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ON THE DISAGGREGATION OF EXCESS DEMAND FUNCTIONS WHEN PRICES AND AGGREGATE INCOME VARY INDEPENDENTLY*

H.M. POLEMARCHAKIS

Columbia University, New York, NY 10027, USA
Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

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Under the assumption that prices and aggregate income can vary independently and that the income distribution scheme is known and homogeneous of degree 1 in prices and aggregate income, I raise the question of the restrictions on aggregate excess demand behavior implied by the postulate of rationality of individual agents. If the number of agents is at least as high as the number of commodities, aggregate excess demand need not satisfy, at a point, any restrictions other than homogeneity of degree 0 and Walras' law. Furthermore, if the number of agents, m, is less than the number of commodities, l, aggregate excess demand can be locally arbitrary when projected on an m-dimensional subspace of the commodity space.

1. Introduction

It has been the subject of current research to identify the restrictions on aggregate excess demand functions implied by the rationality of individual agents [Debreu (1974), McFadden, Mas-Colell, Mantel and Richter (1974), Mantel (1974, 1976), Sonnenschein (1972, 1973a)]. The results have been negative: As long as the number of agents in the economy is at least as high as the number of commodities, Walras' law and homogeneity of degree 0 exhaust the restrictions that can be imposed a priori on aggregate excess demand. If the number of agents, m, is smaller than the number of commodities, l, an arbitrary differentiable function satisfying Walras' law and homogeneity of degree 0 can be decomposed, at a point, into rational individual excess demand functions when projected on an m-dimensional subspace of the commodity space. Furthermore, away from the no-trade point, $m-1$ agents suffice for this latter result [Diewert (1977), Geanakoplos and Polemarchakis (1980), Mantel (1975)]. With the exception of

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Sonnenschein (1973b) and Lau (1977), the analysis has been limited to the case in which the income of an agent is the value of a fixed initial endowment vector. Equivalently, aggregate income has not been allowed to vary independently of prices. If agents' preferences are constant over time, however, it can be argued that one may observe the agents' response to exogenous changes in income not induced by changes in relative prices. It is the purpose of this paper to characterize aggregate excess demand functions when aggregate income may vary independently of the price system.

Two points must be clarified for the question of decomposition of aggregate excess demand in the context of exogenously varying income to be well defined. First, the question of the income distribution scheme. One alternative is to assume that the income distribution scheme is known, and attempt to decompose an arbitrarily given function, which satisfies homogeneity of degree 0 and Walras' law, into rational individual excess demand functions subject to the given income distribution scheme. Another alternative is to consider the distribution scheme as not exactly known but of a given form. The extreme alternative of considering the distribution scheme as unobservable leads to a trivial problem. I shall adopt the first alternative and consider the income distribution scheme as given. Furthermore, I shall allow the distribution of income to depend on prices as well as aggregate income, and I shall only require that it be homogeneous of degree 1 in prices and aggregate income. This homogeneity assumption is intuitively appealing — relative shares should not depend on units of measurement; on the other hand, it is of questionable empirical validity — progressive taxation may be considered as a factual counterexample. Observe, however, that it is satisfied by all distribution schemes considered in the literature cited above. The second point concerns the interpretation to be attached to the exogenously varying income. I shall postulate that it is income given to agents over and above the value of their initial endowment. This is necessary if the latter is to be assumed to be unobservable. And if the initial endowment is observable, it is clear [McFadden, Mas-Colell, Mantel and Richter (1974) and Sonnenschein (1973b)] that the positivity restrictions on the individual agents' final consumption vectors may invalidate the decomposition argument.

The results of the present paper can be summarized as follows: Suppose that prices and aggregate income can vary independently, and that the income distribution scheme is known and homogeneous of degree 1 in prices and aggregate income. Then, if the number of agents is at least as high as the number of commodities, aggregate excess demand need not satisfy, at a point, any restrictions other than homogeneity of degree 0 and Walras' law. Furthermore, if the number of agents, m, is less than the number of commodities, l, aggregate excess demand can be, at a point, arbitrary when projected on an m-dimensional subspace, M, of the commodity space.

