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EXPECTATIONS, DEMAND, AND OBSERVABILITY[1]

BY H. M. POLEMARCHAKIS

Under the assumption that demand behavior depends on intertemporal preferences as well as (point) expectations concerning future prices, it is demonstrated that under plausible conditions rationality imposes no observable restrictions on the demand function and expectations and preferences are observationally indistinguishable.

INTRODUCTION

IN THE CONTEXT of an economy with a complete system of markets, an agent is said to be *rational* if and only if his demand function is derived from the maximization of a preference pre-order subject to the budget constraint. Questions of rationality acquire an additional dimension when exchange takes place sequentially, i.e., when the system of markets is incomplete. Following Grandmont [3], consider a situation where economic activity occurs at two temporally distinct points ($t = 1, 2$) and assume that there exists a unique, costlessly storable medium of exchange referred to as money. An agent at time $t = 1$ is characterized by his endowment vector w, his intertemporal utility function u, and an expectation formation mechanism ϕ, which, for simplicity, I take to be a function mapping current to future price systems. Utility and expectation functions are *unobservable*. The *observable* characteristics of the agent consist—at most—of his demand for current goods and possibly his endowment vector. The issues that can now be raised are twofold: first, whether restrictions are imposed on the demand for current goods by the assumption that it is derived from the maximization of a quasi concave, monotone intertemporal utility function coupled with a well defined expectation function; second, as is important for questions of prediction and welfare, whether preferences u and expectations ϕ are observationally distinguishable. I demonstrate that under plausible conditions the assumption of rationality imposes no observable restrictions on the demand function for current goods and preferences and expectations are observationally indistinguishable: Variations solely in expectations are sufficient to generate the entire range of demand functions for current goods and the same holds for preferences, at least infinitesimally. I term the former the *indeterminacy of preferences*, the latter the *indeterminacy of expectations*.

The implications of the indeterminacy result are clear: Restrictions on agents' expectations (or preferences) alone cannot yield observable restrictions on behavior. The latter require joint restrictions on expectations and intertemporal preferences. In particular, the indeterminacy of expectations suggests that economies in which agents form expectations rationally may be indistinguishable from economies in which rationality fails. The implications of this observation concerning macroeconomic policy have been looked at by Hamdani [5].

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Extensions of the argument are readily suggested: (i) Aspects of the agent's behavior in addition to the demand for current goods may be observable. (ii) Expectations may involve probability measures over future price systems. The results in Polemarchakis and Selden [6] concerning the recoverability of intertemporal von Neumann–Morgenstern cardinal utility indices are related to this point. (iii) If the stochastic nature of the economy is further specified, rationality restrictions may be imposed on the agent's expectations. Observe, however, that such restrictions may affect the indeterminacy of preferences but not that of expectations.

1. THE MODEL

Time periods are indexed by a subscript t, $t = 1,2$. There are l_1 period 1 consumption goods indexed by a superscript h_1, $h_1 = 1, \ldots, l_1$, and l_2 period 2 consumption goods indexed by a superscript h_2, $h_2 = 1, \ldots, l_2$. In addition, there exists a costlessly storable medium of exchange referred to as money, whose price is normalized to equal 1 in both periods. A price system for period t is a vector p_t in $\mathring{\mathbb{R}}^{l_t}_{+}$, $t = 1,2$. At period 1, only markets for period 1 consumption goods and money are open. An agent is characterized by his consumption set X, a convex subset of $\mathbb{R}^{l_1+l_2}$, his endowment w, a point in $\mathbb{R}^{l_1+l_2}$, his intertemporal utility function u defined on X, and his expectation function ϕ assigning to each period 1 price system a unique period 2 price system. The following assumptions are made throughout:

ASSUMPTION 0: $X = \mathring{\mathbb{R}}^{l_1+l_2}_{+}$; $w \in X$.

ASSUMPTION 1: u is a strictly quasi concave and twice continuously differentiable function from X to $\mathbb{R}$. At any $x \in X$, $Du(x) \gg 0$.

