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OBSERVABLE PROBABILISTIC BELIEFS*

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Under the assumption that asset markets are possibly incomplete, conditions are examined under which an agent's asset demand correspondence can be used to recover without ambiguity his subjective joint distribution of returns of the various assets and/or his von Neumann–Morgenstern cardinal utility index.

1. Introduction

Individual choice under uncertainty depends on an agent's attitudes towards risk — typically represented by his von Neumann–Morgenstern cardinal utility index — and probabilistic beliefs — typically summarized by his subjective joint distribution of returns of the available assets. Both of these characteristics are however *unobservable*. What is in principle at least *observable* is the agent's response to alternative opportunity sets — typically described by his asset demand correspondence.

A variety of problems of prediction and welfare require conclusions concerning an agent's unobservable characteristics to be drawn from his observable behavior. Consider for example the case of a firm in an economy with incomplete markets. Computation of its market value under alternative production plans requires knowledge of the marginal shareholder's von Neumann–Morgenstern cardinal utility index. The question is whether the latter can be unambiguously deduced from his choice among existing assets. Alternatively, suppose that an agent is known to possess superior information concerning an uncertain prospect. For a second, less informed, agent to be able to acquire this additional information from market data, it must be the case that the better informed agent's beliefs can be unambiguously recovered from his demand behavior. If it turns out that in plausible environments demand behavior fails to reveal an agent's

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information — probabilistic beliefs — then doubts are cast on the validity of, among others, the efficient markets hypothesis — equilibrium prices are then bound to fail to reveal the relevant information.[1]

This paper analyzes the recoverability of attitudes towards risk and probabilistic beliefs from the asset demand corespondence. The recoverability of the von Neumann–Morgenstern utility index from asset demands under the assumption that the agent's beliefs concerning the joint distribution of returns of the various assets are known has been dealt with elsewhere — Dybvig and Polemarchakis (1981), Green, Lau and Polemarchakis (1979). Consequently, I concentrate here on the recoverability of probabilistic beliefs, as well as the simulataneous recoverability of attitudes towards risk and probabilistic beliefs.

Before specifying formally the agent's allocation problem, a few clarificatory remarks are in order:

(i) The essential aspect of the recoverability problem analyzed in this paper is the incompleteness of markets. Strong regularity assumptions are made concerning the agent's unobservable characteristics, which render recoverability in a complete market setting immediate.
(ii) The central role attached to the incompleteness of markets differentiates the problem at hand from similar problems in which 'experiments' or 'environments' are specifically designed so as to elicit an agent's unobservable characteristics — see, for example, Shapiro and Richter (1978).
(iii) Recoverability is analyzed here in the simplest possible framework — the one-period, one-attribute resource allocation problem. Extensions to more sophisticated choice frameworks are evident.

An agent with von Neumann–Morgenstern cardinal utility index $u(w)$, defined on an interval $\mathscr{D}$, subset of $\boldsymbol{R}$, must allocate his initial wealth, normalized to equal unity, among m assets, $m \geqq 2$, indexed by j, $j \in \{1,\dots,m\}$. States of nature are indexed by s, $s \in S$. The agent's subjective joint distribution of returns of the m assets is described by the probability measure π defined on the measurable space $(S, \mathscr{S})$ — where $\mathscr{S}$ is a σ-field on S — and by the family of random variables $R = \{r_j \mid j = 1,\dots,m\}$ defined on $(S, \mathscr{S}, \pi)$.[2] The pair (π, R) is thus the subjective asset return structure. I use r to denote the vector $r = (r_1,\dots,r_j,\dots,r_m)$. A price system is a vector $p = (p_1,\dots,p_m)$ and a portfolio a vector $x = (x_1,\dots,x_m)$; neither x nor p are required to be non-negative. Faced with prices p, the agent chooses $x(p)$ so as to maximize his

[1]A way out of the impass is to follow a 'generic' approach to the revelation problem — see Radner (1979).

