HOMOTHETICITY AND THE AGGREGATION OF CONSUMER DEMANDS*

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As long as the price mechanism is relied upon for the allocation of resources, the characterization of the response of a group of agents to a change in prices is a necessary step in the development of any dynamic argument, or in the analysis of comparative statics. The theory of consumer behavior has dealt with this problem in the special case in which the group consists of a single agent; the characterization that has been derived can be summarized by the negative semi-definiteness and symmetry of the Slutsky substitution matrix. The question arises whether the same characterization can be derived for the response to prices of larger groups of consumers as well. It is known that this need not be the case. As demonstrated in Geanakoplos and Polemarchakis [1980], the larger the group of agents considered, the higher the degree of "arbitrariness" that aggregate behavior may display; and if the number of agents is equal to (or exceeds) the number of commodities, the aggregate need not obey any restriction other than homogeneity with respect to prices and Walras' law [Debreu, 1974].

That an aggregate (excess) demand function need not satisfy any qualitative restrictions, however, depends on the assumption that the preferences of the individual agents composing the group need not display any particular characteristics other than convexity and monotonicity and that, similarly, the distribution of initial endowments be unrestricted. As a consequence, one can pose the question whether there exist properties, beyond convexity and monotonicity, which, if satisfied by a collection of rational consumers, have as a consequence that the aggregate (excess) demand function can be generated by the maximization of a quasi-concave, monotone utility function subject to the aggregate budget constraint. Chichilnisky and Heal [1979], Chipman [1974], Eisenberg [1961], Gorman [1953], and Samuelson [1956] have given an affirmative answer and have distinguished two cases under which aggregation is possible: first, the case in which individual agents possess (up to possibly different translated origins) *identical* homothetic preferences; i.e., their income expansion paths are *parallel* straight lines; second, the case of possibly diverse ho-

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mothetic preferences and a *fixed* (i.e., independent of prices and aggregate income) distribution of income, or, equivalently, of *collinear* initial endowment vectors. Mantel [1976], on the other hand, demonstrated that as long as the distribution of initial endowment vectors is unrestricted, homotheticity of individual preferences imposes no restrictions on aggregate behavior. More precisely, let $f(p)$ be a twice differentiable function defined on a compact convex subset of the price simplex. Suppose that the second derivatives of $f(p)$ are uniformly bounded and $f(p)$ satisfies Walras' law. Then there exists a collection of agents, each endowed with a strictly concave, *homogeneous* utility function—and thus homothetic preferences—such that the aggregate excess demand function $x(p)$ coincides with $f(p)$.

The purpose of this paper is to make the point complementary to that of Mantel: *Under the assumption that agents are endowed with a fixed share of the aggregate endowment vector, homotheticity of preferences is necessary, as well as sufficient, for aggregation.*

Consider a pure exchange economy consisting of m agents indexed by a superscript k, $k = 1, \ldots, m$. Agent k is characterized by his consumption set Y^k, a convex subset of the commodity space R^l, his endowment vector w^k, a point in Y^k, and his utility function u^k defined on Y^k. I shall assume that all agents are *regular* [Debreu, 1972]. Equivalently, for all k the following conditions are satisfied:

1. The consumption set Y^k is the positive orthant of the commodity space R^l_+.

2. The endowment vector w^k belongs to the interior of the consumption set $\mathring{Y}^k$.

3. The utility function u^k is strictly quasi concave and twice continuously differentiable on $\mathring{Y}^k$; at all $y \in \mathring{Y}^k$, $Du^k(y) \gg 0$, the indifference hypersurface through y has nowhere vanishing Gaussian cuvature, and the closure of the indifference hypersurface through y relative to R^l is contained in $\mathring{Y}^k$.