ASSUMPTION 2: ϕ is a continuous function from $\mathring{\mathbb{R}}^{l_1}_{+}$ to $\mathring{\mathbb{R}}^{l_2}_{+}$.

In situations in which the differentiability of the demand function is of interest I replace Assumptions 1 and 2 by the following stronger versions:

ASSUMPTION 1′: u is a strictly quasi concave and twice continuously differentiable function from X to $\mathbb{R}$. At any $x \in X$, $Du(x) \gg 0$ and the indifference hypersurface through x has nowhere vanishing Gaussian curvature.

ASSUMPTION 2′: ϕ is a continuously differentiable function from $\mathring{\mathbb{R}}^{l_1}_{+}$ to $\mathring{\mathbb{R}}^{l_2}_{+}$.

Assumption 0 is convenient to avoid boundary problems. Assumptions 1 and 2 guarantee the continuity of the demand function. Since the consumption set has been taken to be open, the demand function may not be defined for some price systems; this is of no importance, however, since we shall be dealing with situations in which the demand function is known to be well defined. Assumptions 1′ and 2′ guarantee the differentiability of the demand function for current goods (see Debreu [**2**, pp. 612–613]).

The agent's consumption set X is held constant in the analysis to follow, unlike his endowment vector w, utility function u, and expectation function ϕ. Consequently, I refer to an agent as an ordered triplet (w,u,ϕ). The demand function for current goods by an agent (w,u,ϕ) is derived by projecting from the solution to the following maximization problem:

$$(1)\qquad \underset{(x_1,x_2)\in X}{\mathrm{Max}}\ u(x_1,x_2) \qquad \text{subject to}$$

$$p_1^t x_1 + \phi(p_1)^t x_2 = p_1^t w_1 + p_2^t w_2.$$

By Assumption 1 and the Kuhn–Tucker theorem, $(x_1^*,x_2^*)\in X$ solves (1) if and only if there exists $\lambda^* > 0$ such that

$$(2.1)\qquad D_1 u(x_1^*,x_2^*) = \lambda^* p_1,$$

$$(2.2)\qquad D_2 u(x_1^*,x_2^*) = \lambda^* \phi(p_1),$$

$$(2.3)\qquad p_1^t x_1 + \phi(p_1)^t x_2 = p_1^t w_1 + \phi(p_1)^t w_2,$$

where $D_t u$ is the vector of partial derivatives of u with respect to x_t, $t = 1,2$. By the strict quasi concavity of u, given $p_1 \in \mathring{\mathbb{R}}_+^{l_1}$, the triple (x_1^*,x_2^*,λ^*) satisfying (2.1)–(2.3) is unique—if it exists. I denote it by $x_1(p_1)$, $x_2(p_1)$, $\lambda(p_1)$, which are continuous functions of p_1. Under the stronger regularity Assumptions 1′ and 2′, $x_1(p_1)$, $x_2(p_1)$, and $\lambda(p_1)$ become continuously differentiable functions of p_1.

REMARK 1: At this point a comment is required concerning the structure of the financial market. Following the practice common in the theory of temporary equilibrium, I have postulated the existence of a unique medium of transfer—money. The question follows whether agents are allowed to borrow during the first period in their life, and, if so, at what rate of interest. To restrict agents to nonnegative money holdings is undesirable. For simplicity, I have gone to the other extreme and I have postulated that money can be borrowed costlessly; i.e., the interest rate is fixed and equal to zero. The particular level at which the rate of interest is fixed is, of course, immaterial. Furthermore, the results that I shall derive will depend on the number of current (l_1) relative to the number of future (l_2) commodities. To extend the results to the case of a variable interest rate one only needs to augment the number of current goods by 1 to $(l_1 + 1)$.

