JOURNAL OF ECONOMIC THEORY 25, 323–337 (1981)

Constrained Excess Demand Functions*

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Received January 5, 1979; revised May 23, 1980

1. INTRODUCTION

It is the starting point of non-Walrasian equilibrium theory that agents perceive constraints on the amount they can transact in the various markets. This formulation follows naturally from the observation that, away from a Walrasian equilibrium price system, agents' Walrasian—notional—excess demands are incompatible, and hence, if economic activity is to occur, some agents must be rationed out of their desired level of transaction. The assumption of rationality of individual behavior then implies that agents will express excess demand for the various commodities taking into consideration the prevailing rationing mechanism.

The question I take up in this paper concerns the restrictions that are imposed on individual behavior under quantity constraints by the assumption of rationality:[1] Consider an agent who expresses constrained—effective—excess demand so as to maximize a monotone, strictly quasi-concave preference pre-order, subject to binding quantity constraints in a subset K of the system of markets, as well as a budget constraint. The constrained excess demand vector x^K is a function of the price vector p, and the vector of quantity constraints s, $x^K(p, s)$. Furthermore, the vector x^K can be decomposed into a vector $x^K_{K'}$, of the constrained excess demand for goods in the markets for which the agent is not constrained (K' is the subset complementary to K) and a vector x^K_K of the constrained excess demand for goods in the markets for which the agent is indeed constrained. The function $x^K_K(p, s)$ is trivial, it is the projection $x^K_K(p, s) = s$. What needs to be characterized is the constrained excess demand function in the unconstrained

* This work was supported by National Science Foundation Grant SOC77-06000 at the Institute for Mathematical Studies in the Social Sciences, Stanford University.

† I would like to thank Marie Blanchard for insightful comments. I am grateful to Kenneth Arrow, Jean-Jacques Laffont and Andreu Mas-Colell for very helpful discussions.

[1] In a different framework, the same problem has been taken up by Diamond and Yaari (1972) and Tobin (1952).

0022-0531/81/060323–15$02.00/0

markets—$x^K_{K'}(p, s)$. In particular, one can raise the question whether an arbitrary function can be generated as the constrained excess demand function, in the unconstrained markets, of a rational agent. If the problem is approached in an "infinitesimal" framework, the question of characterization of the function $x^K_{K'}(p, s)$ is reduced to a characterization of the Jacobian matrices $D_p x^K_{K'}(p, s)$ and $D_s x^K_{K'}(p, s)$. The characterization of $D_p x^K_{K'}(p, s)$ is an immediate extension of the characterization of the Jacobian of a notional excess demand function through the decomposition into a substitution matrix and a matrix of income effects. The matrix $D_s x^K_{K'}(p, s)$ on the other hand, has no counterpart in Walrasian theory. It is usually referred to as the *spill-over matrix* and captures the change in the excess demand of an agent as the quantity constraints change. I shall demonstrate that no restrictions (other than adding up, of course) can be imposed on the spill-over matrix;[2] as long as one wants to consider the entire class of monotone strictly quasi-concave preference pre-orders, the spill-over matrix must remain arbitrary.[3]

It is a widely accepted—even if little understood—generalization that the immediate impact of fluctuations in supply and demand is reflected in quantity, rather than price, adjustments. As a consequence, qualitative analysis of short-run dynamics, as well as short-run policy recommendations depend on the properties of the spill-over matrix.[4] Furthermore, restrictions on the spill-over coefficients are necessary for the estimation of multimarket disequilibrium models.[5] In this framework, it is rather unpleasant that consumer theory fails to provide the required structure. It is an open problem to determine plausible restrictions on individual characteristics that yield the desired behavior under quantity constraints. These restrictions must involve statements concerning the nature of the dependence of the marginal utility of goods h, $h \in K'$, on the level of consumption of goods h', $h' \in K$. In other words, they must characterize the off-diagonal elements of the Hessian matrix of the utility function. It is the fact that concavity imposes no restrictions on the off-diagonal elements that leads to the results derived in the present paper.

Remark. That rationality imposes no restrictions on the spill-over matrix for a particular state of the markets (i.e., a particular K and s) does *not*, of course, imply that the spill-over matrices corresponding to different states of the markets can be independently specified. Similarly, the indeterminacy of

[2] Observe that this result does *not* depend on aggregation. It is in the case of notional excess demands that aggregation leads to arbitrary behavior—see Geanakoplos and Polemarchakis (1977) and Mantel (1977) for surveys of this problem.

[3] This result does *not* affect the argument in Laroque (1976, 1978) or Wiesmeth (1977). That the aggregate spill-over matrix has generically fully rank is clearly compatible with the result derived here.

[4] See Malinvaud (1977).

[5] See Ito (1978).

the spill-over matrix for the definition of effective demand adopted here does *not* imply that alternative formulations (e.g., Benassy (1975), Clower (1965), Laroque (1976, 1978), Malinvaud (1977)) display indeterminacy as well.