The intuition is clear: The Jacobian of an individual rational excess demand function is symmetric and negative semi-definite on a subspace of co-dimension 1. This follows from the decomposition into a substitution matrix and a matrix of income effects, the symmetry and negative semi-definiteness of the former, and the collinearity of the columns of the latter. Consequently, the Jacobian of the excess demand function of m agents is not arbitrary; it must be symmetric and negative semi-definite on the orthogonal complement of the space spanned by the columns of the income effect matrices of the m agents. Agreement with an arbitrary homogeneous function satisfying Walras' law and its Jacobian can thus only be attained, and indeed can be attained, on the space spanned by the columns of the income effect matrices of the m agents and the price vector $\bar{p}$, since, by homogeneity, the image of $\bar{p}$ under the linear map defined by the Jacobian is collinear with the aggregate income effect vector. As the individual income effect vectors must add up to the observable aggregate income effect vector, this space can only be guaranteed to have dimensions m.

These results indicate that the infinitesimal disaggregation theorem for excess demand functions with no exogenously varying income [Geanakoplos and Polemarchakis (1980)] extends to the case of exogenously varying income with only a minor modification: In the former case, away from the no-trade point, the subspace M could be taken to be of dimension $m+1$; equivalently, $l-1$ agents could generate infinitesimally arbitrary behavior. In the context of exogenously varying income, m agents are necessary everywhere. This is due to the observability of the aggregate income effect, as demonstrated by an example. Concerning the decomposition theorem in Sonnenschein (1973b), the results here constitute a generalization and extension in that:

- — the income distribution scheme is allowed to depend on prices as well as aggregate income;
- — the connection with the disaggregation results for the case with no exogenous variations in income is clearly brought out;
- — the number of agents necessary and sufficient for locally arbitrary aggregate excess demand behavior is shown to equal the number of commodities;
- — the characterization of aggregate excess demand behavior is completed by the consideration of the case in which the number of agents is lower than the number of commodities.

2. The model and the theorem

An exchange economy with l goods is characterized as a collection of agents i $(i=1,\dots,m)$ together with an income distribution scheme $Q=$

$\{q^i(p,q)\,|\,i=1,\dots,m\}$, where the function $q^i(p,q)\colon \mathring{\mathscr{R}}^l_+ \times \mathscr{R}_+ \to \mathscr{R}_+$ gives the income of agent i as a function of the price system $p \in \mathscr{R}^l_+$ and aggregate income $q \in \mathring{\mathscr{R}}_+$. The distribution scheme is assumed to satisfy the balance equation,

$$\sum_{i=1}^{m} q^i(p,q) = q,$$

everywhere on its domain of definition. It is said to be continuously differentiable and homogeneous of degree 1, if and only if the income distribution functions $q^i(p,q)$ are continuously differentiable and homogeneous of degree 1 for all i. An agent i is characterized by his utility function $u^i(x)$ defined on the strictly positive orthant, $\mathring{\mathscr{R}}^l_+$ and his initial endowment vector w^i, a point in $\mathring{\mathscr{R}}^l_+$. He is said to be *regular* if his utility function $u^i(x)$ is twice continuously differentiable and strictly quasi-concave, and, for all $x \in \mathring{\mathscr{R}}^l_+$, $Du^i(x) \gg 0$ and the indifference hypersurface through x has nowhere vanishing Gaussian curvature, and its closure relative to $\mathscr{R}^l$ is contained in $\mathring{\mathscr{R}}^l_+$. An agent (u^i, w^i) is said to be *rational* if, given prices p and nominal income q, he expresses excess demand, $x^i(p,q)$, by solving the following maximization problem:

$$\begin{aligned} &\underset{x \in \mathring{\mathscr{R}}^l_+ - \{w^i\}}{\text{Max.}} \quad u^i(x+w^i) \\ &\text{s.t.} \quad p^t x = q^i. \end{aligned} \tag{1}$$

As pointed out earlier, the income q^i is interpreted as income over and above the value of the initial endowment. The requirement that $q^i \geqq 0$ can be immediately relaxed to the weaker condition that $p^t w^i + q^i(p,q) > 0$.