2. THE INDETERMINACY OF PREFERENCES

In this section I look into conditions under which variations in the demand function for current goods can be accounted for by variations solely in the expectation function. Let U^* be the family of utility functions u^* such that (i) u^* satisfies Assumption 1, and (ii) given any continuous function $f(p_1)$: $\mathring{\mathbb{R}}_+^{l_1} \to \mathring{\mathbb{R}}_+^{l_2}$

and any initial endowment vector $w = (w_1, w_2) \in \mathring{\mathbb{R}}_+^{l_1+l_2}$, there exists an expectation function ϕ satisfying Assumption 2 such that the agent (w, u^*, ϕ) generates demand for current goods $x_1(p_1)$ equal to $f(p_1)$ everywhere on $\mathring{\mathbb{R}}_+^{l_1}$. The question of indeterminacy of preferences reduces then to the question whether the class U^* is empty or not.

PROPOSITION 1: *U^* is not empty if and only if $l_2 \geq l_1$.*

PROOF: First the proof of sufficiency: Let

$$(3) \qquad u^*(x_1, x_2) = (1/\alpha\beta) \sum_{j=1}^{l_1} \left[(x_1^j)^\alpha + (x_2^j)^\alpha \right]^\beta + (1/\alpha) \sum_{j=l_1+1}^{l_2} (x_2^j)^\alpha,$$

$$0 < \alpha < 1, \quad 0 < \beta < 1,$$

and suppose that $x_1 \in \mathring{\mathbb{R}}_+^{l_1}$, $p_1 \in \mathring{\mathbb{R}}_+^{l_1}$, and $w = (w_1, w_2) \in \mathring{\mathbb{R}}_+^{l_1+l_2}$ are arbitrarily given. I shall first show that there exist $x_2 \in \mathring{\mathbb{R}}_+^{l_2}$, $\phi \in \mathring{\mathbb{R}}_+^{l_2}$, and $\lambda \in \mathring{\mathbb{R}}_+$ such that

$$(4.1) \qquad D_1 u^*(x_1, x_2) = \lambda p_1,$$

$$(4.2) \qquad D_2 u^*(x_1, x_2) = \lambda \phi,$$

$$(4.3) \qquad p_1^t x_1 + p_2^t x_2 = p_1^t w_1 + \phi^t w_2.$$

Pick $k \in (0, \min_j \gamma^j)$ where $\gamma^j = (p_1^j)^{-1} (x_1^j)^{\alpha\beta - 1}$, $j = 1, \ldots, l_1$. Then $D_1^j u^*(x_1, y) = k p_1^j$, $j = 1, \ldots, l_1$, implies that

$$y^j = (x_1^j) \left[(k p^j)^{1/(\beta - 1)} (x_1^j)^{((1-\alpha)/(\beta - 1)) - \alpha} - 1 \right]^{1/\alpha} \qquad (j = 1, \ldots, l_1).$$

Furthermore, if $q^j = (1/k) D_2^j u^*(x_1, y)$,

$$q^j = (p_1^j) \left[(k p_1^j)^{1/(\beta - 1)} (x_1^j)^{((1-\alpha)/(\beta - 1)) - \alpha} - 1 \right]^{(\alpha - 1)/\alpha} \quad \text{and}$$

$$q^j y^j = (p_1^j x_1^j) \left[(k p_1^j)^{1/(\beta - 1)} (x_1^j)^{((1-\alpha)/(\beta - 1)) - \alpha} - 1 \right] \qquad (j = 1, \ldots, l_1).$$

For $j = l_1 + 1, \ldots, l_2$, we set $y^j = w_j^2$ and $q^j = (1/k)(w_j^2)^{\alpha - 1}$.

It is now easy to see that, for some $\hat{k} \in (0, \min_j \gamma^j)$, if one sets $\lambda = \hat{k}$, $\phi = q$, and $x_2 = y$, equations (4.1), (4.2), and (4.3) are satisfied. Observe that (4.1) and (4.2) hold for all $\hat{k} \in (0, \min_j \gamma^j)$ by the construction of q and y. Consequently, we must show that there exists $\hat{k} \in (0, \min_j \gamma^j)$ such that

$$(5) \qquad \sum_{j=1}^{l_1} p_1^j x_1^j + q^j y^j = \sum_{j=1}^{l_1} p_1^j w_1^j + q^j w_2^j.$$