2. Constrained Consumer Excess Demand Functions

Consider an economy with $(l+1)$ commodities indexed by h, $h = 0, 1,..., l$. A consumer is characterized by a consumption set X, a convex subset of $\mathscr{R}^{l+1}$, a strictly quasi-concave utility function u defined on X, and a vector of initial endowments w in X. A price system is a vector p in $\mathring{\mathscr{R}}^{l+1}_{+}$. In addition to the price system, a trader perceives for each commodity h, other than 0, quantitative constraints, $\bar{z}_h \geqslant 0$, $\underline{z}_h \leqslant 0$, that set upper and lower bounds on the amount of commodity h he can trade. It is clear that the consumer must take the quantity constraints into account in expressing his (effective) excess demand; yet, no general agreement has been reached on the price formulation of the consumer's decision problem. In particular, it is not clear whether a trader does (or should) take into consideration the constraints he faces in the market for good k when expressing his excess demand for k. I shall follow here the formulation suggested by Drèze.[6] A trader (X, u, w) expresses excess demand x by solving the problem

$$\max_{x \in X - \{w\}} u(w+u)$$

$$\text{s.t. } p^t x = 0,$$

$$\underline{z}_h \leqslant x_h \leqslant \bar{z}_h, \qquad h = 1,..., l.$$[7]

He is said to be constrained on market k if there exists a consumption bundle $(w + x')$ in X satisfying $p^t x' = 0$ and the quantitative constraint $\underline{z}_h \leqslant x'_h \leqslant \bar{z}_h$ for all h different from 0 and k, which provides a higher utility than x. If $x'_k > \bar{z}_k$ the trader is said to be constrained in the demand for k; if $x'_k < \underline{z}_k$, he is said to be constrained in the supply of k. Let K be the set of markets on which the trader is constrained, and let $s \in \mathscr{R}^{|K|}$ be the vector of quantity constraints—for $h \in K$, s_h is equal to $\bar{z}_h$ or $\underline{z}_h$ depending on whether the trader is constrained in the demand or the supply, respectively, of commodity h. The constrained excess demand x is then the vector in $X - \{w\}$ which maximizes $u(x + w)$ subject to the budget constraint $p^t x = 0$ and the quantity constraints $x_h = s_h$ for $h \in K$. Observe that I am now writing the quantity constraints with equality. This allows the constrained

[6] See Drèze (1975).

[7] For an alternative formulation see Benassy (1975) and Clower (1965).

excess demand function $x^K(p, s)$ to be differentiable with respect to the vector of constraints, s. On the other hand, $x^K(p, s)$ is not necessarily defined for an arbitrary s; furthermore, maximization under the constraint $|x_h| \leqslant |s_h|$ is not necessarily equivalent to maximization under the constraint $x_h = s_h$.[8] These problems need not be explicity dealt with here, however, since I shall only be interested at the infinitesimal characterization of the constrained excess demand function.

I shall make the following assumptions concerning the consumer's characteristics:

ASSUMPTION 1. $X = \mathring{\mathscr{R}}^{l+1}_{+}$.

ASSUMPTION 2. *The utility function u is a twice continuously differentiable and strictly quasi concave function from X to $\mathscr{R}$. At all $x \in X$, $Du(x) \gg 0$, the matrix $D^2u(x)$ is negative definite on the orthogonal complement of $Du(x)$, $[Du(x)]^{\perp}$, and the closure of the indifference hypersurface through x is contained in X.*

Remark. Since $D^2u(x)$ is negative definite on $[Du(x)]^{\perp}$, for any $K' \subset \{0, 1,..., l\}$ the matrix $D^2_{K'K'}u(x)$ is negative definite on $[D_{K'}u(x)]^{\perp}$ and the matrix $\begin{bmatrix} D^2_{K'K'} & D_{K'}u(x) \\ D_{K'}u(x)^t & 0 \end{bmatrix}$ if of full rank.

In the discussion that follows I hold the consumption set fixed and, consequently, I refer to a trader as an ordered pair (u, w).

Consider the decision problem of an agent (u, w) who perceives quantity constraints in the markets for commodities indexed by h, for $h \in K$. K is a subset of index set $\{1,..., l\}$, and K' is the set $\{0, 1,..., l\}/K$. Let $p = (p_0, p_1,..., p_l) \in \mathring{\mathscr{R}}^{l+1}_{+}$ denote the price vector and $s = (s_h \mid h \in K)$ denote the vector of quantity constraints faced by the agent. The agent then solves the problem

$$\begin{aligned} &\operatorname*{Max}_{x \in X - \{w\}} \ u(x + w) \\ &\text{s.t. } x_h = s_h, \qquad h \in K, \\ &\qquad p^t x = 0. \end{aligned} \tag{1}$$

By Assumption 2, given $p \gg 0$ and s such that $(w_h + s_h) > 0$ for all $h \in K$ and $p^t(y - w) < 0$ for some $y \in \mathring{\mathscr{R}}^{l+1}_{+}$ with $y_h = (s_h + w_h)$ for $h \in K$, a solution to (1) exists and is unique. Furthermore, by the Kuhn–Tucker theorem, x^* solves (1) if and only if there exist real numbers λ^*, μ^*_h, $h \in K$ such that

[8] See Laroque ((1976, pp. 10–12)) for a discussion.

$$\begin{aligned} D_h u(x^* + w) - \lambda^* p_h &= 0, & h \in K', \\ D_h u(x^* + w) - \lambda^* p_h + \mu_h^* &= 0, & h \in K, \\ x_h^* &= s_h, & h \in K, \\ p^t x^* &= 0. \end{aligned} \tag{2}$$