By regularity, a unique solution to (1) exists for any $(p,q) \in \mathring{\mathscr{R}}^l_+ \times \mathscr{R}_+$, and is characterized by the first-order conditions

$$Du^i(x) = \lambda^i p, \qquad p^t x = q^i. \tag{2}$$

To derive the local behavior of the excess demand function, we totally differentiate the first-order conditions (2) to get

$$\begin{bmatrix} D^2u^i & -p \\ -p^t & 0 \end{bmatrix} \begin{bmatrix} dx^i \\ d\lambda^i \end{bmatrix} = \begin{bmatrix} \lambda^i \, dp \\ (x^i - D_p q^i)^t \, dp - D_q q^i \, dq \end{bmatrix}. \tag{3}$$

Setting

$$\begin{bmatrix} D^2u^i & -p \\ -p^t & 0 \end{bmatrix}^{-1} = \begin{bmatrix} S^i & -v^i \\ -(v^i)^t & e \end{bmatrix}, \tag{4}$$

which exists by the assumption of non-vanishing Gaussian curvature,

$$D_{p_j} q^i = \gamma^i_j, \qquad j = 1, \dots, l, \tag{5}$$

and

$$D_q q^i = \delta^i, \tag{6}$$

we can write

$$D_p x^i = K^i - v^i (x^i - \gamma^i)', \tag{7}$$

and

$$D_q x^i = v^i \delta^i, \tag{8}$$

where

$$K^i = \lambda^i S^i, \qquad (\gamma^i)' = (\gamma^i_1, \dots, \gamma^i_j, \dots, \gamma^i_l).$$

Concerning the local characterization of the excess demand function of a regular rational agent, the following two propositions give a complete local characterization. Proposition A is well known, while Proposition B follows immediately from Proposition 1 in Geanakoplos and Polemarchakis (1977).

Proposition A. *Let $x(p,q)$ be the excess demand function of a regular agent, (u^i, w^i), subject to the continuously differentiable income distribution function, $q^i(p,q)$. Then, everywhere on $\mathring{\mathscr{R}}^l_+ \times \mathscr{R}_+$, $x(p,q)$ is continuously differentiable, and*

$$D_p x^i(p,q) = K^i(p,q) - v^i(p,q)[x^i(p,q) - \gamma^i(p,q)]',$$

$$D_q x^i(p,q) = v^i(p,q)\delta^i(p \cdot q),$$

such that

(1) $p' x^i(p,q) = q^i(p,q)$,
(2) *$K^i(p,q)$ is symmetric, negative semi-definite, of rank $(l-1)$ and $p' K^i(p,q) = K^i(p,q)p = 0$,*
(3) $p' v^i(p,q) = 1$,

where

$$\gamma^i(p,q) = D_p q^i(p,q) \quad \text{and} \quad \delta^i(p,q) = D_q q^i(p,q).$$

Proposition B. *Let A^i be an $(l \times l)$ matrix, a^i an $(l \times 1)$ vector, and $q^i(p,q)$ a continuously differentiable income distribution function. Let $\bar{K}^i$ be an $(l \times l)$ matrix, $\bar{v}^i, \bar{x}^i, \bar{\gamma}^i$ $(l \times 1)$ vectors, $\bar{\delta}^i$ a scalar and $(\bar{p}, \bar{q})$ a point in $\mathring{\mathscr{R}}^l_+ \times \mathscr{R}$, such that*

$A^i = \bar{K}^i - \bar{v}^i(\bar{x}^i - \bar{y}^i)^t$, $a^i = \bar{v}^i\bar{\delta}^i$, *and*

(1) $\bar{p}^t\bar{x}^i = \bar{q}^i$,

(2) $\bar{K}^i$ *is symmetric, negative semi-definite, of rank* $(l-1)$ *and* $\bar{p}^t\bar{K}^i = \bar{K}^i\bar{p} = 0$,

(3) $\bar{p}^t\bar{v}^i = 1$,

where

$$\bar{q}^i = \bar{q}^i(\bar{p}, \bar{q}), \qquad \bar{y}^i = \mathrm{D}_p q^i(\bar{p}, \bar{q}), \qquad \delta^i = \mathrm{D}_q q^i(\bar{p}, \bar{q}).$$

Then there exists a regular agent (u^i, w^i) *whose excess demand function* $x^i(p, q)$, *subject to the income distribution function* $q^i(p, q)$, *satisfies:*

(a) $x^i(\bar{p}, \bar{q}) = \bar{x}^i$,

(b) $\mathrm{D}_p x^i(\bar{p}, \bar{q}) = A^i$,

(c) $\mathrm{D}_q x^i(\bar{p}, \bar{q}) = a^i$.