[2]If S is discrete $\mathscr{S}$ will be taken to be its power set and R will be a $(|S| \times m)$ matrix and π a vector in $\boldsymbol{R}^{|S|}_{+}$.

von Neumann–Morgenstern objective function $\phi(x)=\mathrm{E}u(rx)$, subject to the budget constraint $px=1$, as well as the feasibility constraint $x\in X$, where $X=\{x\in \boldsymbol{R}^m \mid rx\in \mathscr{D}$ for a.a. $s\in S\}$ is the set of feasible portfolios. An agent is thus fully described by the unobservable characteristics (u,π,R), concerning which I shall make the following regularity assumption:

Assumption

(i) u is a twice continuously differentiable function defined on an interval $\mathscr{D}$ subset of $\boldsymbol{R}$; $u'>0$ and $u''\leqq 0$ everywhere on $\mathscr{D}$;[3,4]
(ii) the set of random variables $R=\{r_j \mid j=1,\dots,m\}$ is linearly independent; $r_j\in\mathscr{D}$ for a.a. $s\in S, j=1,\dots,m$;
(iii) if the utility index u is of class C^k on $\mathscr{D}$, then for any set of non-negative integers $l_1,\dots,l_m,l,\sum_{j=1}^m l_j=l, l\leqq(k-1)$, the expression $\mathrm{E}r_1^{l_1}\dots r_m^{l_m}u^{(l)}(rx)$ is well defined and differentiable for any $x\in X$; furthermore,

$$\frac{\partial}{\partial x_j}\mathrm{E}r_1^{l_1}\dots r_m^{l_m}u^{(l)}(rx)=\mathrm{E}r_1^{l_1}\dots r_j^{(l_j+1)}\dots r_m^{l_m}u^{(l+1)}(rx), \qquad j=1,\dots,m;$$

(iv) all moments and absolute moments of r_j exists, $j=1,\dots,m$; furthermore,

$$\limsup_{n\to\infty}\frac{1}{n}[\mathrm{E}|r_j|^n]^{1/n}=\lambda_j<\infty, \qquad j=1,\dots,m.$$

An agent (u,π,R) is said to be a *regular von Neumann–Morgenstern* agent if his unobservable characteristics satisfy (i)–(iv) above.

The regularity assumption merits some discussion. Assumption (i) is strong but standard, while the linear independence assumption in (ii) excludes redundant assets and is innocuous. That $r_j\in\mathscr{D}$ with probability one can be relaxed to the requirement that the support of r_j be bounded from below (or above) as long as $\mathscr{D}$ is itself bounded from below (or above), but this is a matter of normalization; as such it is natural. Observe that an asset with support unbounded from above as well as from below never enters in a non-trivial way a feasible portfolio of an agent whose cardinal utility index is defined on a domain bounded from below or above. Assumption (iii) assures that the expectation operator and the differential operator commute; it is implied by the remaining regularity assumptions as long as the support of the random variables is compact (even more finite), but this would be too strong a restriction in the present framework. Finally, assumption (iv) is

[3]For $Y\subset\boldsymbol{R}$, $\dot{Y}$ denotes its interior; for a function f, $f^{(k)}$ denotes the kth derivative. The notations f' and $f^{(1)}$ or f'' and $f^{(2)}$ are used interchangably; so are $\boldsymbol{R}_+$ and $[0,\infty)$ or $\dot{\boldsymbol{R}}_+$ and $(0,\infty)$.

[4]If the domain Y of a function f is closed, f is said to be of class C^k on Y if and only if there exists an extension of f to an open neighborhood of Y.

purely technical and its function can be explained as follows: Given a random variable y with distribution function F, we can define the characteristic function $\varphi: \boldsymbol{R} \to C$ by $\varphi(t) \equiv \mathrm{E}(\mathrm{e}^{ity})$. Furthermore, since distinct distributions have distinct characteristic functions, to recover the distribution function F it suffices to recover the characteristic function φ. If in addition

$$\limsup_{n\to\infty} \frac{1}{n} [\mathrm{E}(|y|^n)]^{1/n} = \lambda < \infty,$$

the characteristic function is analytic in a neighborhood of the real axis and hence determined by its power series about the origin. Since $\varphi^{(n)}(0) = \mathrm{i}^n \mathrm{E}(y^n)$, the characteristic function — and hence the distribution function F as well — is completely determined by the moments of the random variable.[5] It is this latter property that we use in the argument for recoverability of probabilistic beliefs.