Totally differentiating the system of equations (2) we get

$$\begin{bmatrix} D^2u & -p & e^{h_1+1} & \cdots & e^{h_{|K|}+1} \\ -p^t & 0 & 0 & \cdots & 0 \\ (e^{h_1+1})^t & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (e^{h_{|K|}+1})^t & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} dx^* \\ d\lambda^* \\ d\mu_{h_1}^* \\ \vdots \\ d\mu_{h_{|K|}}^* \end{bmatrix} = \begin{bmatrix} \lambda^* dp \\ (dp)^t x^* \\ ds_{h_1} \\ \vdots \\ ds_{h_{|K|}} \end{bmatrix}, \tag{3}$$

where $\{h_1, ..., h_{|K|}\} = K$ in the natural order, and e^i is the ith unit vector in $\mathscr{R}^{l+1}$.Without loss of generality we may assume that $K = \{k, ..., l\}$ for some $k \in \{1, ..., l\}$. Furthermore, letting

$$D^2u = \begin{bmatrix} D_{K'K'} & D_{K'K} \\ D_{KK'} & D_{KK} \end{bmatrix} \quad \text{and} \quad p = \begin{bmatrix} p_{K'} \\ p_K \end{bmatrix}$$

we can rewrite (3) as

$$\begin{bmatrix} D^2_{K'K'}u & D^2_{K'K}u & -p_{K'} & 0 \\ D^2_{KK'}u & D^2_{KK}u & -p_K & I \\ -p^t_{K'} & -p^t_K & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} dx^*_{K'} \\ dx^*_K \\ d\lambda^* \\ d\mu^* \end{bmatrix} = \begin{bmatrix} \lambda^* dp_{K'} \\ \lambda^* dp_K \\ (dp)^t x^* \\ ds \end{bmatrix}. \tag{3'}$$

It is easy to see, by elementary row and column operations, that the matrix

$$A = \begin{bmatrix} D^2_{K'K'}u & D^2_{K'K}u & -p_{K'} & 0 \\ D^2_{KK'}u & D^2_{KK}u & -p_K & I \\ -p^t_{K'} & -p^t_K & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}$$

is similar to the matrix

$$B = \begin{bmatrix} D^2_{K'K'}u & -p_{K'} & 0 \\ -p^t_{K'} & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Consequently, the matrix A is invertible if and only if the matrix B is invertible. Furthermore, the matrix B is invertible if and only if the matrix

$$C = \begin{bmatrix} D^2_{K'K'}u & -p_{K'} \\ -p^t_{K'} & 0 \end{bmatrix}$$

if of full rank. But, from (2), $\lambda^* p_{K'} = D_{K'}u$. Since, by Assumption (2), $Du \gg 0$ everywhere on $\mathring{\mathscr{R}}^{l+1}_+$, $\lambda^* > 0$, and hence,

$$\det[C] = \left(\frac{1}{\lambda^*}\right)^2 \det[D],$$

where

$$D = \begin{bmatrix} D^2_{K'K'}u & D_{K'}u \\ D_{K'}u & 0 \end{bmatrix}.$$

But, by definition, $\det[D]$ is the Gaussian curvature of the hypersurface through $(w + x^*)$ of the restriction of u to the coordinate subspace $T(K, w, s) = \{y \in \mathring{\mathscr{R}}^{l+1}_+ \mid y_h = w_h + s_h$ for all $h \in K\}$. By Assumption (2) then, the matrix D has non-zero determinant, and hence is invertible, which implies the invertibility of the matrix A.[9]

Consider now the solution to (1) as a function of p and s, $x^K(p, s)$. It is a well-defined and continuous function from $S(K, w)$ to $\mathscr{R}^{l+1}$, where $S(K, w) = \{(p, s) \in \mathring{\mathscr{R}}^{l+1}_+ \times \mathscr{R}^{|K|} \mid (w_h + s_h) > 0$ for $h \in K$, and $p^t(y - w) < 0$ for some $y \in \mathring{\mathscr{R}}^{l+1}_+$ with $y_h = (s_h + w_h)$ for $h \in K\}$. Furthermore, the invertibility of the matrix A and the continuous dependence of the inverse on (p, s) are sufficient to guarantee that the constrained excess demand function is continuously differentiable. We have demonstrated the following:[10]

PROPOSITION 1. *Under Assumptions* (1) *and* (2) *the constrained excess demand function* $x^K(p, s)\colon S(K, w) \to \mathscr{R}^{l+1}$ *is continuously differentiable.*

Remark. The function $x^K(p, s)$ is, of course, homogeneous of degree 0 in p.

Remark. The functions $\lambda^K(p, s)$ and $\mu^K_h(p, s)$, $h \in K$, are, similarly, continuously differentiable; they are homogeneous of degree (-1) in p.

The discussion has, up to this point, been limited to the formulation of the agent's constrained maximization problem and the demonstration of the fact

[9] Given a matrix A with rows $i = 1, \dots, l_1$ and columns $j = 1, \dots, l_2$ and subsets $K_1 \subset \{1, \dots, l_1\}$, $K_2 \subset \{1, \dots, l_2\}$, A_{K_1,K_2} denotes the obvious submatrix. Similarly, given a vector $y = (y_1, \dots, y_l)$ and $K \subset \{1, \dots, l\}$, y_k denotes the obvious vector in $\mathscr{R}^{|K|}$.

