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DISAGGREGATION OF EXCESS DEMAND UNDER ADDITIVE SEPARABILITY*

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A differentiable function satisfying Walras' law and homogeneity of degree zero can be disaggregated, at a point, into excess demand functions, each derived by the maximization of a regular and additively separable utility function subject to the budget constraint.

It is well known[1] that, given an arbitrary function f defined on the price domain $\mathring{\mathscr{R}}^l_+$, which satisfies Walras' law, homogeneity of degree zero and a boundedness condition, there exists a set of l agents with monotone, convex preferences, whose aggregate excess demand function coincides with f. It is the purpose of the present paper to demonstrate that this disaggregation result holds, at a point,[2] even if agents are required to possess preferences representable by *additively separable* utility functions.

The result is of interest for two reasons. First, it demonstrates that the restriction of individuals' preferences to the class representable by additively separable utility functions does not in any way provide a way out of the possibility of arbitrary aggregate excess demand behavior, at least at a point. Second, and more important, an appropriate interpretation of the variables reveals that the aggregate excess demand of risk averse, von Neumann–Morgenstern investors for elementary securities need not satisfy — at a point — any properties beyond homogeneity and Walras' law.

The extension of the result to open neighborhoods is as usual an open question. Furthermore, the interpretation of the results in terms of the maximization of von Neumann–Morgenstern utility functions captures only a very special case of the problem. In particular, the extension to complex securities is not immediate; neither is the extension to the combined consumption portfolio problem.

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[1]The problem was first posed and analyzed by Sonnenschein (1972, 1973). It was further elaborated on by Debreu (1974), Geanakoplos and Polemarchakis (1980), McFadden, Mas-Colell, Mantel and Richter (1974), and Mantel (1974, 1975, 1976).

[2]Since the analysis is local, a differentiability assumption will be made on f.

An agent is characterized by a utility function u defined on the consumption set $\mathring{\mathscr{R}}^l_+$ and an endowment vector w, a point in $\mathring{\mathscr{R}}^l_+$. The utility function is said to be *regular and additively separable* if and only if the following hold:

— $u(y)=\sum_{j=1}^l u_j(y_j)$,

— u_j is a twice continuously differentiable function and satisfies $u'_j>0$, $u''_j<0$, $j=1,\dots,l$,

— the closure of the indifference hypersurface through y relative to $\mathscr{R}^l$ is contained in $\mathring{\mathscr{R}}^l_+$, for all $y\in\mathring{\mathscr{R}}^l_+$.

A regular and additively separable utility function is said to be *stationary* if and only if it is for the form

$$u(y)=\sum_{j=1}^{l} v(y_j).$$

Faced with prices $p\in\mathring{\mathscr{R}}^l_+$, the agent (u,w) expresses excess demand $x(p)$ by solving the following problem:

$$\max_{x\in\mathring{\mathscr{R}}^l_+-\{w\}} \quad u(x+w)$$

$$\text{s.t.} \quad p^t x=0. \tag{1}$$

A solution to (1) exists, is unique and is given by

$$u'_j(w_j+x_j)=\lambda p_j, \qquad j=1,\dots,l,$$

$$p^t x=0, \tag{2}$$

where

$$\lambda>0. \tag{3}$$

The excess demand function is continuously differentiable. To compute the Jacobian we totally differentiate (2),

$$\begin{aligned} &u''_j(w_j+x_j)\mathrm{d}x_j-p_j\mathrm{d}\lambda=\lambda\,\mathrm{d}p_j, \qquad j=1,\dots,l,\\ &-p^t\mathrm{d}x=x^t\mathrm{d}p. \end{aligned} \tag{4}$$

Consequently,

$$\mathrm{D}x=K-vx^t, \tag{5}$$

where

$$\begin{bmatrix} S & -v \\ -v^t & e \end{bmatrix} = \begin{bmatrix} D^2u & -p \\ -p^t & 0 \end{bmatrix}^{-1}, \tag{6}$$

$$K = \lambda S, \tag{7}$$

and

$$D^2u = \begin{bmatrix} u_1'' & & \\ & \ddots & \\ & & u_l'' \end{bmatrix}. \tag{8}$$

The following propositions are well known[3] and they provide the necessary and sufficient conditions for a function $x(p)$ to be, at least at a point, the excess demand function derived by the maximization of a regular and additively separable utility function, subject to the budget constraint.