The aggregate excess demand function of agents $\{(u^i, w^i) \mid i = 1, \ldots, m\}$, subject to the income distribution scheme $Q = \{q^i(p, q) \mid i = 1, \ldots, m\}$, is by definition equal to $x(p, q) \equiv \sum_{i=1}^m x^i(p, q)$, where $x^i(p, q)$ is the excess demand function of agent (u^i, w^i) subject to the income distribution function $q^i(p, q)$. A necessary condition for a function $x(p, q)$ to be the aggregate excess demand function is given by the following:

Proposition C. Let $x(p, q)$ *be the aggregate excess demand function of regular agents* $\{(u^i, w^i) \mid i = 1, \ldots, m\}$, *subject to the continuously differentiable income distribution scheme* $Q = \{q^i(p, q) \mid i = 1, \ldots, m\}$. *Then, everywhere on* $\mathring{\mathscr{R}}^l_+ \times \mathscr{R}_+$, $\mathrm{D}_p x(p, q)$ *defines a symmatric, negative semi-definite quadratic form on* $[y^1(p, q), \ldots, y^m(p, q)]^L$ *and a symmatric, negative definite quadratic form on* $[y^1(p, q), \ldots, y^m(p, q), p]$, *where*

$$y^i(p, q) \equiv (x^i(p, q) - \mathrm{D}_p q^i(p, q)), \qquad i = 1, \ldots, m.$$

Proof. By definition,

$$\begin{aligned} \mathrm{D}_p x(p, q) &= \sum_{i=1}^m \mathrm{D}_p x^i(p, q) \\ &= \sum_{i=1}^m K^i(p, q) - v^i(p, q)[x^i(p, q) - \gamma^i(p, q)]^t \\ &= \sum_{i=1}^m K^i(p, q) - v^i(p, q) y^i(p, q)^t. \end{aligned}$$

Let $z \in [y^1(p,q),\ldots,y^m(p,q)]^L$. Then

$$z^t \mathrm{D}x(p,q)z = zt\left[\sum_{i=1}^{m} K^i(p,q)\right]z + z^t\left[\sum_{i=1}^{m} v^i(p,q)y^i(p,q)^t\right]z$$

$$= z^t\left[\sum_{i=1}^{m} K^i(p,q)\right]z < 0.$$

Symmetry follows from the symmatry of $K^i(p,q)$, $i=1,\ldots,m$. For $z \in [y^1(p,q),\ldots,y^m(p,q),p]^L$ and $z \neq 0$,

$$z^t K^i(p,q)z < 0, \qquad i=1,\ldots,m;$$

hence

$$z^t \mathrm{D}x(p,q)z < 0. \qquad \text{Q.E.D.}$$

To complete the local characterization of the aggregate excess demand function of m agents for l goods when aggregate income and prices vary independently and the income distribution scheme is known, it must be shown that the necessary restrictions of Proposition C are sufficient as well.

Theorem. Let A be an $(l \times l)$ matrix, a an $(l \times 1)$ vector, $\bar{x}$ an $(l \times 1)$ vector, $(\bar{p},\bar{q})$ a point in $\mathring{\mathscr{R}}^l_+ \times \mathscr{R}_+$, $Q=\{q^i(p,q)\,|\,i=1,\ldots,m\}$ a continuously differentiable and homogeneous of degree 1 income distribution scheme, where $m \leqq l$, such that

(1) $\bar{p}^t A = -\bar{x}$, $\quad A\bar{p} = -\bar{q}a$, $\quad \bar{p}^t a = 1$,
(2) $\bar{p}^t \bar{x} = \bar{q}$.

Then there exists a subspace M of $\mathscr{R}^l$ of dimension m, and m agents $\{(u^i,w^i)\,|\, i=1,\ldots,m\}$, such that the aggregate excess demand function $x(p,q) = \sum_{i=1}^m x^i(p,q)$ derived by utility maximization subject to the income distribution scheme Q satisfies

(1) $\mathrm{proj}_M x(\bar{p},\bar{q}) = \mathrm{proj}_M \bar{x}$,
(2) $\mathrm{D}_p x(\bar{p},\bar{q})y = Ay$ *for all* $y \in M$,
(3) $\mathrm{D}_q x(\bar{p},\bar{q}) = a$.