[10] This is, of course, an immediate extension of the argument in Debreu (1972).

that the regularity properties imposed on individuals preferences by Assumption (2) are sufficient to guarantee the differentiability of the constrained excess demand function $x^K(p, s)$. The question follows whether the regularity assumptions have any further qualitative implications concerning the agent's response to a change either in prices or in the quantity constraints it perceives.

The Jacobian of the function $x^K(p, s)$ is determined by the matrix A^{-1}. Since the matrix A is symmetric, A^{-1} is symmetric as well, and hence with no loss of generality we may write

$$A^{-1} = \begin{bmatrix} S_{K'K'} & S_{K'K} & -v_{K'} & R_{K'} \\ (S_{K'K})^t & S_{KK} & -v_K & R_K \\ (-v_{K'})^t & (-v_K)^t & b_{00} & -b_K^t \\ (R_{K'})^t & (R_K)^t & -b_K & B_K \end{bmatrix}.$$

Since the prouct $AA^{-1} = I$, the identity matrix, the following conditions must be satisfied:

(a) $S_{K'K} = 0$; $S_{KK} = 0$.

(b) $v_K = 0$.

(c) $p_{K'}^t(S_{K'K'}) = (S_{K'K'})\,p_{K'} = 0$.

(d) $p_{K'}^t v_{K'} = 1$.

(e) $R_K = I$.

(f) $p_{K'}^t R_{K'} = -p_K^t$.

Furthermore, the following argument shows that the matrix $S_{K'K'}$ is negative semi-definite and has rank $(|K'| - 1)$. Since $A^{-1}A = I$, $(D^2_{K'K'}u)(S_{K'K'}) + p_{K'}v_{K'}^t = I$. Let y be any non-zero eigenvector of $S_{K'K'}$ and suppose $(S_{K'K'})\,y = \mu y$, for some $\mu \neq 0$. Observe that y is in the span of the columns of $S_{K'K'}$,$[S_{K'K'}]$,and hence $y^t p_{K'} = 0$. Consequently, $y^t[(D^2_{K'K'}u)(S_{K'K'})]\,y = y^t y$, and hence $\mu y^t(D^2_{K'K'}u)\,y = y^t y$. Since the matrix D^2u is negative definite on the orthogonal complement of Du, the matrix $D^2_{K'K'}u$ is negative definite on the orthogonal complement of $D_{K'}u$, $[D_{K'}u]^{\perp}$. But, from (2), $[D_{K'}u] = [p_{K'}]$, and, since $y^t p_{K'} = 0$, $y \in [D_{K'u}]^{\perp}$. As a result, $y^t(D^2_{K'K'}u)\,y < 0$, and, since $y^t y > 0$, the eigenvalue μ of $S_{K'K'}$ must be negative. It remains to show that rank $(S_{K'K'}) = (|K'| - 1)$. But this follows from the fact that $p_{K'}^t S_{K'K'} = 0$, while for all $y \in [p_{K'}]^{\perp}$, $y^t[(D^2_{K'K'}u)(S_{K'K'})]\,y = y^t y$ and $y^t(D^2_{K'K'}u) \neq 0$. Finally, since the matrix $S_{K'K'}$ is negative semi-definite, has rank $(|K'| - 1)$ and satisfies $p_{K'}^t(S_{K'K'}) = 0$ with $p_{K'}^t \neq 0$, $S_{K'K'}$ is negative definite when restricted to the complement of $[p_{K'}]$. We have thus demonstrated the following:

Proposition 2. *Let $x^K(p, s)$ be the constrained excess demand function of an agent (u, w) satisfying Assumptions* (1) *and* (2). *Then everywhere on $S(K, w)$, $p^t x^K(p, s) = 0$, $x_K^K(p, s) = s$, and the following conditions are satisfied*:

(1) $D_{p_{K'}} x_{K'}^K = \lambda S_{K'K'} - v_{K'}(x_{K'}^K)^t$; *the matrix $S_{K'K'}$ is symmetric, negative semi-definite of rank $(|K'| - 1)$ and satisfies $p_{K'}^t S_{K'K'} = S_{K'K'} p_{K'}^t = 0$; the scalar λ is strictly positive; the vector $v_{K'}$ satisfies $p_{K'}^t v_{K'} = 1$;*

(2) $D_{p_{K'}} x_K^K = 0$; $D_{p_K} x_{K'}^K = -v_{K'} x_K^t$; $D_{p_K} x_K^K = 0$;

(3) $D_s x_{K'}^K = R_{K'}$; *the matrix $R_{K'}$ satisfies $p_{K'}^t R_{K'} = -p_K^t$;*

(4) $D_s x_K^K = I$.

Remark. Homogeneity of degree zero with respect to p of the function $x^K(p, s)$ holds, if $dp = \alpha p$, $\alpha > 0$ and $ds = 0$ imply $dx^K = 0$. But, since $dx_{K'}^K = \lambda S_{K'K'} dp_{K'} - v_{K'}(x^K)^t dp + R_{K'} ds$ while $dx_K^K = ds$, and furthermore $S_{K'K'} p_{K'} = 0$ and $(x^K)^t p = 0$, this is indeed the case.

