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Recovering Cardinal Utility

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Individual choice under uncertainty depends on an agent's attitudes towards risk, typically represented by his von Neumann–Morgenstern cardinal utility function, u. This function is, of course, an unobservable characteristic. What is, at least in principal, observable is the agent's demand for alternative assets. Problems of prediction and welfare often require conclusions to be drawn concerning the unobservable characteristics based on observable behaviour. To predict the response to a change in the stochastic production plan of a firm, or to determine the compensation necessary so as to avoid a loss in welfare due to a change in the capital tax structure, knowledge of an agent's attitude towards risk is required. It is the purpose of the present paper to demonstrate that, as long as the joint distribution of returns of the available assets is known, the cardinal utility function can be recovered without ambiguity from asset demands, provided that the set of available assets contains a riskless asset.

The distinguishing characteristic of the recoverability problem as posed here is the possible incompleteness of markets. In a framework of complete markets, if nominal income and prices vary independently, the range of the demand correspondence can be assumed to cover an open subset of the consumption set. The utility function can then be immediately recovered, up to a monotone transformation, over this range, given mild regularity conditions on demand behaviour. Given incomplete markets, however, this argument fails. Since the choice set is a lower dimensional subspace of state space, we learn only about the agent's preferences within this subspace. We show that the existence of a riskless asset (and some risky asset) implies recoverability of von Neumann–Morgenstern preferences, even if markets are incomplete. In the presence of a riskless asset, any positive, riskless level, α, of wealth is demanded for some asset prices. Using arguments from Pratt (1964), the ratio $u''(\alpha)/u'(\alpha)$ can be recovered without ambiguity for all positive (resp. non-negative) values of α, and hence, an integration argument can be used to recover the cardinal utility function, u, up to a positive affine transformation.

Beyond the argument based on knowledge of the asset demand correspondence, we explore an alternative approach to recoverability. We demonstrate that knowledge of the portfolio indifference surfaces permits recovery of the cardinal utility function, provided there is a riskless asset. The intuition here is that the agent's risk aversion is directly related to the curvature of the indifference curve around the riskless point and the variance of the risky asset(s). That this approach is in general equivalent to the demand correspondence approach is well known—we give a brief proof for twice continuously differentiable and strictly monotone utility functions.

We point out two results concerning the informational requirements of the argument for recoverability. Specifically, the argument does not require complete knowledge of the distribution of returns independently of the asset demands. Instead, it suffices to know the mean and variance of the returns to just one asset other than the riskless, and the return of the riskless asset. Furthermore, if the demand for assets is known only on a subset of the domain of prices, the cardinal utility function can still be recovered on a subset of its domain of definition, provided an interval of the amount invested in the riskless asset is in the interior of the supported set.

It has been demonstrated by Green, Lau and Polemarchakis (1979) that if the demand for risky assets is determined by the maximization of a cardinal utility function which is analytic at the origin, and if these demands are known as a function of prices, then the utility function can be recovered without ambiguity. The question follows whether the existence of a riskless asset is a necessary condition for recoverability if utility is not analytic on the closed non-negative real line. Following McLennan (private correspondence), we give an example to demonstrate that in the absence of restrictions on the distribution of returns, one can generate pairs of cardinal utility functions which are not positive linear transformations of each other and which generate functionally identical demands for appropriately chosen assets. These utility functions, however, display either an unbounded derivative at the origin or non-concavity. We conjecture that, in the absence of a riskless asset, the cardinal utility can be recovered provided it is strictly monotone, concave and continuous on the closed non-negative real line and, in addition, the right derivative at the origin is bounded. Recoverability in the absence of a riskless asset would thus depend on the behaviour of the marginal utility at the origin. We conclude by demonstrating that the boundedness of the marginal utility at the origin is itself an observable property, and so has observable content in principle (subject to the observability of the entire demand curve). This would be consistent with some of the recoverability results by Dybvig (1979), which also depend critically on the slope at the origin.

Without later elaboration we offer a simple application of our results to mean-variance theory. When mean-variance theory is motivated by multi-variate normality, our results imply that preferences (over mean and variance) are determined by preferences in a neighbourhood of the riskless axis. This illustrates vividly the severity of the restrictions on preferences over mean and variance implied by multivariate normality.

Consider an investor who must divide his initial wealth among $m+1$ assets indexed by a subscript j, $j=0, 1, \ldots, m$. Each asset, j, has a random gross return, r_j. The distribution of the random variable $r=(r_0, r_1, \ldots, r_m)$ describes their joint distribution, and hence the distribution of the value of any portfolio, $r \cdot x$.