Given a price vector $p \in \mathring{R}^l_+$, agent k expresses excess demand $x^k(p)$ by solving the following problem:

$$\max_{w^k + x \in Y^k} u^k(w^k + x) \qquad \text{subject to } p^t x = 0. \tag{1}$$

By the assumption of regularity, the excess demand function $x^k(p)$ is well defined and continuously differentiable everywhere on $\mathring{R}^l_+$. The aggregate excess demand function $x(p)$ is defined as $x(p) = \Sigma_{k=1}^m x^k(p)$. It is clearly well defined and continuously differentiable.

To solve the problem of aggregability, we must first answer the following question: What are the conditions that must be satisfied by a function $f(p)$ so that it can be derived as the excess demand function of a regular agent? The answer to this question is well-known: Let $f(p)$ be continuously differentiable and satisfy homogeneity ($Df(p)p \equiv 0$) and Walras' law ($p^tDf(p) = -f(p)$). For a given $\bar{p} \in \mathring{\mathbf{R}}^l_+$, suppose that there exists a $(l \times l)$ matrix K, which is symmetric, negative semi-definite, of rank $(l-1)$, with $K\bar{p} = 0$, and a $(l \times 1)$ vector v with $\bar{p}^tv = 1$, such that $Df(\bar{p}) = K - vf(\bar{p})^t$. Then there exists a regular agent (X,w,u) whose excess demand function $x(p)$ agrees at $\bar{p}$ with $f(p)$; i.e., $f(\bar{p}) = x(\bar{p})$ and $Df(\bar{p}) = Dx(\bar{p})$. Thus, given a collection of regular agents $\{(X^k,w^k,u^k), k = 1, \ldots, m\}$ with aggregate excess demand function $x(p)$, we say *aggregation is possible at* $\bar{p} \in \mathring{\mathbf{R}}^l_+$, if and only if $Dx(\bar{p}) = K(\bar{p}) - v(\bar{p})x(\bar{p})^t$, where

(1) $K(\bar{p})$ is symmetric, negative semi-definite, of rank $(l-1)$, and $p^tK(\bar{p}) = 0$;
(2) $\bar{p}^tv(\bar{p}) = 1$.

Aggregation is possible[1] if and only if it is possible everywhere on $\mathring{\mathbf{R}}^l_+$.

Characteristics in addition to regularity refer either to the distribution of initial endowments or to the preferences or demand functions of individual agents. For a collection of agents $\{(X^k,w^k,u^k), k = 1, \ldots, m\}$ if there exist strictly positive constraints, δ^k, $k = 1, \ldots, m$, with $\Sigma_{i=1}^m \delta^k = 1$, such that $w^k = \delta^k w$, where w is the aggregate endowment vector, the initial endowment vectors are said to be *collinear*. Equivalently, the distribution of income is independent of prices and aggregate income. A regular agent (X,w,u) with excess demand function $x(p)$ is said to be *homothetic at* $\bar{p} \in \mathring{\mathbf{R}}_+$, if and only if $Dx(\bar{p}) = K(\bar{p}) - v(\bar{p})x(\bar{p})^t$, where

(1) $K(\bar{p})$ is symmetric, negative semi-definite, of rank $(l-1)$, and $\bar{p}^tK(\bar{p}) = 0$;
(2) $\bar{p}^tv(\bar{p}) = 1$;
(3) $v(\bar{p}) = (1/\bar{p}^tw)(x(\bar{p}) + w)$.

He is said to be *homothetic*, if and only if he is homothetic everywhere on $\mathring{\mathbf{R}}^l_+$.

Equivalently, a preference pre-order on $\mathring{\mathbf{R}}^L_+$ is said to be homothetic if and only if to any x and y in $\mathring{\mathbf{R}}^l_+$ and any $\lambda > 0$ x is indifferent to y if and only if λx is indifferent to λy.

1. That aggregation be possible is of course necessary but not sufficient for integrability [Debreu, 1972]. A necessary condition, however, is sufficient for the present argument.