Remark. Observe that the only restriction claimed on the spill-over matrix $R_{K'}$ is the condition $p_{K'}^t R_{K'} = -p_K^t$. This restriction of course only says that the consumer stays on the budget hyperplane; it is equivalent to $\sum_{h' \in K'} p_{h'}(\partial x_{h'}^K / \partial s_h) + p_h = 0$, for all $h \in K$.

Proposition 2 establishes a set of necessary conditions for the function $x^K(p, s)$ to be the constrained excess demand function of a rational agent. The question I want to take up now is whether the set of restrictions imposed by Proposition 2 is not only necessary but also sufficient for rationality. It is clear that since conditions (1) to (4) of Proposition 2 refer only to the Jacobian of constrained excess demand function, the strongest result that can be hoped for is that they are sufficient for the function $x^K(p, s)$ to be infinitesimally the constrained excess demand function of a rational agent. The following argument shows that this is indeed the case.

Suppose we are given a vector $\bar{x}^K$ in $\mathscr{R}^{l+1}$, a vector $\bar{s}$ in $\mathscr{R}^{|K|}$, a vector $\bar{p}$ in $\mathring{\mathscr{R}}_+^{l+1}$, a $(|K'| \times |K'|)$ matrix $\bar{S}_{K'K'}$, a vector $\bar{v}_{K'}$, in $\mathscr{R}^{|K'|}$ and a $(|K'| \times |K|)$ matrix $\bar{R}_{K'}$, such that $\bar{x}_K^K = \bar{s}$, $\bar{p}^t \bar{x}^K = 0$ and the following conditions are satisfied:

(a) The matrix $\bar{S}_{K'K'}$ is symmetric, negative semi-definite of rank $(|K'| - 1)$ and satisfies $\bar{p}_{K'}^t \bar{S}_{K'K'} = \bar{S}_{K'K'} \bar{p}_{K'} = 0$; the vector $\bar{v}_{K'}$ satisfies $\bar{p}_{K'}^t \bar{v}_{K'} = 1$.

(b) The matrix $\bar{R}_{K'}$, satisfies $\bar{p}_{K'}^t \bar{R}_{K'} = -\bar{p}_K^t$.

Consider the matrix

$$E=\begin{bmatrix} \bar{S}_{K'K'} & 0 & -\bar{v}_{K'} & \bar{R}_{K'} \\ 0 & 0 & 0 & I \\ -\bar{v}^t_{K'} & 0 & b_{00} & -b^t_K \\ \bar{R}^t_{K'} & I & -b_K & B_K \end{bmatrix},$$

where the scalar b_{00}, the vector b_K and the matrix B_K are chosen in a way to be specified shortly. Since $\bar{p}_{K'}\bar{S}_{K'K'}=0$ while $\bar{p}^t_{K'}\bar{v}_{K'}=1$, $\bar{v}_{K'}\notin[\bar{S}_{K'K'}]$. Consequently, the matrix

$$F=\begin{bmatrix} \bar{S}_{K'K'} & -\bar{v}_{K'} \\ -\bar{v}^t_{K'} & 0 \end{bmatrix}$$

has full rank, $(|K'|+1)$. By elementary column and row operations it is easy to see that the matrix E is similar to the matrix

$$G=\begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix}$$

and, hence, is of full rank. Let E^{-1} be the inverse matrix: By symmetry, E^{-1} can be written as follows:

$$E^{-1}=\begin{bmatrix} \bar{U}''_{K'K'} & \bar{U}''_{K'K} & -q_{K'} & d^k_{K'} & \cdots & d^l_{K'} \\ (\bar{U}''_{K'K})^t & \bar{U}''_{KK} & -q_K & d^k_K & \cdots & d^l_K \\ -q^t_{K'} & -q^t_K & c_{00} & & c^t_K & \\ (d^k_{K'})^t & (d^k_K)^t & & & & \\ \vdots & \vdots & c_K & & C_K & \\ (d^l_{K'})^t & (d^l_K)^t & & & & \end{bmatrix}.$$

Since the product $EE^{-1}=I$, the identity matrix, the following conditions must be satisfied:

(1) $d^h_{K'}=0$, $h=k,\ldots,l$; $d^h_K=e^{h-k+1}$, the $(h-k+1)$st unit vector in $\mathscr{R}^{|K|}$, $h=k,\ldots,l$.

(2) $C_K=0$.

Since the matrix $S_{K'K'}$ is singular, the cofactor of b_{00} is zero and hence $c_{00}=0$. Similarly, all the elements of b_K have cofactors equal to zero, since the minor obtained by deleting the row and the column intersecting at an element of b_k contains a zero row (or a zero column).