But as k tends to $(\min_j \gamma^j)$, the left hand side of (5) tends to a finite limit and the right hand side to infinity, while the reverse holds as k tends to 0. Since the dependence on k is continuous, (5) is indeed satisfied for some $\hat{k} \in (0, \min_j \gamma^j)$. To complete the argument for sufficiency, it must be shown that the solution $\phi(p_1)$ is a continuous function of p_1; equivalently, that $\hat{k}$ is a continuous function of p_1, $\hat{k}(p_1)$. Considering the left hand side and right hand side of (5) as function, $L(k; p_1)$ and $R(k; p_1)$, respectively, we see that

$$\frac{\partial L}{\partial k} = \left[1/(\beta - 1)\right] \sum_{j=1}^{l_1} (p^j)^2 (x^j)^{(\beta(1-\alpha))/(\beta-1)} (kp^j)^{(2-\beta)/(\beta-1)}$$

while

$$\frac{\partial R}{\partial k} = \left[(\alpha - 1)/(\alpha(\beta - 1))\right] \sum_{j=1}^{l_1} (p_1^j)^2 (w_2^j)(x_1^j)^{((1-\alpha)/(\beta-1))-\alpha}$$
$$\times (kp_1^j)^{(2-\beta)/(\beta-1)} \left((kp_1^j)^{1/(\beta-1)} (x_1^j)^{((1-\alpha)/(\beta-1))-\alpha} - 1\right)^{-1/\alpha}$$

Since $0 < \alpha$, $\beta < 1$, and $k \in (0, \min_j \gamma^j)$, for all p_1 and k, $(\partial L/\partial k) < 0$ while $(\partial R/\partial k) > 0$. Consider now the solution $\hat{k}(p_1)$ to the equation $L(k; p_1) = R(k; p_1)$. The solution exists by the preceding argument and is unique by monotonicity. Furthermore, the curves $L(k; p_1)$ and $R(k; p_1)$ intersect transversally, since the first is an increasing while the second is a decreasing function of the variable k. The uniqueness of the intersection, combined with the continuity in p_1 of both curves, implies the continuity of the solution $\hat{k}(p_1)$. The argument for sufficiency is now complete, since the utility function u^* clearly satisfies Assumption 1.

The argument for necessity is as follows: For fixed $x_1 \in \mathring{\mathbb{R}}^{l_1}_+$ consider $D_1 u(\bar{x}_1, x_2)$ as a continuously differentiable function from $\mathring{\mathbb{R}}^{l_2}_+$ to $\mathring{\mathbb{R}}^{l_1}_+$. Since $l_2 < l_1$, there exists $p_1 \in \mathbb{R}^{l_1}_+$ such that the equation $D_1 u(\bar{x}_1, x_2) = \lambda p_1$ has at most a discrete set of solutions; this is an immediate consequence of Sard's theorem applied to the function $[(1/\lambda) D_1 u(\bar{x}_1, x_2)]$: $\mathring{\mathbb{R}}^{l_1+1}_+ \to \mathring{\mathbb{R}}^{l_1}_+$. Consequently, there exist values for $w = (w_1, w_2)$ for which the system of equations (4) has no solution. *Q.E.D.*

COROLLARY 1: *If $l_1 \geqslant 2$, U^* contains no additively separable utility functions.*