Proposition A. Let $x(p)$ be the excess demand function derived by the maximization of a regular, additively separable utility function u, subject to the budget constraint $p^t x(p) = 0$. Then

$$\mathrm{D}x(p) = \mu(p)[v(p)v(p)^t - V(p)P^{-1}] - v(p)x(p)^t, \tag{9}$$

where $\mu(p)$ is a strictly positive scalar, $v(p)$ is a vector in $\mathring{\mathscr{R}}^l_+$ such that $p^t v(p) = 1$ and $V(p)$ and P are the diagonal matrices formed out of the vectors $v(p)$ and p, respectively.

Remark 1. The interpretation of the variables above is clear. The vector $v(p)$ is the vector of income effects; the scalar $\mu(p)$ is equal to the ratio $\lambda(p)/e(p)$, where $\lambda(p) > 0$ is the marginal utility of income and $-e(p) < 0$ its first derivative. Furthermore, $u_j' = \lambda p_j$ while $u_j'' = -ep_j/v_j$; thus $v(p) \in \mathring{\mathscr{R}}^l_+$.

Proposition B. Let $\bar{x}$ be a vector in $\mathscr{R}^l$, and $\bar{p}$ and $\bar{v}$ vectors in $\mathring{\mathscr{R}}^l_+$, such that $\bar{p}^t\bar{x} = 0$, $\bar{p}^t\bar{v} = 1$, $\bar{\mu}$ a strictly positive scalar, and $\bar{V}$ and $\bar{P}$ the diagonal matrices formed out of the vectors $\bar{v}$ and $\bar{p}$, respectively. Then there exists a regular and additively separable utility function, u, such that the excess demand function $x(p)$ derived by the maximization of u, subject to the budget constraint $p^t x(p) = 0$, satisfies

(a) $$x(\bar{p}) = \bar{x}, \tag{10}$$

(b) $$\mathrm{D}x(\bar{p}) = \bar{\mu}[\bar{v}\bar{v}^t - \bar{V}\bar{P}^{-1}] - \bar{v}\bar{x}^t. \tag{11}$$

[3]See Houthakker (1960, theorem 1). Observe, however, that there is a typographical error in the last equation in the statement of the theorem — the correct formula is (13) on p. 248.

Remark 2. As an extension of Proposition B, one wants to pose the question whether the utility function u that rationalizes at a point the observed behavior can be chosen to be not only regular and additively separable but also stationary, i.e., to satisfy the additional restriction that $u_j \equiv u_k$ for all j, k. If $\bar{p}_j \neq \bar{p}_k$ for all $j \neq k$, the following argument shows that this is indeed possible. From (2), $u_j'(w_j + \bar{x}_j) \neq u_k'(w_k + \bar{x}_k)$ for all $j \neq k$, and hence, with no loss of generality, we may assume that $u_j'(w_j + \bar{x}_j) > u_k'(w_k + \bar{x}_k)$ if and only if $j < k$, for all j, k. Since the initial endowment vector, w, is assumed to be unobservable, it may be chosen so that $0 < w_1 + \bar{x}_1 < \ldots < w_l + \bar{x}_l$. The argument is then completed by constructing a regular function v with the property that at $v'(w_j + \bar{x}_j) = u_j'(w_j + \bar{x}_j)$ and $v''(w_j + \bar{x}_j) = u_j''(w + \bar{x})$, $j = 1, \ldots, l$. If, for some $j \neq k$, $\bar{p}_j = \bar{p}_k$, then from (2), $u_j'(w_j + \bar{x}_j) = u_k'(w_k + \bar{x}_k)$. Consequently for v to exist it is necessary that $u_j''(w_j + \bar{x}_j) = u_k''(w_k + \bar{x}_k)$. From (6), however, it follows that $u_j''(w_j + \bar{x}_j) = -ep_j/\bar{v}_j$ and $u_k''(w_j + \bar{x}_k) = -ep_k/\bar{v}_k$. Stationarity then requires that $\bar{v}_j = \bar{v}_k$ whenever $\bar{p}_j = \bar{p}_k$.That this condition is not only necessary but sufficient as well follows from the previous argument.