On a different level, it can be argued that there is some arbitrariness involved in the specification of the asset return structure (π, R).[6] Agents' preferences are, strictly speaking, defined over distributions of returns and portfolios are chosen based on the distribution of returns they generate. The point is well taken and the answer straightforward. Clearly, given (π, R), the distribution of returns of any feasible portfolio — rx for $x \in X$ — can be computed without ambiguity. On the other hand, if (π, R) is not known, we can recover at best the equivalence class of return structures which generate the same distribution for any feasible portfolio. I use the term distinct asset return structures to mean that they generate distinct distribution functions for some feasible portfolio. Equivalently, the asset return structure will be said to be recoverable without ambiguity if the distribution of any feasible portfolio can be determined without ambiguity.

The observable characteristics of the agent (u, π, R) consist of the demand correspondence $x(p)$ defined as the solution to the problem

$$\max_{x \in X} \mathrm{E}u(rx) \quad \text{s.t.} \quad px = 1. \tag{1}$$

A solution to (1) may of course not exist for an arbitrary $p \in \boldsymbol{R}^m$. We denote by P the subset of $\boldsymbol{R}^m$ on which a solution to (1) exists. Knowledge of the demand correspondence entails knowledge of P.

The question of recoverability can now be formulated as follows: Is the information contained in the demand correspondence $x(p)$ sufficient to recover without ambiguity the unobservable characteristics (u, π, R)? Recoverability without ambiguity requires that the generating characteristics

[5]For proofs connected with this argument, see Feller (1966, vol. 2, p. 487).

[6]I wish to thank L. Selden for emphasizing this issue and helping me in clarifying it.

be unique and that they be computable — in some way — from the demand correspondence. A milder requirement would involve the first but not necessarily the second.

It is of course conceivable that the demand correspondence has been derived from a more general class of characteristics than those satisfying the regularity assumption. Furthermore, the objective function need not be von Neumann–Morgenstern. This is not an issue I want to consider; here I raise the question of recoverability of the unobservable characteristics under the assumption that the latter satisfy the conditions of regularity and in particular that the objective function maximized is indeed von Neumann–Morgenstern and state independent.

The requirement of recoverability is excessively strong — i.e., it fails, as examples demonstrate later on. Consequently we are led to examine conditions under which independent knowledge of *some* of the unobservable characteristics, in addition to the demand correspondence, suffices to recover unambiguously the remaining ones.

2. From probabilistic beliefs to cardinal utility

This problem was handled in Green, Lau and Polemarchakis (1979) and Dybvig and Polemarchakis (1981). There it was demonstrated that given (π, R) recoverability of u holds under two alternative sets of conditions:

Proposition 1 [Green, Lau and Polemarchakis (1979)]. *If the utility index u of a regular von Neumann–Morgenstern agent is analytic on $\mathcal{D}=[0,\infty)$ and if the asset return characteristics (π, R) are known, the utility index u can be recovered without ambiguity — within the class of utility indices analytic on* $[0,\infty)$.

Proposition 2 [Dybvig and Polemarchakis (1981)]. *If a linear combination of assets is riskless and if the variance of the distribution of returns of a risky asset, as well as the return of the riskless, are known, the utility index u of a regular von Neumann–Morgenstern agent can be recovered without ambiguity on $\dot{\mathcal{D}}$.*

The above Propositions 1 and 2 leave open two questions: First whether recoverability can be attained in the absence of analyticity as well as of a riskless asset; second whether recoverability can be attained in the absence of knowledge of the variance of the distribution of returns of a risky asset or the return of the riskless one. Concerning the first question, a class of examples[7] were given in Dybvig and Polemarchakis (1981) to show that

[7]These examples generalized the counterexample to recoverability developed and communicated privately by A. McLennan and H. Sonnonschein — see McLennan (1979).

recoverability may fail if no linear combination of the assets is known to be riskless, even under the assumption of regularity of the agent's characteristics. The second question refers to the possibility of simultaneous recoverability of attitudes towards risk and probabilistic beliefs. It is taken up in section 4.