The following propositions are well-known: the first establishes the possibility of aggregation under homothetic preference and collinear initial endowments vectors; the second establishes that in the absence of restrictions on the distribution of initial endowments homotheticity imposes no restriction on aggregate demand.

PROPOSITION. Let $\{(X^k, w^k, u^k), k = 1, \ldots, m\}$ be a collection of regular homothetic agents with collinear initial endowment vectors. Then aggregation is possible.

Proof. See Chipman [1974].

The intuition behind the proof is clear: Regularity imposes no restrictions on the vector of income effects. Homotheticity, on the other hand, implies that it is collinear with the vector of demands. This restriction, combined with the collinearity of the initial endowment vectors, yields the symmetry and negative semi-definiteness of the aggregate substitution matrix. The possibility of aggregation follows.

PROPOSITION. Let A be an $(l \times l)$ matrix, $\bar{x}$ a vector in $\mathbf{R}^l$, and $\bar{p}$ a vector in R^l_+, such that

(1) $\bar{p}^t\bar{x} = 0$;

(2) $\bar{p}^t A = -\bar{x}^t$, $A\bar{p} = 0$.

Then there exist l regular homothetic agents $\{(X^k, w^k, u^k), k = 1, \ldots, m\}$ such that the aggregate excess demand function satisfies the following:

a) $x(\bar{p}) = \bar{x}$

b) $Dx(\bar{p}) = A$.

Proof. See Mantel [1976].[2]

We now come to the result of the paper: homotheticity of individual preferences is necessary for aggregation under a collinear distribution of initial endowments.

THEOREM. Let $\{(X^k, w^k, u^k), k = 1, \ldots, m\}$ be a collection of regular agents with collinear initial endowment vectors, and suppose that $l > 2$. If $(X^{k'}, w^{k'}, u^{k'})$ is not homothetic at $\bar{p}$, for some $\bar{p} \in \mathbf{R}^l_+$ and some $k' \in \{1, \ldots, m\}$, aggregation need not be possible at $\bar{p}$.

Proof. By the collinearity of the initial endowment vectors, there exist $\delta = (\delta^1, \ldots, \delta^m) \in \mathring{\mathbf{R}}^l_+$, such that $\sum_{k=1}^m \delta^k = 1$ and $w^k = \delta^k w$, $k = 1, \ldots, m$. By the regularity of the individual agents $Dx^k(p) = K^k(p) - v^k(p)x^k(p)^t$ everywhere on $\mathring{\mathbf{R}}^l_+$, where $K^k(p)$ is symmetric, negative

2. This result is weaker than Mantel's [1976], but it suffices for our purposes.

semi-definite of rank $(l-1)$ and $p^t K^k(p) = 0; \bar{p}^t v^k(p) = 1, k = 1, \ldots, m$. Let $y^k(p) = x^k(p) + w^k$, $k = 1, \ldots, m$, $y(p) = \Sigma_{k=1}^m y^k(p) = (\Sigma_{k=1}^m x^k(p)) + w$ and define $v(p) = \Sigma_{k=1}^m \delta^k v^k(p)$. Then

$$Dx(p) = \left[\sum_{k=1}^m K^k(p)\right] - \left[\sum_{k=1}^m v^k(p) x^k(p)^t\right].$$

Equivalently,

$$Dx(p) = \left[\sum_{k=1}^m K^k(p)\right] + \left[v(p)y(p)^t - \sum_{k=1}^m v^k(p) y^k(p)^t\right] - [v(p)x(p)^t].$$

The matrix $v(p)x(p)^t$ is the aggregate income effect matrix; the matrix $[\Sigma_{k=1}^m K^k(p) + v(p)y(p)^t - \Sigma_{k=1}^m v^k(p)y^k(p)^t]$ is the aggregate substitution matrix.