Assume that r satisfies:

Assumption R

(1) The support of the random variable r_j is contained in a compact subset of the (open) positive real line, $\mathring{\mathbb{R}}_+$, for all j;

(2) r_j is not riskless, for at least one j;

(3) $Er_j^l < \infty$ for $l = 1, 2$, for all j;

(4) $r_0 = 1$ with probability 1.

By virtue of these assumptions, we can choose units of measurement of each asset so that

$$Er_j = 1 \quad \text{for each } j. \tag{1}$$

The investor is assumed to be an expected utility maximizer. He is assumed to know the distribution of r and to have a cardinal utility function u.

Assume that u satisfies:

Assumption U

(1) u is defined over the domain of positive real numbers, $\mathring{\mathbb{R}}_+$;

(2) u is twice continuously differentiable;

(3) $u'(\alpha) > 0$ everywhere on $\mathring{\mathbb{R}}_+$;

(4) $u''(\alpha) \leqq 0$ everywhere on $\mathring{\mathbb{R}}_+$.

Faced with prices $p=(p_0, p_1, \ldots, p_m)$ in $\mathring{\mathbb{R}}_+^{m+1}$ the investor chooses $x \in X$, where $X=\{x \mid r \cdot x>0$ with probability one$\}$. Without loss of generality, the investor's initial wealth can be taken to be unity. Thus his problem is

$$\max_{x\in X} Eu(r \cdot x) \quad \text{s.t. } p \cdot x \leqq 1. \tag{2}$$

At some prices, the maximum may fail to exist.

Let the demand correspondence be denoted by $\xi: \mathcal{P} \to \mathbb{R}^{m+1}$, where $\mathcal{P}$ is the set of prices in $\mathring{\mathbb{R}}_+^{m+1}$ for which a solution to (2) exists. The correspondence ξ is observable. We want to demonstrate that the investor's cardinal utility function u, can be deduced from ξ, of course up to a positive linear transformation. In addition to ξ, we will assume knowledge of the joint distribution of r. Given ξ and r, a quantity will be called observable if it is determined without ambiguity.

Theorem 1. *Let both u and v satisfy Assumption U and generate the asset demand correspondence ξ for assets satisfying R. Then $u \equiv v$ up to a positive linear transformation.*

Proof. By the concavity of the utility function u—Assumption U(4)—the objective function $Eu(r \cdot x)$ is concave in x. Its domain of definition, X, includes all x for which $r \cdot x$ is in the domain of definition of u with probability one. Assumptions R(1), R(4) and U(1) guarantee this whenever $x \in X^* \equiv \mathring{\mathbb{R}}_+ \times \mathbb{R}_+^m$; i.e. whenever the investor puts a positive amount of his initial wealth on the riskless asset, and holds non-negative quantities of the other assets. Furthermore, by the differentiability of u—Assumption U(2)—and Leibniz' rule—Dieudonné (1969), p. 177—the derivative of the objective function $Eu(r \cdot x)$ with respect to x_j, $j=0, \ldots, m$, exists and is given by

$$\frac{\partial Eu(r \cdot x)}{\partial x_j}=Er_j u'(r \cdot x) \tag{3}$$

everywhere on X^*. We can thus define the marginal rate of substitution of asset j for asset k by

$$s_{jk}(x)=\left(\frac{\partial Eu(r \cdot x)}{\partial x_j} \Big/ \frac{\partial Eu(r \cdot x)}{\partial x_k}\right). \tag{4}$$

By the differentiability, monotonicity and concavity of u, and by the positivity and finiteness of the mean returns of the different assets, $s_{jk}(x)$ is a positive real number for all pairs of assets j and k and all $x \in X^*$. Note that $X^* \subseteq X$.

Notice now that for each $x \in X^*$ there exists a unique of $p \in \mathring{\mathbb{R}}_+^{m+1}$ such that x solves the investor's maximization problem (2) at prices p. Given $x \in X^*$, existence of p follows by choosing $p \in \mathring{\mathbb{R}}_+^{m+1}$ so that $(p_0, p_1, \ldots, p_m)$ is proportional to $(1, s_{01}(x), \ldots, s_{0m}(x))$ and so that $p \cdot x=1$. Uniqueness follows from the differentiability of u and the fact that at least x_0 is positive. As a consequence, the marginal rate of substitution between any two assets j and k, $s_{jk}(x)$, is an observable function of x for $x \in X^*$. Furthermore, since the function $s_{jk}(x)$ is observable on X^*, so are its derivatives, provided, of course, that they exist. Consider the expression

$$\frac{Er_k u'(r \cdot x) Er_i r_j u''(r \cdot x)-Er_j u'(r \cdot x) Er_i r_k u''(r \cdot x)}{[Er_k u'(r \cdot x)]^2}.$$

If well defined it would be the value of $\partial s_{jk}(x)/\partial x_i$.