3. From cardinal utility to probabilistic beliefs

Reversing point of view from the previous section we now suppose that — in addition to the asset demand correspondence $x(p)$ — the agent's cardinal utility index u is known. We want to recover his asset return characteristics (π, R). I shall first give two examples in which recoverability fails. Let $u(w)=w^{\frac{1}{2}}$ and $S=\{1,2,3,4\}$, and suppose that the asset returns are known to be

$$R^t=\begin{bmatrix}1 & 1 & 4 & 9\\ 1 & 2 & 8 & 18\end{bmatrix}.$$

Then the objective function is

$$\begin{aligned}\phi(x_1,x_2)&=\pi_1(x_1+x_2)^{\frac{1}{2}}+\pi_2(x_1+2x_2)^{\frac{1}{2}}+\pi_3(4x_1+8x_2)^{\frac{1}{2}}+\pi_4(9x_1+18x_2)^{\frac{1}{2}}\\ &=\pi_1(x_1+x_2)^{\frac{1}{2}}+(\pi_2+2\pi_3+3\pi_4)(x_1+2x_2)^{\frac{1}{2}}.\end{aligned}$$

Consequently, two state probability measures π and π' will be indistinguishable on the demand level as long as

$$\pi_1=\pi_1' \quad \text{and} \quad (\pi_2+2\pi_3+3\pi_4)=(\pi_2'+2\pi_3'+3\pi_4').$$

On the other hand, π and π' may generate different wealth distributions and would not lead to identical demand behavior for a different cardinal utility index. Recoverability thus fails. Similarly, for the same cardinal utility index and state space, suppose that the measure of state probabilities is known and equal to

$$\pi=(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}),$$

while asset returns are of the form

$$R^t=\begin{bmatrix}1 & \alpha^2 & \beta^2 & \gamma^2\\ 1 & 2\alpha^2 & 2\beta^2 & 2\gamma^2\end{bmatrix} \quad \text{for some } \alpha,\ \beta,\ \gamma.$$

Then the objective function is

$$\begin{aligned}\phi(x_1,x_2)&=\tfrac{1}{4}(x_1+x_2)^{\frac{1}{2}}+\tfrac{1}{4}(\alpha^2x_1+2\alpha^2x_2)^{\frac{1}{2}}+\tfrac{1}{4}(\beta^2x_1+2\beta^2x_2)^{\frac{1}{2}}+\tfrac{1}{4}(\gamma^2x_1+2\gamma^2x_2)^{\frac{1}{2}}\\ &=\tfrac{1}{4}(x_1+x_2)^{\frac{1}{2}}+\tfrac{1}{4}(\alpha+\beta+\gamma)(x_1+2x_2)^{\frac{1}{2}}.\end{aligned}$$

Consequently, two distinct asset returns R and R' will be indistinguishable on the demand level as long as $(\alpha+\beta+\gamma)=(\alpha'+\beta'+\gamma')$. Since R and R' may clearly generate different wealth distributions, recoverability again fails.

Having given examples in which recoverability fails, I now want to demonstrate that recoverability can be obtained in the presence of a riskless asset with a known rate of return. Equivalently, that if, in addition to regularity, it is assumed that a given linear combination of the available assets is riskless and its return is known, then the asset demand correspondence $x(p)$ in conjunction with the cardinal utility index $u(w)$ is roughly speaking sufficient to recover the agent's subjective asset return characteristics without ambiguity:

Proposition 3. If a linear combination of assets is riskless with known return and if the utility index u of a regular von Neumann–Morgenstern agent is known, of class C^∞ on $\mathscr{D}$ and $u^{(n)}(w)\not\equiv 0$ on $\dot{\mathscr{D}}$, $n=1,2,\ldots$, the subjective asset return characteristics can be recovered without ambiguity.

Proof. For simplicity, suppose that there are only two assets, $j=1,2$ and that $r_1=1$ for a.a. $s\in S$ — the asset $j=1$ is riskless with return of unity. Let P^* denote the subset of P on which the solution to the maximization problem (1) is characterized by the first-order conditions

$$\begin{aligned} &\mathrm{E}u'(rx)=\lambda p_1, \\ &\mathrm{E}r_2u'(rx)=\lambda p_2, \qquad (2)\\ &p_1x_1+p_2x_2=1. \end{aligned}$$

