1, ..., A$ so that their commodity demands remain unchanged; to insulate g, perturb the endowment of individual $h=0$; finally, to maintain the budget constraint of individual $h=0$, simply adjust his endowment of commodity $l=1$ in state $s=1$ the excess demand for which does not appear in the function g.

From the transversal density theorem it follows now that there exists an open set $V \subseteq E \times W$ of full lebesque measure for which $g_{(e,w)} \pitchfork 0$ and also $\gamma_{(e,w)} \pitchfork 0$; from the boundary behavior of individual $h=0$, V is an open set. Let $W^* = \text{proj}_w V$; it is an open set of full lebesque measure. Consider an economy $\bar{w} \in W^*$. Pick $\bar{e} \in E$ such that $(\bar{e}, \bar{w}) \in V$. From the existence argument, $g^{-1}_{(\bar{e},\bar{w})}(0) \neq \varnothing$. On the other hand, $\gamma^{-1}_{(\bar{e},\bar{w})}(0) = \varnothing$: $\gamma_{(\bar{e},\bar{w})} \pitchfork 0$ and the dimension of the range exceeds the dimension of the domain. From the implicit function theorem it follows that there exists a neighborhood $V(\bar{w}) \subseteq E$ and a continuously differentiable parameterization $(p(e), e)$ of competitive equilibria with domain $V(\bar{w})$. Since $g_{\bar{w}}(p(e), e) = 0$ while $\det(y^1(e), ..., y^A(e)) \neq 0$, $(p(\bar{e}), \bar{e})$ and $(\bar{\bar{p}}), \bar{\bar{e}})$ are distinct competitive equilibria as long as $\bar{e}$ and $\bar{\bar{e}}$ are distinct exchange rates.

Special Cases and Extensions

In the case of single currency, $N=1$; the economy then displays, generically, $S-1$ degrees of indeterminacy.[8] In the case of pure currencies, $A=N$; the economy then displays, generically, the maximal degree of indeterminacy,[9] $A(S-A)$. If, addition to the A nominal assets there are B real assets, with $A+B<S$, the economy displays, generically, $N(S-B)-A(N-1)-N$ degrees of indeterminacy. This follows from a straightforward extension of our argument.[10] It is an open question to develop the argument in economies which extend over infinite time and in which currencies, fiat money, need not be in zero net supply.

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[8] For the case of a single currency, Geanakoplos and Mas-Colell [10] have extended the characterization of undeterminacy to allow for matrices of asset returns which are not in general position. An analogous extension in the case of multiple currencies should be straightforward.

[9] Variations in exchange rates generate in this case the entire grassmanian of A-planes in S-dimensional space.

[10] To avoid the (non-generic) failure of existence which occurs when real assets are denominated in multiple commodities we may assume for simplicity that $L=1$; there is just a single, aggregate consumption good.