At $\bar{\alpha}=(\alpha, 0, \ldots, 0)$ in X^* we have $s_{jk}(\bar{\alpha})=1$, and hence

$$\frac{\partial s_{jk}(\bar{\alpha})}{\partial x_i}=\frac{u''(\alpha)}{u'(\alpha)}(Er_j r_i-Er_k r_i). \tag{5}$$

For it to be well defined, it is required that the two expectations on the R.H.S. exist. But by Hölder's inequality—Royden (1968), p. 113—we have

$$(Er_jr_i)^2 \leqq Er_j^2 Er_i^2$$

which is finite by virtue of Assumption R(4). We are now ready to complete the proof of identifiability of the cardinal utility function u.

Let v be a cardinal utility function satisfying Assumptions U(1)–U(4) and suppose v generates the demand correspondence ξ. Then the marginal rates of substitution functions implied by v must be the same as those observable through knowing ξ. At $\bar{\alpha} = (\alpha, 0, \ldots, 0)$ in X^* we have

$$\frac{\partial s_{jk}(\bar{\alpha})}{\partial x_i} = \frac{v''(\alpha)}{v'(\alpha)}[Er_jr_i - Er_kr_i]. \tag{6}$$

Setting $j = 0$ and $i = k \neq 0$, we get from (6)

$$\frac{\partial s_{0k}(\bar{\alpha})}{\partial x_k} = \frac{v''(\alpha)}{v'(\alpha)}(Er_k - Er_k^2). \tag{7}$$

Furthermore, since k can be chosen so that r_k is *not* a safe asset—Assumption R(2)—its variance is positive, which implies that

$$[Er_k - Er_k^2] = [(Er_k)^2 - Er_k^2] = -\text{var}\,(r_k) < 0. \tag{8}$$

Consequently, the function $v''(\alpha)/v'(\alpha)$ is an observable function, $q(\alpha)$, *on* $\mathring{\mathbb{R}}_+$:

$$\frac{v''(\alpha)}{v'(\alpha)} = q(\alpha) \equiv \frac{\partial s_{0k}(\bar{\alpha})}{\partial x_k}(Er_k - Er_k^2)^{-1}. \tag{9}$$

From this point on, we can follow the argument in Pratt (1964). Integrating $q(\alpha)$ gives $\log v'(\alpha) + c$. Exponentiating and integrating again gives $e^c v(\alpha) + d$. The constants of integration are immaterial since the cardinal utility functions $e^c v(\alpha) + d$ and $v(\alpha)$ are positive affine transformations of each other. ‖

Remark 1. Of course, the riskless asset need only be achieved as a linear combination of other assets. Also, compact support of the returns is only used to guarantee the existence of derivatives. If the latter exist, recovery is possible even without compact support bounded away from zero.

Theorem 1 establishes the recoverability of the cardinal utility u satisfying Assumption U for assets satisfying Assumption R from knowledge of the asset demand correspondence. An alternative approach consists of assuming knowledge of the portfolio indifference correspondence, I. For any $x \in X$ let $I(x) = \{x' \in X \mid Eu(r \cdot x) = Eu(r \cdot x')\}$. Assume that the correspondence I is observable. We shall demonstrate that, under Assumptions U and R, this is sufficient to recover the cardinal utility u up to a positive linear transformation. Our argument here is motivated by an argument of H. Sonnenschein (private correspondence).

Theorem 2. *Let both u and v satisfy Assumption U and generate the indifference correspondence I for assets satisfying Assumption R. Then $u \equiv v$ up to a positive linear transformation.*