Since $EE^{-1}=I$, $c_{00}=0$ and $c^t_K=0$, $q^t_{K'}S_{K'K'}=0$ and $q^t_{K'}\bar{v}_{K'}=1$. Since $[\bar{S}_{K'K'}]^L=[\bar{p}_{K'}]$, $q_{K'}=\alpha\bar{p}_{K'}$, for some $\alpha\in\mathscr{R}$. Since $q_{K'}\bar{v}=1=\bar{p}_{K'}\bar{v}_{K'}$,

$q_{K'} = \bar{p}_{K'}$. Furthermore, since $-q^t_{K'}\bar{R}_{K'} - q^t_K = 0$, $q^t_K = -\bar{p}^t_{K'}\bar{R}_{K'} = \bar{p}^t_K$ by assumption. The matrix E^{-1} can now be written as follows:

$$E^{-1} = \begin{bmatrix} \bar{U}''_{K'K'} & \bar{U}''_{K'K} & -p_{K'} & 0 \\ \bar{U}''_{KK'} & \bar{U}''_{KK} & -p_K & I \\ -\bar{p}^t_{K'} & -\bar{p}^t_K & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

It can now be shown that, given any $\pi_K \in \mathring{\mathbb{R}}^K_+$, by appropriately choosing B_K we can guarantee that the matrix

$$\bar{U}'' = \begin{bmatrix} \bar{U}''_{K'K'} & U''_{K'K} \\ U''_{KK'} & \bar{U}''_{KK} \end{bmatrix}$$

is negative definite on the orthogonal complement of the vector $(\bar{p}^t_{K'}, \pi^t_K)$, $[(\bar{p}^t_{K'}, \pi^t_K)]^{\perp}$. Consider the matrices

$$M = \begin{bmatrix} \bar{U}''_{K'K'} & \bar{U}''_{K'K} & 0 \\ \bar{U}''_{KK'} & \bar{U}''_{KK} & I \\ 0 & I & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} \alpha\bar{p}_{K'}\bar{p}^t_{K'} & \alpha\bar{p}_{K'}\pi^t_K & 0 \\ \alpha\pi_K\bar{p}^t_{K'} & \alpha\pi_K\pi^t_K & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} \bar{S}_{K'K'} - \bar{v}_{K'}\bar{v}^t_{K'} & 0 & \bar{R}_{K'} - \bar{v}_{K'}b^t_K \\ 0 & 0 & I \\ \bar{R}^t_{K'} - b_K\bar{v}^t_{K'} & I & B_K - b_Kb^t_K \end{bmatrix} \equiv \begin{bmatrix} P_1 & 0 & P_2 \\ 0 & 0 & I \\ P^t_2 & I & P_3 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & \alpha\bar{p}_{K'}(\bar{p}_K - \pi_K)^t \\ \alpha(\bar{p}_K - \pi_K)\, v^t_{K'} & 0 & \alpha(\bar{p}_K - \pi_K)\, b^t_K + \alpha\pi_K(\bar{p}_K - \pi_K)^t \\ 0 & 0 & 0 \end{bmatrix},$$

and observe that for $\alpha = (b_{00} - 1)$, $(M + N)P = I + Q$. Furthermore, since $\bar{v}_{K'} \notin [\bar{S}_{K'K'}]$, the matrix P_1 is negative definite. Let Y be the set of vectors defined by $Y = \{(y_1, y_2, y_3) \in \mathscr{R}^{l+1+|K|} \mid y_2 = z_3 - P_3y_3 - P^t_2y_1,\ z_3 \in [y_3]^{\perp},\ (y_1, y_3) \neq 0\}$. For $y = (y_1, y_2, y_3) \in Y$, $\alpha = (b_{00} - 1)$, $b_K = 0$, $B_K = \delta I$, $y^tP^t(M + N)Py = y^t_1P_1y_1 - \delta y^t_3Iy_3 - 2\alpha y^t_3(\pi_K - \bar{p}_K)\,\bar{v}^t_{K'}y_1 + \alpha y^t_3(\pi_K - \bar{p}_K)(\pi_K - \bar{p}_K)^ty_3$. Consequently, there exists $\bar{\delta} > 0$ such that, for $\delta > \bar{\delta}$, $B_K = \delta I$, $b_K = 0$, $\alpha = (b_{00} - 1)$, $y^tP^t(M + N)Py < 0$ for all $y \in Y$. Consider the set $\bar{Y}$ of vectors defined by

$$\bar{Y} = \{(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \mathscr{R}^{l+1+|K|} \mid \bar{y}_2 \in [\bar{y}_3]^{\perp}, (\bar{y}_1, \bar{y}_2) \neq 0\},$$

and observe that, given $\bar{y} \in \bar{Y}$ there exists a unique $y \in Y$ such that $\bar{y} = Py$: Explicitly,

$$(y_1, y_2, y_3) = (P_1^{-1}(\bar{y}_2 - P_2 \bar{y}_2), \bar{y}_3 - P_3 y_3 - P_2^t y_1; \bar{y}_2).$$

But then, $\bar{y}^t(M+N)\bar{y} < 0$ for all $\bar{y} \in \bar{Y}$. Equivalently, the matrix M is negative definite on the set $Y^* = \bar{Y} \cap [\bar{p}_{K'}, \pi_K, 0]^\perp$. But this is equivalent to the negative definiteness of $\bar{U}''$ on $[p_{K'}, \pi_K]^\perp$: for given $(y_1, y_2) \in [\bar{p}_{K'}, \pi_K]^\perp$ we can find y_3 such that $y = (y_1, y_2, y_3) \in Y^*$, and, furthermore, $y^t My = (y_1^t, y_2^t)\, \bar{U}'' \binom{y_1}{y_2}$. To complete the argument to the effect that Proposition (2) gives a set of not only necessary but also infinitesimally sufficient conditions, it suffices to construct an agent (u, w), satisfying Assumptions (1) and (2), and such that $D^2 u(\bar{x}^K) = \bar{U}''$, $Du(\bar{x}^K)^t = (p_{K'}^t, \pi_K^t)$, and $(\bar{x}^K + w) \in X$. But this is evident