Propositions A and B give the necessary and sufficient conditions for a function to be derived as the excess demand function of an agent with a regular and additively separable utility function. The question then is whether there are restrictions that can be imposed a-priori on the *aggregate* excess demand function based on the individual agents' regularity and additive separability. The following theorem demonstrates that beyond homogeneity of degree zero and Walras' law no such restrictions exist, at least at a point:

Theorem. Let $\bar{x}$ be a vector in $\mathscr{R}^l$, $\bar{p}$ a vector in $\mathring{\mathscr{R}}^l_+$ and A an $(l \times l)$ matrix such that

$$\bar{p}^t \bar{x} = 0, \tag{1}$$

$$\bar{p}^t A = -\bar{x}, \qquad A\bar{p} = 0. \tag{2}$$

Then, there exist l regular and additively separable utility functions $\{u^i | i = 1, \ldots, l\}$ such that, if $x(p) = \sum_{i=1}^{l} x^i(p)$ where $x^i(p)$ is the excess demand function derived by the maximization of u^i subject to the budget constraint $\bar{p}^t x^i = 0$, the following hold:

(a) $x(\bar{p}) = \bar{x}$,

(b) $\mathrm{D}x(\bar{p}) = A$.

Proof. Let $(s^1,\ldots,s^l)$ be an orthonormal basis for $\mathscr{R}^l$ such that $s^l=\bar{p}/\|p\|$ — this is possible since $\bar{p}\in\mathscr{R}^l_+$. Observe that, by orthonormality, $S^t=S^{-1}$ and inner products are preserved. Choose vectors $\bar{v}^i$ in $\mathscr{R}^l$, $i=1,\ldots,l-1$, such that $\bar{v}^i\in[\bar{s}^i,\bar{p}/\|\bar{p}\|]$, $\bar{v}^i$ is strictly positive but not collinear with $\bar{p}/\|\bar{p}\|$, and $\bar{p}^t\bar{v}^i=1$. Finally let $\bar{v}^l=\bar{p}/\|\bar{p}\|^2$ — *i.e.*, $\bar{v}^l$ is collinear with $\bar{p}/\|\bar{p}\|$ and satisfies $\bar{p}^t\bar{v}^l=1$. Let a subscript s denote vectors and matrices expressed in the new bases $(s^1,\ldots,s^l)$. Then

$$\bar{x}_s=S^t\bar{x}=(\hat{x}_1,\ldots,\hat{x}_{l-1},0)^t,$$

$$\bar{p}_s=S^t\bar{p}=(0,\ldots,0,\|\bar{p}\|)^t,$$

$$\bar{v}^i_s=S^tv^i=(0,\ldots,\hat{v}_1,\ldots,0,1/\|\bar{p}\|)^t,\qquad i=1,\ldots,l-1,$$

$$\bar{v}^l_s=\bar{S}^t\bar{v}^l=(0,\ldots,0,1/\|\bar{p}\|)^t,$$

$$As=S^tAS=\begin{bmatrix}\hat{A} & 0\\ -\hat{x}/\|p\| & 0\end{bmatrix}.$$

Consider the matrices defined by

$$C^i_s=S^t[\bar{V}^i\bar{P}^{-1}]S,\qquad i=1,\ldots,l,$$

where $\bar{V}^i$ and $\bar{P}$ are the diagonal matrices formed out of the vectors $\bar{v}^i$ and $\bar{p}$, respectively, and let

$$K^i_s=[\bar{v}^i_s(\bar{v}^i_s)^t-C^i_s],\qquad i=1,\ldots,l.$$

Finally define

$$K_s=\sum_{i=1}^{l}K^i_s\quad\text{and}\quad B_s=A_s-K_s.$$

Since $C^i_s\bar{p}_s=\bar{v}^i_s$, $\bar{p}^t_sC^i_s=(\bar{v}^i_s)^t$ and $\bar{p}^t_s\bar{v}^i_s=1$, $i=1,\ldots,l$, the matrix B_s has the form

$$B_s=\begin{bmatrix}(b^1_s)^t & 0\\ (b^{l-1}_s)^t & 0\\ -\hat{x}^t/\bar{p} & 0\end{bmatrix}$$