Proof. Let $(\bar{s}^1,\ldots,\bar{s}^{l-1}, \bar{p}/\|\bar{p}\|)$ be an orthonormal basis for $\mathscr{R}^l$, such that, if $\bar{x} \notin [\bar{p}]$, $\bar{x}^t\bar{s}^k \neq 0$ $(k=1,\ldots,l-1)$. This can be done as follows: Let $(\bar{e}^1,\ldots,\bar{e}^{l-1}, \bar{p}/\|\bar{p}\|)$ be an orthonormal basis for $\mathscr{R}^l$ — this is possible since $p \in \mathring{\mathscr{R}}^l_+$ — and let ρ be the rotation taking $\mathrm{proj}_{[\bar{e}^1,\ldots,\bar{e}^{l-1}]}\,\bar{x}$ to

$$\frac{\bar{e}}{\|\bar{e}\|}\|\text{proj}_{[\bar{e}^1,\ldots,\bar{e}^{l-1}]}\bar{x}\| \quad \text{where} \quad \bar{e}=\sum_{k=1}^{l-1}\bar{e}.$$

Then $\bar{s}^k$ can be defined as $\rho^{-1}(\bar{e}^k)$. Let S denote the orthonormal transformation taking the standard basis $(e^1,\ldots,e^l)$ of $\mathscr{R}^l$ to $(\bar{s}^1,\ldots,s^{l-1},\bar{p}/\|\bar{p}\|)$; a subscript S denotes vectors and matrices expressed in the new basis: Let $A_S=S^{-1}AS$, $a_S=S^{-1}a$, $\bar{x}_S=S^{-1}\bar{x}$, $\bar{p}_S=S^{-1}\bar{p}$, $\bar{\gamma}^i_S=\bar{S}-1\bar{\gamma}^i$. Then

$$a^t_S=(\hat{a}_1,\ldots,\hat{a}_{l-1},1/\|\bar{p}\|),$$

$$\bar{\gamma}^i_S=(\gamma^i_S,\gamma^i_{S,l}),$$

$$\bar{x}^t_S=(\hat{x}_1,\ldots,\hat{x}_{l-1},q/\|p\|),$$

$$\bar{p}^t_S=(0,\ldots,0,\|\bar{p}\|),$$

and

$$A_S=\begin{bmatrix}\hat{A} & -\hat{a}q/\|\bar{p}\| \\ -\hat{x}/\|\bar{p}\| & -q/\|\bar{p}\|^2\end{bmatrix}.$$

Consider first the case in which $\bar{x}\notin[\bar{p}]$ and hence $\hat{x}_k\neq 0$ $(k=1,\ldots,l-1)$; furthermore, since $\sum_{i=1}^m\bar{\delta}^i=1$ we may, without loss of generality, assume that $\bar{\delta}^m\neq 0$. The following are then well defined:

$$\bar{x}^i_S=(0,\ldots,\hat{x}_i,\ldots,\bar{q}^i/\|\bar{p}\|)+(\hat{\gamma}^i_S,0), \qquad i=1,\ldots,m-1,$$

$$\bar{x}^m_S=(0,\ldots,0,\bar{q}^m/\|\bar{p}\|)+\hat{\gamma}^m_S,0),$$

$$\bar{K}^i_S=\frac{1}{m}\begin{bmatrix}-I & 0\\ 0 & 0\end{bmatrix}, \qquad \bar{K}_S=\begin{bmatrix}-I & 0\\ 0 & 0\end{bmatrix},$$

$$B_S=A_S-\bar{K}_S=\begin{bmatrix}\hat{B} & -\hat{a}q/\|\bar{p}\| \\ -\hat{x}/\|\bar{p}\| & -q/\|\bar{p}\|^2\end{bmatrix}$$

$$\bar{v}^i_S=\begin{bmatrix}\hat{B}_i/\hat{x}_i \\ 1/\|\bar{p}\|\end{bmatrix}, \qquad i=1,\ldots,m-1,$$

$$\bar{v}^m_S=\left[a_S-\sum_{i=1}^{m-1}\bar{\delta}^i\bar{v}^i_S\right]\Big/\bar{\delta}^m.$$

By construction, the following conditions are satisfied:

(1) $\bar{p}_S^t x_S^i = \bar{q}^i$,
(2) $\bar{p}_S^t \bar{K}_S^i = \bar{K}_S^i \bar{p}_S = 0$, and $\bar{K}^i$ is symmetric, negative semi-definite and of rank $(l-1)$, for all i,
(3) $\bar{p}_S^t \bar{v}_S^i = 1$, for all i.