Remark. To satisfy the system of equations (1) I have set $\lambda = 1$. Furthermore, since the vector π_K was chosen arbitrarily, no contradiction can arise concerning the sign of the multipliers μ_h, $h \in K$, when the equality constraints in (1) are replaced by the corresponding inequalities.

Remark. In the proof, b_{00} was left unspecified. By setting $b_{00} = 1$ (and hence $\alpha = 0$) we see that $M + N = M$. But then, for $b_{00} = 1$, $b_K = 0$, $\underline{B_K} = \delta I$ the matrix M is negative definite on $\bar{Y}$. Equivalently the matrix $\bar{U}''$ is negative definite.

We have thus demonstrated the following:

PROPOSITION 3. *Let $\bar{x}^K$ be a vector in $\mathscr{R}^{l+1}$, $\bar{s}$ a vector in $\mathscr{R}^{|K|}$, $\bar{p}$ a vector in $\mathring{\mathscr{R}}_+^{l+1}$, $\bar{S}_{K'K'}$ a $(|K'| \times |K'|)$ matrix, $\bar{v}_K$, a vector in $\mathscr{R}^{|K'|}$ and $\bar{R}_{K'}$ a $(|K'| \times |K|)$ matrix, such that $\bar{x}_K^K = \bar{s}$, $\bar{p}^t x^K = 0$ and the following conditions are satisfied:*

(1) *$\underline{\bar{S}_{K'K'}}$ is symmetric, negative semi-definite of rank $(|K'| - 1)$ and $\bar{p}_{K'}^t \bar{S}_{K'K'} = \bar{S}_{K'K'} \bar{p}_{K'} = 0$;*

(2) $\bar{p}_{K'}^t \bar{v}_{K'} = 1$;

(3) $\bar{p}_{K'}^t \bar{R}_{K'} = -\bar{p}_K^t$.

Then there exists an agent (u, w) satisfying Assumptions 1 and 2 whose constrained excess demand function $x^K(p, s)$ satisfies the following:

(a) $x^K(\bar{p}, \bar{s}) = \bar{x}^K$;

(b) $D_{p_{K'}} x_{K'}^K(\bar{p}, \bar{s}) = \bar{S}_{K'K'} - \bar{v}_{K'}(\bar{x}_{K'}^K)^t$;

(c) $D_s x_{K'}^K(\bar{p}, \bar{s}) = \bar{R}_{K'}$.

The following corollary follows immediately:

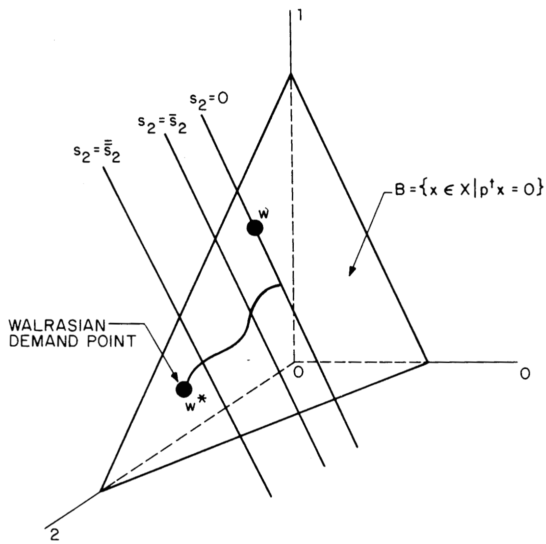

FIG. 1. The path (on B) to w^* is locally arbitrary.

COROLLARY 1. *The only restrictions imposed by Assumptions* 1 *and* 2 *on the spill-over matrix* $R_{K'}$ *is the adding up restriction*—$p_K^t R_{K'} = -p_{K'}^t$.

It was argued in the introduction that the indeterminacy of the spill-over matrix $R_{K'}$ results from the lack of restrictions on the off-diagonal elements of the Hessian matrix D^2u. I now want to pursue this line of reasoning for the purpose of determining the type of properties that, if satisfied by the Hessian matrix, have observable (qualitative) implications for the spill-over matrix. For simplicity I shall consider the case of three commodities, $h = 0, 1, 2$, and in particular the state of the market in which the consumer is constrained in the market for good 2, $K = \{2\}$ and unconstrained in the markets for goods 0 and 1, $K' = \{0, 1\}$.