If aggregation is to be possible at $\bar{p} \in \mathring{\mathbf{R}}_+^l$, the aggregate substitution matrix at $\bar{p}$ must be symmetric. Since the matrix $\Sigma_{k=1}^m K^k(\bar{p})$ is indeed symmetric, aggregation requires that the matrix $\phi(\bar{p})$ defined by $\phi(\bar{p}) = v(\bar{p})y(\bar{p})^t - \Sigma_{k=1}^m v^k(\bar{p})y^k(\bar{p})^t$ be symmetric. Since, by definition, $v(\bar{p}) = \Sigma \delta^k v^k(\bar{p})$, the matrix $\phi(\bar{p})$ can be written as

$$\phi(\bar{p}) = \left[\left(\sum_{k=1}^m \delta^k v^k(\bar{p})\right)\left(\sum_{k=1}^m y^k(\bar{p})^t\right)\right] - \left[\sum_{k=1}^m v^k y^k(\bar{p})^t\right].$$

Furthermore, since $\bar{p}^t y^k(\bar{p}) = \delta^k \bar{p}^t w$, $\bar{p}^t \phi(\bar{p}) = \phi(\bar{p})\,\bar{p} = 0$. Consequently, symmetry of $\phi(\bar{p})$ is equivalent to symmetry of the matrix

$$\hat{\phi}(\bar{p}) = \left[\left(\sum_{k=1}^m \delta^k \hat{v}^k(\bar{p})\right)\left(\sum_{k=1}^m \hat{y}^k(\bar{p})^t\right)\right] - \left[\sum_{k=1}^m \hat{v}^k(\bar{p})\hat{y}^k(\bar{p})^t\right],$$

where carets denote the projection of a vector in $\mathbf{R}^l$ to its first $(l-1)$ components. To prove the proposition, we must demonstrate that, for $l > 2$, unless $v^k(\bar{p}) = (1/\delta^k \bar{p}^t w) y^k(\bar{p})$, $\phi(\bar{p})$ need not be symmetric. Since $\bar{p}^t v^k(\bar{p}) = 1$, while $\bar{p}^t y^k(\bar{p}) = \delta^k \bar{p}^t w$, it suffices to show that unless $v^k(\bar{p}) = a^k y^k(\bar{p})$—some a^k—, $k = 1, \ldots, m$, $\phi(\bar{p})$ need not be symmetric. Finally, since $y^k(\bar{p}) \in \mathring{\mathbf{R}}_+^l$ and $l > 2$, it suffices to show that unless $\hat{v}^k(\bar{p}) = a^k \hat{y}^k(\bar{p})$, $k = 1, \ldots, m$, $\hat{\phi}(\bar{p})$ need not be symmetric. Since the points $\{\hat{y}^1(\bar{p}), \ldots, \hat{y}^m(\bar{p})\}$ can be chosen independently of each other and arbitrarily in $\mathring{\mathbf{R}}_+^{l-1}$, we may suppose that $\hat{y}^k(\bar{p})$ is arbitrarily close to 0 for $k \geqslant 2$, which we denote by $\hat{y}^k(\bar{p}) \sim 0$, $k = 2, \ldots, m$. But then

$$\hat{\phi}(\bar{p}) \sim (\delta^1 - 1)\hat{v}^1(\bar{p})\hat{y}^1(\bar{p})^t + \left[\sum_{k=2}^m \delta^k \hat{v}^k(\bar{p})\right] \hat{y}^1(\bar{p})^t.$$

Since $\hat{v}^k(\bar{p})$ is derived from the characteristics of agent k, (X^k, w^k, u^k), which, by assumption, can be specified independently of the characteristics $(X^{k'},w^{k'},u^{k'})$ of agent k', $k' \neq k$, the symmetry of $\hat{\phi}(\overline{p})$ requires that $\hat{v}^k(\overline{p}) \sim 0$, $k = 2,\dots,m$. It follows that $\hat{\phi}(\overline{p}) \sim (\delta^1 - 1)\cdot \hat{v}^1(\overline{p})\hat{y}^1(\overline{p})^t$. Since $\delta^1 \in (0,1)$, symmetry of $\hat{\phi}(\overline{p})$ then requires that $(\hat{v}^1(\overline{p})y^1(\overline{p})^t)$ be a symmetric matrix. Equivalently, that $\hat{v}^1(\overline{p}) = a^1 y^1(\overline{p})$. The same argument applied to any $k \in \{1,\dots,m\}$ implies that $\hat{v}^k(\overline{p}) = a^k \hat{y}^k(\overline{p})$, $k = 1,\dots,m$, as required.