For each $i = 1, \dots, m$, let the triplet $(\bar{K}^i, \bar{v}^i, \bar{x}^i)$ be defined by

$$\bar{K}^i = S\bar{K}_S^i S^{-1}, \qquad \bar{v}^i = S\bar{v}_S^i, \qquad \bar{x}^i = S\bar{x}_S^i.$$

Since S is an orthonormal matrix, $S^t = S^{-1}$. Consequently, $\bar{K}^i$ is symmetric, negative, semi-definite and of rank $(l-1)$. Since S is orthonormal it preserves inner products, and hence $\bar{p}^t \bar{K}^i = \bar{K}^i \bar{p}^t = 0$, $\bar{p}^t \bar{x}^i = \bar{q}^i$ and $\bar{p}^t \bar{v}^i = 1$, for all i. Proposition B yields then the existence of m regular agents $\{(u^i, w^i), i = 1, \dots, m\}$ whose excess demand function, subject to the income distribution scheme, satisfies

$$x^i(\bar{p}, \bar{q}) = \bar{x}^i, \qquad \mathrm{D}_p x^i(\bar{p}, \bar{q}) = \bar{K}^i - \bar{v}^i(\bar{x}^i - \bar{y}^i)^t, \qquad \mathrm{D}_q x^i(\bar{p}, \bar{q}) = \bar{\delta}^i \bar{v}^i.$$

Consider now the aggregate behavior of the economy composed of the m agents described above, i.e., the aggregate excess demand function,

$$x(p, q) \equiv \sum_{i=1}^{m} x^i(p, q),$$

subject to the income distribution scheme Q. By construction the following hold: Let M be the space spanned by $[s^{-1}, \dots, \bar{s}^{m-1}, \bar{p}/\|\bar{p}\|]$. Then

$$\mathrm{proj}_M x(\bar{p}, \bar{q}) = \mathrm{proj}_M \bar{x},$$

since,

$$\sum_{i=1}^{m} \bar{y}_S^i = \sum_{i=1}^{m} \bar{y}^i = 0.$$

Furthermore,

$$\sum_{i=1}^{m} \delta^i \bar{v}_S^i = a_s,$$

and hence

$$\sum_{i=1}^{m} \delta^i \bar{v}^i = a.$$

Finally, consider the Jacobian with respect to p of the aggregate excess demand function,

$$\mathrm{D}_p x(\bar{p},\bar{q}) \equiv \sum_{i=1}^{m} \mathrm{D}_p x^i(\bar{p},\bar{q}).$$

To demonstrate that

$$Ay = \mathrm{D}_p x(\bar{p},\bar{q})y \quad \text{for all} \quad y \in M,$$

is equivalent to demonstrating that the first $(m-1)$ columns of B_S as well as the last coincide with the corresponding columns of

$$\Gamma_S = -\sum_{i=1}^{m} \bar{v}_S^i(\bar{x}_S^i - \bar{y}_S^i)'.$$

Agreement on the first $(m-1)$ columns follows immediately by construction. Agreement on the last column follows by construction and the homogeneity of degree 1 of the income distribution functions and is demonstrated as follows: By homogeneity,

$$p'y^i + q\delta^i \equiv q^i \quad \text{and hence} \quad \bar{y}_{S,l}^i \|\bar{p}\| = q^i - q\delta^i.$$

Consequently,

$$(\bar{x}_S^i - \bar{y}_S^i)_l = \bar{q}_i/\|\bar{p}\| - \bar{y}_{S,l}^i = \bar{q}\delta^i/\|p\|.$$

But the lth column of Γ_S is equal to

$$\Gamma_S^l = -\sum_{i=1}^{m} (\bar{x}_S^i - y_S^i)_l \bar{v}_S^i = \sum_{i=1}^{m-1} (\bar{x}_S^i - y_S^i)_l \bar{v}_S^i - (\bar{x}_S^m - \bar{y}_S^m)_l \bar{v}_S^m$$

$$= -\sum_{i=1}^{m-1} \frac{\bar{q}}{\|\bar{p}\|} \delta^i \bar{v}_S^i - \frac{q}{\|\bar{p}\|} \delta^m \left(\bar{a}_s - \sum_{i=1}^{m-1} \delta^i \bar{v}_S^i \right) \Big/ \delta^m$$

$$= -\bar{a}_S q/\|\bar{p}\|,$$

which is precisely the last column of B_S.