By the regularity assumption on the characteristics (u,π,R) there exists an open set X^*, containing $\dot{\mathscr{D}}\times[0]$, such that for $\bar{x}\in X^*\bar{x}=x(\bar{p})$ for a unique price system $\bar{p}\in P^*$. The inverse demand function $p(x)$ is thus well defined on $\dot{\mathscr{D}}\times[0]$ and hence observable — observability follows from the observability of the demand correspondence $x(p)$ on P^*. Consider now the marginal rate of substitution function $m(x)=(p_2/p_1)(x)$. The function is well defined on $\dot{\mathscr{D}}\times[0]\subset X^*$, and, furthermore, it follows from (2) that

$$\mathrm{E}r_2u'(rx)=m(x)\mathrm{E}u'(rx). \qquad (3)$$

By (iii) of the regularity assumption, if the utility index $u(w)$ is of class C^∞ on $\mathscr{D}$, so is the function $m(x)$ on $\dot{\mathscr{D}}\times[0]\subset X^*$. For brevity of notation, let

$$u^{(k)}(w)=\mathrm{d}^ku(w)/\mathrm{d}w^k \quad \text{and} \quad m_j^{(k)}(x)=\partial^km(x)/\partial x_j^k, \qquad j=1,2,$$

and

$$m^{(k_1,k_2)}(x) = \partial m^{k_1+k_2}/\partial x_1^{k_1}\,\partial x_2^{k_2}.$$

Successive differentiation of (3) with respect to x_2 then gives for any $l \geqq 0$,

$$\mathrm{E}r^{l+1}u^{(l+1)}(rx) = \sum_{k=0}^{l}\binom{l}{k} m_2^{(k)}(x)\mathrm{E}r_2^{l-k}u^{(l+1-k)}(rx). \tag{4}$$

Evaluating (4) at $x=(\alpha,0)$, $\alpha \in \dot{\mathscr{D}}$, and rearranging, we get for any $l \geqq 1$,

$$u^{(l+1)}(\alpha)\mathrm{E}[r_2^{l+1} - m(\alpha,0)r_2^l] = \sum_{k=1}^{l}\binom{l}{k} m_2^k(\alpha,0)u^{(l+1-k)}(\alpha)\mathrm{E}r_2^{l-k}. \tag{5}$$

For example,

$$u^{(1)}(\alpha)\mathrm{E}[r_2 - m(\alpha,0)] = 0,$$

$$u^{(2)}(\alpha)\mathrm{E}[r_2^2 - m(\alpha,0)r_2] = m_2^{(1)}(\alpha,0)u^{(1)}(\alpha),$$

$$u^{(3)}(\alpha)\mathrm{E}[r_2^3 - m(\alpha,0)r_2^2] = 2m_2^{(1)}(\alpha,0)u^{(2)}(\alpha)\mathrm{E}r_2 + m_2^{(2)}(\alpha,0)u^{(1)}(\alpha), \quad \text{etc.}$$

It then follows that as long as, for any $l \geqq 0$, $u^{(l+1)}(\alpha) \neq 0$ for some $\alpha \in \dot{\mathscr{D}}$, (5) can be used to recover without ambiguity the $(l+1)$st moment of the distribution of the random variable r_2, $\mathrm{E}r_2^{l+1}$, $l=0,1,\ldots$. Consider now the characteristic function

$$\varphi_2(t) \equiv \mathrm{E}(\mathrm{e}^{\mathrm{i}r_2 t}). \tag{6}$$

Knowledge of the distribution of the random variable r_2 is equivalent to knowledge of the characteristic function $\varphi_2(t)$. But the regularity assumption (iv) guarantees [see Feller (1966, p. 487)] that the characteristics function $\varphi_2(t)$ is completely determined by its derivatives at the origin. Since $\varphi_2^{(l+1)}(0) = \mathrm{i}^{(l+1)}\mathrm{E}r_2^{l+1}$, $l=0,1,\ldots$, the distribution of r_2, and hence of any feasible portfolio, is determined. To complete the argument we must handle the case of multiple risky assets. The objective is to determine the distribution of returns of any feasible portfolio rx. Since $rx \in \mathscr{D}$ for a.a. $s \in S$, we can repeat the preceding argument with r_2 replaced by rx. Q.E.D.

The requirement that no derivative of the utility index u vanish identically in the argument for the recoverability of the complete distribution of any portfolio is not surprising. If for example the utility index u is polynomial, only finitely many moments of the distribution of the risky assets enter the agents' objective function and hence the possibility of distinct distributions generating the same asset demand function cannot be excluded.