PROOF: Let $\bar{p}_1$ and $\bar{\bar{p}}_1$ be two linearly independent vectors in $\mathring{\mathbb{R}}^{l_1}_+$, and let $\bar{x}_1(p_1)$ and $\bar{\bar{x}}_1(p_1)$ be two continuous functions from $\mathring{\mathbb{R}}^{l_1}_+$ to $\mathring{\mathbb{R}}^{l_1}_+$ such that $\bar{x}_1(\bar{p}_1) = \bar{\bar{x}}_1(\bar{\bar{p}}_1)$. Suppose U^* contains an additively separable utility function—i.e., $u^* \in U^*$ and $u^*(x_1, x_2) = u_1^*(x_1) + u_2^*(x_2)$. It follows that for any $\bar{x}_2$, $\bar{\bar{x}}_2$ in $\mathring{\mathbb{R}}^{l_2}_+$, $D_1 u^*(\bar{x}_1(\bar{p}_1), \bar{x}_2) = D_1 u^*(\bar{\bar{x}}_1(\bar{\bar{p}}_1), \bar{\bar{x}}_2)$. But a necessary condition for u^* to generate $\bar{x}_1(p_1)$ and $\bar{\bar{x}}_2(p_2)$ is that $D_1 u^*(\bar{x}_1(\bar{p}_1), \bar{x}_2) = \bar{\lambda}\bar{p}_1$ and $D_1 u^*(\bar{\bar{x}}_1(\bar{\bar{p}}_1), \bar{\bar{x}}_2) = \bar{\bar{\lambda}}\bar{\bar{p}}_2$. It follows that $\bar{\lambda}\bar{p}_1 = \bar{\bar{\lambda}}\bar{\bar{p}}_1$. Since $\bar{p}_1$ and $\bar{\bar{p}}_1$ were chosen not to be colinear this implies that $\bar{\lambda} = \bar{\bar{\lambda}} = 0$ which violates Assumption 1. *Q.E.D.*

Proposition 1 shows that if the number of future commodities (l_2) is at least as great as the number of current commodities (l_1) indeterminacy of preferences prevails. Since the future period can be interpreted to represent all future transactions of an agent, it is reasonable to suppose that l_2 is indeed at least as great as l_1. Furthermore, no restriction was imposed on $x_1(p_1)$—the demand function for current goods—other than continuity. Consequently, the question whether rationality imposes any restrictions on the demand function for current goods is answered negatively by the following proposition:

PROPOSITION 2: *If $l_2 \geq l_1$, any continuous function $x_1(p_1)$: $\mathring{\mathbb{R}}^{l_1}_{+} \rightarrow \mathring{\mathbb{R}}^{l_2}_{+}$ is the demand function for current goods of an agent (w, u, φ) satisfying Assumptions* 0, 1, *and* 2.

PROOF: It follows from Proposition 1. *Q.E.D.*

REMARK 2: In [4] Gausch demonstrated the sufficiency part of Proposition 1 for the special case $l_1 = l_2 = 1$.

3. THE INDETERMINACY OF EXPECTATIONS

It was the object of the previous section to demonstrate that any variation in the demand function for current goods can be attributed to variations in the expectation function. In this section I take the opposite point of view. I look into the conditions under which at least any infinitesimal variation in the observable characteristics of an agent can be attributed solely to variations in the intertemporal utility function u. Let Φ^* be the family of expectation functions ϕ^* such that (i) ϕ^* satisfies Assumption 2′, and (ii) given any continuous function $f(p_1)$: $\mathbb{R}^{l_1}_{+} \rightarrow \mathbb{R}^{l_2}_{+}$, any initial endowment vector $w = (w_1, w_2) \in \mathring{\mathbb{R}}^{l_1+l_2}$, and any $\bar{p}_1 \in \mathring{\mathbb{R}}^{l_1}_{+}$ there exists a utility function u satisfying Assumption 1′ such that the agent (w, u, ϕ^*) generates demand for current goods $x_1(p_1)$ equal to $f(p_1)$ infinitesimally at $\bar{p}_1$, i.e., $f(\bar{p}_1) = x_1(\bar{p}_1)$ and $Df(\bar{p}_1) = Dx_1(\bar{p}_1)$. The question of (infinitesimal) indeterinacy of expectations reduces then to the question whether Φ^* is empty or not.

PROPOSITION 3: *If $l_2 \geq l_1 + 1$, Φ^* is not empty. An expectation function ϕ^* satisfying Assumption 2′ lies in Φ^* if and only if $D\phi^*(p_1)$ has rank l_1 everywhere on* $\mathbb{R}^{l_1}_{+}$.

PROOF: For the proof of Proposition 3 I need the following lemmas:

LEMMA 1: *Let A be a symmetric negative definite matrix of order $(n_1 \times n_1)$, and B an arbitrary matrix of order $(n_2 \times n_1)$, n_1, $n_2 \geqslant 1$. There exists a symmetric matrix C of order $(n_2 \times n_2)$ such that the matrix M defined by*

$$M = \begin{bmatrix} A & B^t \\ B & C \end{bmatrix}$$

is symmetric and negative definite.