To complete the proof of the theorem, I must now consider the case in which $\bar{x} = c\bar{p}$, and $c \in \mathscr{R}$. Proceed as before. Now, however, $\bar{x}_S^t = (0,\ldots,0,\bar{q}/\|\bar{p}\|)$, and so we define

$$\bar{x}_S^i = (0,\ldots,0,1,0,\ldots,\bar{q}^i/\|\bar{p}\|) + (\bar{y}_S^i, 0), \qquad i = 1,\ldots,m,$$

$$\bar{x}_S^m = (-1,\ldots,-1,0,\ldots,\bar{q}^m/\|\bar{p}\|) + (\bar{y}_S^m, 0).$$

Let

$$v_S^i = (\hat{v}_S^i, 1/\|\bar{p}\|), \quad i = 1, \ldots, m,$$

and choose $\hat{v}_S^i$ such that

$$(\hat{v}_S^i)_k - (\hat{v}_S^m)_k = (\hat{B}_S^i)_k, \qquad i = 1, \ldots, m-1, \quad k = 1, \ldots, l-1,$$

and

$$\sum_{i=1}^{m} (\hat{v}_S^i)_k = \hat{a}_k, \qquad k = 1, \ldots, l-1.$$

This is possible since it is equivalent to solving a system of linear equations of the form $Cy = z$, where

$$C = \begin{bmatrix} I & & & -I \\ & \ddots & & \vdots \\ & & I & -I \\ I & \ldots & I & I \end{bmatrix}$$

and hence of full rank. The argument can then be completed as in the previous case. Q.E.D.

Corollary. Let A be an $(l \times l)$ matrix, a an $(l \times 1)$ vector, $\bar{x}$ an $(l \times 1)$ vector, $(\bar{p}, \bar{q})$ a point in $\mathring{\mathscr{R}}^l \times \mathscr{R}_+$, $Q = \{q^i(p,q) \mid i = 1, \ldots, l\}$ a continuously differentiable and homogeneous of degree 1 income distribution scheme, such that

(1) $\bar{p}^t A = -\bar{x}, \quad A\bar{p} = -\bar{q}a, \quad \bar{p}^t a = 1,$

(2) $\bar{p}^t \bar{x} = \bar{q}$.

Then there exist l agents $\{(u^i, w^i) \mid i = 1, \ldots, l\}$, such that the aggregate excess demand function,

$$x(p,q) = \sum_{i=1}^{l} x^i(p,q),$$

derived by utility maximization subject to the income distribution scheme Q satisfies

(1) $x(\bar{p}, \bar{q}) = \bar{x}$,

(2) $D_p x(\bar{p}, \bar{q}) = A$,

(3) $D_q x(\bar{p}, \bar{q}) = a$.

I shall conclude with some remarks.

Remark 1. The analogue of the theorem for the case of no exogenous variation in income was demonstrated in Geanakoplos and Polemarchakis (1977). The only difference between the two cases turns out to be that, if aggregate income does not vary independently of prices and if $\bar{x} \neq 0$, $(m-1)$ agents suffice. That this is not possible in the present context is demonstrated by the following example. Let $m=1$, $l=2$. Then clearly

$$q^1(p,q) \equiv q, \qquad \gamma^1 \equiv 0, \qquad \delta^1 = 1.$$

Let

$$\bar{q}=2, \qquad \bar{x}' = (1,1), \qquad \bar{p}' = (1,1),$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad a' = (1,0).$$

Then the (single) agent's substitution matrix, $\bar{K}^1$, must satisfy

$$\bar{K} = A + \bar{v}\bar{x}'.$$

Since $\bar{v}$ must equal a which is assumed to be known,

$$\bar{K} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix},$$

which is not negative semi-definite. In other words, it is the observability, in the present framework, of the aggregate income effect, which requires one additional agent for the decomposition argument.

Remark 2. As a consequence of Remark 2 we see that m agents are not only sufficient but also necessary for the local agreement of the aggregate excess demand function with an arbitrary function on a subspace of dimension m.

Remark 3. The space M in the theorem could have been specified exogenously, provided $\bar{p} \in M$. The modifications in the proof necessary for this stronger version of the theorem are evident.

Remark 4. The analogue of the theorem for open neighborhoods is yet to be derived. A first attempt towards a counterargument is undertaken in Biddard (1980) and Schafer and Sonnenschein (1981). Attention must be paid, however, to distinguish the argument from the lack of decomposability due to the positivity restrictions on individual demand behavior when initial endowments are observable.

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