The assumption of existence of a riskless asset can be replaced by the assumption that there exists an asset — $j=1$ for simplicity — such that the distribution of the random variable r_1 is known and its support is contained in $[0,\infty)$, and that the domain of definition of the utility index, $\mathscr{D}$, contains the origin, i.e., $[0,\infty)\subset\mathscr{D}$:

Proposition 4. If the distribution of returns of one asset is known and supported on $[0,\infty)$ *and if the utility index, u, of a regular von Neumann–Morgenstern agent is known, of class* C^∞ *on* $[0,\infty)$ *and* $u^{(n)}(0)\neq 0$, $n=1,2,\ldots$, *the subjective asset return characteristics can be recovered without ambiguity.*

Proof. I shall only outline the argument since it is very much along the lines of the proof of Proposition 3. Without loss of generality we may suppose again that there is only one asset, $j=2$, other than the one whose distribution of returns is known.

From the first-order conditions for a maximum, it follows that

$$\mathrm{E}r_2u'(rx)=m(x)\mathrm{E}r_1u'(rx), \tag{7}$$

where the marginal rate of substitution function is well defined and of class C^∞ on $\boldsymbol{R}_+\times[0]\subset X^*$. Successive differentiation of (7) with respect to x_1 and x_2, and evaluation at $x=0$ yields the following systems of equations:

$$\begin{aligned}
&\mathrm{E}r_2=m(0)\mathrm{E}r_1,\\
&\mathrm{E}r_2r_1u^{(2)}(0)=m^{(1,0)}(0)\mathrm{E}r_1u^{(1)}(0)+m(0)\mathrm{E}r_1^2u^{(2)}(0),\\
&\mathrm{E}r_2^2u^{(2)}(0)=m^{(0,1)}(0)\mathrm{E}r_1u^{(1)}(0)+m(0)\mathrm{E}r_1r_2u^{(2)}(0),\\
&\mathrm{E}r_2r_1^2u^{(3)}(0)=m^{(2,0)}(0)\mathrm{E}r_1u^{(1)}(0)+2m^{(1,0)}(0)\mathrm{E}r_1^2u^{(2)}(0)+m(0)\mathrm{E}r_1^3u^{(3)}(0),\\
&\mathrm{E}r_2^2r_1u^{(3)}(0)=m^{(1,1)}(0)\mathrm{E}r_1r_2u^{(1)}(0)+m^{(1,0)}(0)\mathrm{E}r_1r_2u^{(2)}(0)\\
&\qquad\qquad+m^{(0,1)}\mathrm{E}r_1^2u^{(2)}(0)+m(0)\mathrm{E}r_1^2r_2u^{(3)}(0),\\
&\mathrm{E}r_2^2r_1u^{(3)}(0)=m^{(3,0)}(0)\mathrm{E}r_1u^{(1)}(0)+m^{(2,0)}(0)\mathrm{E}r_1^2u^{(2)}(0)\\
&\qquad\qquad+m^{(1,0)}\mathrm{E}r_1r_2u^{(2)}(0)+m(0)\mathrm{E}r_1^2r_2u^{(3)}(0),\\
&\mathrm{E}r_2^3u^{(3)}(0)=m^{(3,0)}\mathrm{E}r_1u^{(1)}(0)+2m^{(0,1)}\mathrm{E}r_1r_2u^{(2)}(0)+m(0)\mathrm{E}r_1r_2^2u^{(3)}(0).
\end{aligned} \tag{8}$$

Continuing in this fashion we can derive $\mathrm{E}(r_1^{l_1}, r_2^{l_2})$, $k_1 \geqq 0$, $l_2 \geqq 0$, as long as $u^{l+1}(0) \neq 0$, $l \geqq 0$. By an argument exactly as before this suffices — given the regularity assumption (iii) — to recover the joint distribution of the random variables r_1 and r_2. Q.E.D.