PROOF OF LEMMA 1: For $\alpha > 0$, let the matrix $M(\alpha)$ be defined by

$$M(\alpha) = \begin{bmatrix} A & B^t \\ B & -\alpha I \end{bmatrix}.$$

It is clearly symmetric, and, as the following argument shows, it is also negative definite provided α is large. For any $y \in \mathbb{R}^{n_1+n_2}$ let $y = (y_1, y_2)$, where $y_1 \in \mathbb{R}^{n_1}$ and $y_2 \in \mathbb{R}^{n_2}$. To demonstrate that $M(\alpha)$ is negative definite, it suffices to show that $y^t M(\alpha) y < 0$ for all y in $S^{n_1+n_2-1}$, the $(n_1 + n_2 - 1)$ dimensional sphere. Consequently, we can assume that $\|(y_1, y_2)\| = 1$. The product $y^t M(\alpha) y$ can be written as $y_1^t A y_1 + 2y_1^t B^t y_2 - \alpha \|y_2\|$. There exist constants $\eta > 0$ and $\theta > 0$ such that $2y_1^t B^t y_2 \leq \eta \|y_1\| \cdot \|y_2\| \leq \eta \|y_2\|$ and, for $y_1 \neq 0$, $y_1^t A y_1 < -\theta \|y_1\|^2$. Observe that if $\|y_2\|$ is small $\|y_1\|$ must be close to 1 in order that $\|(y_1, y_2)\| = 1$. Hence, there exists $\bar{y}_2$ such that $\eta\|y_2\| - \theta\|y_1\| < 0$ for all y_2 with $\|y_2\| \leq \|\bar{y}_2\|$. Choosing α large enough such that $-\alpha\|y_2\|^2 + \eta\|y_2\| < 0$ for all y_2 with $\|\bar{y}_2\| \leq \|y_2\| \leq 1$ proves that $M(\alpha)$ is negative definite for α large. *Q.E.D.*

LEMMA 2: *Let A be a symmetric negative definite matrix of order $(n_1 \times n_1)$ and B a matrix of order $(n_2 \times n_1)$ such that, for a given vector $(a^t, b^t) \in \mathring{\mathbb{R}}_+^{n_1+n_2}$, $a'A + b'B = 0$, $n_1 \geq 1$, $n_2 \geq 2$. There exists a symmetric matrix C of order $(n_2 \times n_2)$ such that the matrix M defined by*

$$M = \begin{bmatrix} A & B^t \\ B & C \end{bmatrix}$$

is symmetric and negative semi-definite, and $(a^t, b^t)M = 0$.

PROOF OF LEMMA 2: Let $\hat{B}$ be the matrix of order $((n_2 - 1) \times n_1)$ obtained from B by deleting the last row. By Lemma 2, there exists a symmetric matrix $\hat{C}$ of order $((n_2 - 1) \times (n_2 - 1))$ such that the matrix $\hat{M}$ defined by

$$\hat{M} = \begin{bmatrix} \hat{A} & \hat{B}^t \\ \hat{B} & \hat{C} \end{bmatrix}$$

is symmetric and negative definite. Given $\hat{M}$, the symmetry and adding up constraints uniquely define the matrix

$$M = \begin{bmatrix} A & B^t \\ B & C \end{bmatrix}.$$

Since M is symmetric, it is negative semi-definite if and only if $|M_k|(-1)^k \geq 0$ for $k = 1, \ldots, (n_1 + n_2)$, where M_k denotes the kth principal minor of M. Now for $k \leq (n_1 + n_2 - 1)$, $|M_k| = |\hat{M}_k|$ and hence the inequality holds by the negative definiteness of $\hat{M}$. Since $(a^t, b^t)M \neq 0$ and $(a^t, b^t) \neq 0$, $|M| \equiv |M_{n_1+n_2}| = 0$. *Q.E.D.*