By the choice of $\bar{v}^i$, $\hat{v}_i\neq 0$, $i=1,\ldots,l-1$, and, hence, the following are well defined:

$$(\vec{x}_s^i)^t = ((b_s^i)^t/\hat{v}_i), \qquad i=1,\ldots,l-1,$$

$$(\vec{x}_s^l)^t = \vec{x}_s - \sum_{i=1}^{l} (\vec{x}_s^i)^t.$$

It then follows by construction that

$$A_s = \sum_{i=1}^{l} (K_s^i - \vec{v}_s^i(\vec{x}_s^i)^t),$$

$$\bar{x}_s = \sum_{i=1}^{l} \vec{x}_s^i \quad \text{and} \quad \vec{p}_s^t\vec{x}_s^i = 0, \qquad i=1,\ldots,l.$$

Consequently,

$$A = \sum_{i=1}^{l} (K^i - \vec{v}^i(\vec{x}^i)^t) \quad \text{and} \quad \bar{x} = \sum_{i=1}^{l} \vec{x}^i.$$

By Proposition B, to complete the proof we must show that

$$K^i = v^i(\vec{v}^i)^t - \bar{V}^i\bar{P}^{-1}.$$

But this is clear since

$$K = SK_s^iS^t = S(\vec{v}_s^i(\vec{v}_s^i)^t)S^t - SC_s^iS^t$$

$$= \vec{v}^i(\vec{v}^i)^t - \bar{V}^i\bar{P}^{-1}. \qquad \text{Q.E.D.}$$

Remark 3. The Theorem can be immediately extended to demonstrate that, given an integer m smaller than the number of commodities in the economy, l, agreement of the Jacobian, at $\bar{p}$, of the aggregate excess demand derived by the maximization of m regular and additively separable utility functions with a matrix A, satisfying $A\bar{p}=0$ and $\bar{p}^tA = -\bar{x}$, can be attained on a subspace of the commodity space of dimension m.

Remark 4. In the absence of the restriction of additive separability, an infinitesimal disaggregation theorem was demonstrated in Geanakoplos and Polemarchakis (1980). In this earlier work the point was made that, as long as $\bar{x}\neq 0$, local disaggregation can be attained with only $l-1$ agents. This is not, however, possible under the restriction of additive separability, as the following example demonstrates. It suffices to consider the case of two goods ($l=2$) and one agent, Let

$$\bar{x}^t = (1, -2), \qquad \bar{p}^t = (2, 1), \qquad A = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}.$$

If $A = \bar{v}\bar{v}^t - VP^{-1}$ a direct computation shows that $v^t = [0, 1]$, which contradicts the regularity of the agent's utility function.

Remark 5. Having demonstrated that infinitesimal disaggregation is attainable under additive separability, we would like to know whether the result holds under the additional requirement of stationarity. As long as $\bar{p}_j \neq \bar{p}_k$ for all $j \neq k$, this is indeed the case, as follows immediately from the Theorem and Remark 2. The following example demonstrates the disaggregation with l agents and regular, additively separable and stationary utility functions may, in general, fail. It suffices to consider the case of two goods ($l = 2$) and an arbitrary number of agents (l').

Let

$$x^t = (0, 0), \qquad \bar{p}^t = (1, 1), \qquad A = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}.$$

It follows from (9) that for disaggregation to hold it must be the case that

$$\sum_{i=1}^{l'} \mu^i \bar{v}_1^i \bar{v}_2^i - \sum_{i=1}^{l'} \bar{v}_1^i \bar{x}_2^i = -a \quad \text{for some} \quad \mu^i > 0, \quad i = 1, \ldots, l',$$

while

$$\sum_{i=1}^{l'} \bar{x}_2^i = 0.$$

By stationarity however (Remark 2),

$$\bar{v}_2^i = 1/l, \qquad i = 1, \ldots, l'.$$

Consequently,

$$-a = (1/l)^2 \sum_{i-1}^{l'} \mu^i,$$

and hence recoverability fails for $a \geqq 0$.

Remark 6. Consider the following interpretation of the results. Let good j be an elementary security with a return of unity in state of nature j and zero in all other states. The initial endowment vector w gives the investor's initial holdings of the various securities. The utility function u_j is the von Neumann–Morgenstern, state-dependent utility of wealth. The Theorem then states that the aggregate excess demand for securities is locally arbitrary. Furthermore, even though state dependence of the von Neumann–Morgenstern utility functions cannot be excluded (Remark 5), it can be reduced to a scalar; in particular, to the subjective probability of occurrence of the various states of nature.

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