By the definition of the matrices A and A^{-1},

$$R_{K'}^t D_{K'K'}^2 u + D_{KK'}^2 u + b_K p_{K'}^t = 0, \tag{4}$$

which can be written in terms of partial derivatives as follows: for all $h' \in K'$ and $h \in K$,

$$\sum_{h'' \in K'} \frac{\partial x_{h''}}{\partial s_h} \frac{\partial^2 u}{\partial x_{h''} \partial x_{h'}} + \frac{\partial^2 u}{\partial s_h \partial x_h'} + b_h p_{h'} = 0. \tag{5}$$

In the special case I am here considering, we get

$$\begin{aligned}\frac{\partial^2 u}{\partial x_0^2}\frac{\partial x_0}{\partial s_2}+\frac{\partial^2 u}{\partial x_0 \partial x_1}\frac{\partial x_1}{\partial s_2}+\frac{\partial^2 u}{\partial x_0\,\partial s}&=\frac{\partial \lambda^*}{\partial s_2}p_0,\\ \frac{\partial^2 u}{\partial x_0 \partial x_1}\frac{\partial x_0}{\partial s_2}+\frac{\partial^2 u}{\partial x_1^2}\frac{\partial x_1}{\partial s_2}+\frac{\partial^2 u}{\partial x_1\,\partial s_2}&=\frac{\partial \lambda^*}{\partial s_2}p_1.\end{aligned} \tag{5'}$$

Multiplying both sides of the first equation by $\partial u/\partial x_1$ and of the second equation by $\partial u/\partial x_0$, and subtracting the second equation from the first we get that

$$\frac{\partial x_0}{\partial s_2}\frac{\partial \sigma_{0,1}}{\partial x_0}+\frac{\partial x_1}{\partial s_2}\frac{\partial \sigma_{0,1}}{\partial x_1}+\frac{\partial \sigma_{0,1}}{\partial s_2}=0, \tag{6}$$

where $\sigma_{0,1}$ is the marginal rate of substitution between goods 0 and 1, or $(\partial u/\partial x_0)/(\partial u/\partial x_1)$. Equation (6) is precisely the equation we want. The spill over matrix is now the vector $(\partial x_0/\partial s_2, \partial x_1/\partial s_2)$. Qualitative restrictions on the spill over effects can be derived by restricting the term $\partial \sigma_{0,1}/\partial s_2$, which captures the effects of a change in the consumption of good 2 on the marginal rate of substitution between goods 0 and 1.

3. Extensions: Constrained Production Decisions

The results which I derived in the previous section can be immediately extended to the theory of production decisions. Here I shall only sketch the argument.[11]

The production plan of a producer is denoted by $y \in \mathscr{R}^{l+1}$—a commodity, h, is an output (resp. an input) if $y_h > 0$ (resp. if $y_h < 0$). The production possibilities of the firm are described by a function f: a plan y is feasible if and only if $f(y) \leqslant 0$. Given a price system p the firm chooses y so as to maximize $\pi = p^t y$ subject to the quantity constraints $y_h = s_{h'}$ for $h \in K$. Since the problem degenerates if the firm is constrained in all markets, I shall assume that $K = \{k,..., l\}$ for some $k \geqslant 1$. The interpretation here of the 0th commodity is, of course, different from its interpretation in the theory of consumption. I shall assume that the function f is twice continuously differentiable, and strictly increasing, $f(0)=0$, the production set $\{y \mid f(y) \leqslant 0\}$ is convex, and the hypersurface $\{y \mid 0=f(y),\ y_h = s_{h'},\ h=k,..., l\}$ has nowhere vanishing Gaussian curvature. The first order

[11] Here I follow, more or less, the analogous sketch in the Appendix in Laroque (1976).

necessary and sufficient conditions for a constrained maximum then becomes

$$\begin{aligned} p_{K'} - \lambda^* D_{K'} f(y^*) &= 0, \\ p_K - \lambda^* D_K f(y^*) - \mu^* &= 0, \\ f(y^*) &= 0, \\ y_K^* &= s, \end{aligned} \tag{6}$$

for some λ^*, μ_h^*, $h \in K$. To determine the Jacobian of the function $y^K(p, s)$ we totally differentiate (4):

$$\begin{bmatrix} \lambda^* D^2_{K'K'} f & \lambda^* D^2_{K'K} f & D_{K'} f & 0 \\ \lambda^* D^2_{KK'} f & \lambda^* D^2_{KK} f & D_K f & I \\ (D_{K'} f)^t & (D_K f)^t & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} dy_{K'} & & dp_{K'} \\ dy_K & = & dp_K \\ d\lambda^* & & 0 \\ d\mu^* & & ds \end{bmatrix}.$$

Setting

$$A = \begin{bmatrix} \lambda^* D^2_{K'K'} f & \lambda^* D^2_{K'K'} f & D_{K'} f & 0 \\ \lambda^* D^2_{KK'} f & \lambda^* D^2_{KK} f & D_K f & I \\ (D_{K'} f)^t & (D_K f)^t & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

we see that the function $y^K(p, s)$ is completely described locally by the matrix

$$A^{-1} = \begin{bmatrix} S_{K'K'} & 0 & v^{K'} & R_{K'} \\ 0 & 0 & 0 & I \\ v^t_{K'} & 0 & b & b^t_K \\ R^t_{K'} & I & b_K & B_K \end{bmatrix}.$$

The matrix $R_{K'}$ is—as in the problem of the consumer—the spill-over matrix. The analogue of Propositions 1, 2, and 3 and of Corollary 1 can now be easily derived.

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