LEMMA 3: *Let*

$$\bar{K} = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix}$$

be a $((l_1 + l_2) \times (l_1 + l_2))$ *matrix,* $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\bar{w} = (\bar{w}_1, \bar{w}_2)$, *and* $\bar{p} = (\bar{p}_1, \bar{p}_2)$ *vectors in* $\mathring{\mathbb{R}}^{l_1+l_2}_+$, $\bar{v} = (\bar{v}_1, \bar{v}_2)$ *a vector in* $\mathring{\mathbb{R}}^{l_1+l_2}$, *and* $\bar{F}$ *a* $(l_2 \times l_1)$ *matrix such that:* (i) $\bar{K}$ *is symmetric, negative semi-definite, of rank* $(l_1 + l_2 - 1)$, *and* $(\bar{p}_1^t, \bar{p}_2^t)\bar{K} = 0$; (ii) $\bar{p}_1^t\bar{v}_1 + \bar{p}_2^t\bar{v}_2 = 1$; (iii) $\bar{p}_1^t\bar{x}_1 + \bar{p}_2^t\bar{x}_2 = \bar{p}_1^t\bar{w}_1 + \bar{p}_2^t\bar{w}_2$, $l_1 \geqslant 1$, $l_2 \geqslant 2$. *Then there exists an agent* (w, u, ϕ) *satisfying Assumptions* 0, 1, *and* 2 *whose demand function* $x(p_1) = (x_1(p_1), x_2(p_2))$ *and expectation mechanism satisfy the following conditions:*

(a) $x(\bar{p}_1) = (\bar{x}_1, \bar{x}_2)$;

(b)
$$Dx_1(\bar{p}_1) = \bar{K}_{11} - \bar{K}_{12}\bar{F} - \bar{v}_1(\bar{x}_1 - \bar{w}_1)^t - \bar{v}_1(\bar{x}_2 - \bar{w}_2)^t\bar{F},$$
$$Dx_2(\bar{p}_1) = \bar{K}_{21} + \bar{K}_{22}\bar{F} - \bar{v}_2(\bar{x}_1 - \bar{w}_2)^t - \bar{v}_2(\bar{x}_2 - \bar{w}_2)^t\bar{F};$$

(c) $\phi(\bar{p}_1) = \bar{p}_2$;

(d) $D\phi(\bar{p}_1) = \bar{F}$.

PROOF OF LEMMA 3: Let $\bar{e}$ be an arbitrary real number and consider the matrix

$$D = \begin{bmatrix} \bar{K} & -\bar{v} \\ -\bar{v}^t & \bar{e} \end{bmatrix}.$$

Observe that since $\bar{p}^t\bar{v} = 1$ while $\bar{p}^tK = 0$, $\bar{v} \notin [\bar{K}]$. Consequently, D is invertible. Let

$$D^{-1} = \begin{bmatrix} \bar{U}'' & -\bar{q} \\ -\bar{q}^t & \bar{z} \end{bmatrix}$$

—the form of D^{-1} follows from the symmetry of D. Since rank $(\bar{K}) = l_1 + l_2 + 1$, $\bar{z} = 0$. Since $\bar{q}^t\bar{K} = 0$, $\bar{q}^t \in [\bar{K}]^{\perp} = [\bar{p}]$, and hence $\bar{q} = k\bar{p}$ for some k. But $\bar{q}^t\bar{v} = 1$ implies that $k\bar{p}^t\bar{v} = 1$ and hence $k = 1$; so $\bar{q} = \bar{p}$ by construction. I shall now show that, restricted to $[\bar{p}]^{\perp}$, $\bar{U}''$ defines a negative definite quadratic form. By a theorem of Debreu [1], it suffices to demonstrate that the quadratic form $[\bar{U}'' - \mu\bar{p}\bar{p}^t]$ is negative definite for some $\mu \in \mathbb{R}$. Let $\mu = 1 - \bar{e}$. Then

$$[\bar{U}'' - \mu\bar{p}\bar{p}^t][\bar{K} - \bar{v}\bar{v}^t] = \bar{U}''\bar{K} - \bar{U}''\bar{v}\bar{v}^t + \mu\bar{p}\bar{v}^t$$
$$= I - \bar{p}\bar{v}^t + \bar{e}\bar{p}\bar{v}^t - \mu\bar{p}\bar{v}^t = I.$$