Proof. Consider the plane, Π, in portfolio space, R^{m+1}, defined by $\Pi = \{x \in \mathbb{R}^{m+1} \mid x_k = 0, k = 1, \ldots, j-1, j+1, \ldots, m\}$. In Π consider any point $\bar{x} = \{\bar{x}_0, 0, \ldots, 0, \bar{x}_j, 0, \ldots, 0\}$ with $\bar{x} \in \mathring{X}$. By the Implicit Function Theorem and a corollary concerning differentiability—Dieudonné (1969), pp. 270–272—along with Assumptions R(1), U(2), U(3) and the fact that $Eu(r \cdot x)$ is a twice continuously differentiable function on $\mathring{X}$, it follows that, in some neighbourhood V of $(\bar{x}_0, \bar{x}_j)$ in Π, x_0 can be written as a unique twice continuously differentiable function $x_0 = f(x_j)$ such that

$$Eu(f(x_j)r_0 + x_j r_j) \equiv \bar{u} \tag{10}$$

everywhere on V. This is a parametric form of the agent's indifference curve passing through $\bar{x}$ restricted to the plane Π, and is, therefore, observable. Totally differentiating equation (10) with respect to x_j, we get

$$f'(\bar{x}_j) = -\frac{Er_j u'(\bar{x}_0 r_0 + \bar{x}_j r_j)}{Er_0 u'(\bar{x}_0 r_0 + \bar{x}_j r_j)} \tag{11}$$

and

$$f''(\bar{x}_j) = -\frac{Ef'(\bar{x}_j) r_0 + r_j^2 u''(\bar{x}_0 r_0 + \bar{x}_j r_j)}{Er_0 u'(\bar{x}_0 r_0 + \bar{x}_j r_j)}. \tag{12}$$

But if we look at $(\bar{x}_0, \bar{x}_j)$ with $\bar{x}_0 > 0$ and $\bar{x}_j = 0$, then $\bar{x}_0 r_0 + \bar{x}_j r_j = \bar{x}_0$ with probability one. Therefore, from (12) and the fact that $Er_j = 1$ and $r_0 = 1$ with probability one,

$$-\frac{u''(\bar{x}_0)}{u'(\bar{x}_0)} = \frac{f''(0)}{\text{var}\,(r_j)}. \tag{13}$$

Since the indifference map is observable, so is $f''(x_j)$; therefore, $-[u''(\bar{x}_0)/u'(\bar{x}_0)]$ is observable for all $\bar{x}_0 > 0$. A simple integration argument as in Pratt (1964) can then be used to recover the cardinal utility u up to a positive linear transformation. ||

Remark 2. As before, all that is really important is that some portfolio of assets be riskless. Also, compact support of the returns is needed only to guarantee existence of derivatives.

The two recoverability results contained in Theorems 1 and 2 were based on identical assumptions concerning the asset returns and the cardinal utility, but on different assumptions concerning the observable characteristics of the agent. That both results hold together should not be surprising since it is rather straightforward to derive the agent's demand correspondence from knowledge of the indifference correspondence, and the following simple argument outlines how the restriction of the indifference correspondence to the plane Π can be derived from the knowledge of the restricted demand correspondence.

As argued earlier, observability of the demand correspondence implies observability of the marginal rate of substitution function, s_{j0}; furthermore, the latter is continuously differentiable on $\mathring{X}$. Consider, then, the differential equation

$$\frac{d}{dt}(x_0, x_j) = (-s_{j0}, 1). \tag{14}$$

By Cauchy's Existence Theorem—Dieudonné (1969), p. 283—this differential equation has a unique solution for t in a neighbourhood of 0, once we are given the boundary condition $(x_0, x_j) = (\bar{x}_0, \bar{x}_j)$ at $t = 0$. As t varies, the solution to (14) maps out the indifference curve in a neighbourhood of $(\bar{x}_0, \bar{x}_j)$. This can be seen by noting that

$$\frac{dEu(x_0 r_0 + x_j r_j)}{d_t} = \frac{dx_0}{d_t}\frac{\partial Eu(\cdot)}{\partial x_0} + \frac{dx_j}{d_t}\frac{\partial Eu(\cdot)}{\partial x_j} = 0 \tag{15}$$

and that, since $dx_j/dt = 1$,

$$x_j(t) = \bar{x}_j + t. \tag{16}$$

Consequently, $t = x_j - \bar{x}_j$, and $g(x_j) \equiv x_0(t + \bar{x}_j)$ is an implicit function such that

$$Eu(g(x_j)r_0 + x_j r_j) = Eu(\bar{x}_0 r_0 + \bar{x}_j r_j) \tag{17}$$

for x_j in a neighbourhood of $\bar{x}_j$. That such a $g(\cdot)$ is unique and maps out the indifference curve follows from the Implicit Function Theorem.

Concerning the informational requirements for recoverability, Green, Lau and Polemarchakis (1979) posed the following questions:

—Can the asset demand correspondence be employed to reveal information about the distribution of returns of the different assets as well as the investor's cardinal utility function?

—What is it that can be said about recoverability if the demand correspondence is known only at a subset of the price domain?