4. To cardinal utility as well as probabilistic beliefs

It was demonstrated in the two preceding sections that as long as an agent's characteristics are regular and a riskless asset is available observation of his asset demand correspondence is, roughly speaking, sufficient to render his subjective joint distribution of asset returns recoverable without ambiguity from independent knowledge of his cardinal utility index, and vice versa. As a matter of fact, however, independent knowledge either of an investor's attitudes towards risk, or of his probabilistic beliefs is rather unlikely. We are thus led naturally to tackle the problem of simultaneous recoverability of the characteristics (u, π, R), from the asset demand function. The following corollary to Proposition 2 and 3 is clear:

Corollary 1. If a linear combination of assets is riskless, if the variance of the distribution of returns of a risky asset, as well as the return of the riskless asset, are known and if the utility index, u, is of class C^∞ on $\mathscr{D}$ and $u^{(n)}(w) \neq 0$ on $\dot{\mathscr{D}}$, $n = 1, 2, \ldots$, the triple of characteristics (u, π, R) of a regular von Neumann–Morgenstern agent can be recovered without ambiguity.

That knowledge of the distribution of returns cannot be altogether dispensed with is evident by the following argument: Even if markets are complete, the agent with characteristics (u, π, R) generates the same asset demand function as the agent $(u^*, \pi, (1/k)R)$, where $u^*(w) = u(kw)$, any $k > 0$. An increase in the mean and variance of returns can be compensated for by a reciprocal reduction in the agent's risk aversion so as to leave the observable demand behavior unchanged. The only remaining question is whether recoverability can be attained without knowledge of the variance of the returns of any risky asset. To see that this is not the case, consider the agent with cardinal utility index $u(w) = -\mathrm{e}^{-\rho w}$, some $\rho > 0$, and suppose he must allocate his initial wealth of unity between a riskless asset with return equal to $\bar{r}_1$, $\bar{r}_1 > 0$, and a risky asset whose return is normally distributed, with mean $\bar{r}_2$, $\bar{r}_2 > 0$, and variance σ_2^2, $\sigma_2^2 > 0$. A simple computation shows that the demand function for the risky asset is given by $x_2(p_1, p_2) = [\bar{r}_2 - \bar{r}_1(p_2/p_1)]/(\rho\sigma_2^2)$. As a result, if either the return of the riskless asset, $\bar{r}_1$, or the variance of the return of the risky asset σ_2^2 is not known, recoverability fails.

Since that knowledge of the variance of the return of some risky asset is necessary for recoverability, we would like to describe the indeterminacy that would prevail in its absence. This is the object of the following:

Proposition 5. If a linear combination of assets is riskless, if the return of the riskless asset is known and if two distinct triples (u, π, R) *and* $(\hat{u}, \hat{\pi}, \hat{R})$ *of regular von Neumann–Morgenstern characteristics generate the same asset demand correspondence, then* $u^{(1)}(w) = [\hat{u}^{(1)}(w)]^{\delta}$, *some* $\delta > 0$, *everywhere on* $\mathscr{D}$.

Proof. The argument is straightforward. Denoting as before, by $m(x)$ the marginal rate of substitution between the riskless and a (the) risky asset, differentiating the first-order conditions with respect to x and evaluating at $(x_1, x_2) = (\alpha, 0)$, $\alpha \in \dot{\mathscr{D}}$, we get

$$u^{(2)}(\alpha)/u^{(1)}(\alpha) = m_2^{(1)}(\alpha, 0)/\sigma_2^2, \tag{9}$$

where $\sigma_2^2 = \mathrm{E}(r_2^2) - [\mathrm{E}(r_2)]^2$. But if the triple $(\hat{u}, \hat{\pi}, \hat{R})$ generates the same demand function as (u, π, R), it generates the same marginal rate of substitution function as well and hence

$$\hat{u}^{(2)}(\alpha)/\hat{u}^{(1)}(\alpha) = m_2^{(1)}(\alpha, 0)/\hat{\sigma}_2^2. \tag{10}$$

From (9) and (10), setting $\delta = \hat{\sigma}_2^2/\sigma_2^2$ and integrating gives the desired result. Q.E.D.

To conclude the discussion of simultaneous recoverability of probabilistic beliefs and attitudes towards risk, the following must be pointed out: The example, using exponential utility and normally distributed returns for the risky asset, employed to demonstrate that knowledge of variance of some risky asset is necessary for recoverability, depends on the non-finiteness of the state space. It remains an open question whether — under the additional assumption that the support of the distribution of some risky asset is finite — the variance can be recovered as well. Even if true, however, the result would be of limited interest, since the state space is not, after all, an exogenous structure, but part of the subjective probabilistic beliefs of the agent.

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