Since $\bar{K}$ is negative on $[\bar{p}]^{\perp}$ and $\bar{v} \notin [\bar{K}]$, $[\bar{K} - v\bar{v}^t]$ is negative definite. Since

$[\bar{U}'' - \mu\bar{p}\bar{p}^t] = [\bar{K} - \bar{v}\bar{v}^t]^{-1}$, $[\bar{U}'' - \mu\bar{p}\bar{p}^t]$ is negative definite as desired. To complete the proof I must demonstrate that there exists a utility function u satisfying Assumption 1′ and an expectation function ϕ satisfying Assumption 2′ such that $D^2u(\bar{x}) = \bar{U}''$, $\phi(\bar{p}_1) = \bar{p}_2$, and $D\phi(\bar{p}_1) = \bar{F}$. But this is evident. *Q.E.D.*

PROOF OF PROPOSITION 3: It suffices to demonstrate that conditions (a) through (d) in Lemma 3 impose no restrictions on the Jacobian Dx_1. By Lemma 2, (a) and (b) are satisfied if $K_{11}(\bar{p}_1)$ is a negative definite matrix of order $(l_1 \times l_1)$ and $K_{21}(\bar{p}_1)$ is a matrix of order $(l_2 \times l_1)$ such that $p_1^t K_{11}(\bar{p}_1) + \phi(p_1)^t K_{21}(\bar{p}_1) = 0$. With no loss of generality let

$$D\phi(\bar{p}_1) = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

where $D\phi(\bar{p}_1)$ is of order $(l_2 \times l_1)$ and I is the identity matrix of order $(l_1 \times l_1)$. Let $v_1(\bar{p}_1) = 0$, and let A be an arbitrary matrix of order $(l_1 \times l_1)$. Define $K_{12}(\bar{p}_1)$ by $K_{12}(\bar{p}_1) = A - K_{11}(\bar{p}_1)$ and derive $K_{12}(\bar{p}_1)$ by adjoining $(l_2 - l_1)$ rows to $K_{12}(\bar{p}_1)$ so as to satisfy $p_1^t K_{11}(\bar{p}_1) + \phi(p_1)^t K_{12}(\bar{p}_1) = 0$. This is clearly possible since $l_2 \geq l_1 + 1$ and $(p_1^t, \phi(p_1)^t) \in \mathring{\mathbb{R}}_+^{l_1+l_2}$. Then $Dx_1(\bar{p}_1) = A$—as desired. *Q.E.D.*

Proposition 3 characterizes a class of expectation functions, Φ^*, which can be employed to yield the infinitesimal indeterminacy of expectations.

REMARK 3: The assumption that $D\phi(p_1)$ has everywhere full rank is essential for Proposition 3. In particular, consider the static expectation function $\phi(p_1) = \bar{p}_2$, where $D\phi(p_1) = 0$ for all p_1. Then the Jacobian $Dx_1(p_1)$ is negative definite on the orthogonal complement of $(x_1 - w_1)$ and hence it is not arbitrary.

REMARK 4: The assumption that $l_2 \geq l_1 + 1$ is essential. In the case $l_1 = l_2$, the following argument shows that one rational agent may not suffice to locally generate an arbitrary current demand function. Consider the economy with $l_1 = l_2 = 1$. If $\phi(p_1)$ is specified, and since the initial endowment is assumed to be observable, the observed $x_1(p_1)$ fully defines $x_2(p_1)$ by

$$x_2(p_1) = w_2 - \frac{p_1}{\phi(p_1)}(x_1(p_1) - w_1)$$

and reveals the individual's offer curve. Since the income effect vanishes at the initial endowment point, $Dx_1(p)$ is *not* arbitrary.

The global indeterminacy of expectations is an open question.

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