Q.E.D.

Remark. The preceding argument depends crucially on the fact that restrictions on preferences are imposed agent by agent and not on the collection of characteristics $\{(X^k,w^k,u^k),\ k = 1,\dots,m\}$. As Shafer [1977] has demonstrated, in the latter case homotheticity is not a necessary condition for aggregation to be possible under the assumption of collinear endowments.

Remark. That the argument requires the number of commodities in the economy to be at least equal to three is not surprising. In the case of just two goods, the symmetry restrictions on the Slutsky substitution matrix disappear.

Remark. The issue of aggregation can be raised as well in risky settings, such as the standard one-period portfolio allocation problem. As is well-known, the only class of smooth one-period von Neumann-Morgenstern utility functions $u: \mathbf{R}_+ \rightarrow \mathbf{R}$, which yield preferences (ordinally) homogeneous of degree one, is the class of functions of the form $u(x) = (1/\alpha)x^\alpha$, $\alpha < 1$ where α is the degree of constant relative risk aversion. Consider now a collection of individuals with von Neumann-Morgenstern utility functions of the general form specified above, but with diverse values of the parameter α. Suppose furthermore that the agents agree on the probability of occurrence of the various states of nature, while the distribution of income is fixed and independent of prices and aggregate income. Then, even though aggregation is ordinally possible, the aggregate utility function is *not* additively separable and hence not von Neumann-Morgenstern; and it does *not* display constant relative risk aversion.

Recent work by Polemarchakis *et al.* [1981] has demonstrated, however, that the von Neumann-Morgenstern agent with constant relative risk aversion equal to the income-weighted harmonic mean of the degrees of relative risk aversion of the individuals aggregated is a good *approximate aggregator*. The possibility of approximate aggregation in the case of heterogenous probability beliefs as well as attitudes toward risk needs further investigation.

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REFERENCES

Chichilnisky, G., and G. Heal, "Preference Aggregation, Community Preference and Social Choice," mimeo, Columbia University, 1979.
Chipman, J. S., "Homothetic Preferences and Aggregation," *Journal of Economic Theory*, VIII, No. 1 (1974), 26–38.
Debreu, G., "Smooth Preferences," *Econometrica*, XL, No. 4 (1972), 603–15.
——, "Excess Demand Functions," *Journal of Mathematical Economics*, I, No. 1 (1974).
Eisenberg, E., "Aggregation of Utility Functions," *Management Science*, VII, No. 4 (1961).
Geanakoplos, J. D., and H. M. Polemarchakis, "On the Disaggregation of Excess Demand Functions," *Econometrica*, XLVIII, No. 2 (1980), 315–31.
Gorman, W. M., "Community Preference Fields," *Econometrica*, XXI, No. 1 (1953), 63–80.
Mantel, R., "Homothetic Preferences and Community Excess Demand Functions," *Journal of Economic Theory*, XII, No. 2 (1976), 197–201.
Polemarchakis, H., L. Selden, P. Zipkin, and L. Pohlman, "Approximate Aggregation under Uncertainty," Discussion Paper no. 101, Columbia University, 1981.
Samuelson, P. A., "Social Indifference Curves," this *Journal*, LXX, No. 1 (1956), 1–22.
Shafer, W. J., "Conditions for the Aggregation of the Symmetry and Negative Semi-Definiteness of Individual Demand Theory," mimeo, 1977.