The argument on analyticity required knowledge of the demand correspondence on an unbounded domain as well as complete knowledge of the distribution of returns of at least one asset. On the other hand, the argument develops above based on the existence of a riskless asset and Assumptions U and R entails much milder informational assumptions.

Proposition 1. *If the demand for assets is known only on a subset of the domain of prices, the cardinal utility can be recovered on a subset of its domain of definition, provided an interval of the amount invested in the riskless asset is in the interior of the supported set.*

Proposition 2. *Complete knowledge of the distribution of returns of the different assets is not necessary. It suffices to know the mean and variance of the returns to just one asset other than the riskless one.*

Proof. Both propositions follow immediately from the proofs of Theorems 1 and/ or 2.

Remark 3. Proposition 2 is not to be misinterpreted as implying that asset demands depend only on the first and second moments of the asset distribution. The point is simply that no independent knowledge of the asset distribution, beyond the first two moments, is required, in addition to the asset demands, for recoverability.

It is an open question whether the existence of a riskless asset is a necessary assumption in order to dispense with analyticity on the closed non-negative real line as a condition for recoverability of the cardinal utility. The following example, which generalizes an example developed by A. McLennan and communicated privately, serves as a partial answer to this question.

Consider an investor with utility $u(\alpha)$ satisfying Assumption U, who must allocate his initial wealth between two assets, $j = 1, 2$, with returns in four states of nature. The returns to asset 1 are given by $(c, c\mu, d, d\mu)$ and to asset 2 by $(d, d\mu, c, c\mu)$, where $c > 0$, $d > 0$, $\mu > 0$, $c \neq d$. The objective function of the investor is then

$$\begin{aligned} V(x_1, x_2) = {} & \pi_1 u(cx_1 + dx_2) + \pi_2 u(\mu(cx_1 + dx_2)) \\ & + \pi_3 u(dx_1 + cx_2) + \pi_4 u(\mu(dx_1 + cx_2)) \end{aligned} \tag{18}$$

where π_i is the probability of occurrence of state i, $i = 1, \ldots, 4$, and x_j is the amount invested in asset j, $j = 1, 2$.

Now let $f(\alpha)$ be a function defined on the same domain as $u(\alpha)$ and such that

$$\tilde{u}(\alpha) \equiv u(\alpha) + f(\alpha) \quad \text{satisfies } U,$$

and

$$f(\lambda\alpha) \equiv -\lambda^m f(\alpha) \quad \text{for some } \lambda > 0 \text{ and some } m > 0.$$

Then, if $\pi_1 = \lambda^m \pi_3$ while $\pi_2 = \lambda^m \pi_4$, and $\lambda = \mu$, the investor with cardinal utility $\tilde{u}$ has the same objective function as the investor with cardinal utility u. Hence, knowledge of the asset demands $x_j(p)$, $j = 1, 2$, and the distribution of asset returns *does not* permit the recoverability of the cardinal utility—u and $\tilde{u}$ are *not* positive linear transformations of each other, yet lead to identical behaviour for the particular problem considered.

To complete the example, we must show that there exist functions $u(\alpha)$ and $f(\alpha)$ as described above. Taking

$$u(\alpha) = \frac{k}{l}\alpha^l, \qquad 0 < l < 1, \quad k > 0. \tag{19}$$

$$\tilde{u}(\alpha) = \frac{k}{l}\alpha^l + \alpha^l \sin(ln\alpha), \qquad 0 < l < 1, \quad k > 0. \tag{20}$$

we see that U is satisfied by both $u(\alpha)$ and $\tilde{u}(\alpha)$ for k large. Furthermore,

$$f(\lambda\alpha) = -\lambda^l f(\alpha) \quad \text{for } \lambda = e^{\pi}.$$

Observe now that u and $\tilde{u}$ as defined by (19) and (20) satisfy much stronger regularity conditions than Assumptions U. Namely, in addition to monotonicity and concavity, they are analytic on the open positive real line and continuous on the closed non-negative real line. Furthermore, there is a trade-off between concavity and the degree of differentiability at the origin of the functions u and $\tilde{u}$ that yield counterexamples to recoverability. Namely, setting the constant l arbitrarily high, we make u and $\tilde{u}$ defined by (19) and (20) differentiable of arbitrarily high degree at the origin, while maintaining monotonicity. Concavity is, however, violated for $l \geqq 1$. We are thus led to the following

Conjecture. *Let both u and v be continuous, strictly monotone and concave on the closed non-negative real line, and possess bounded right derivatives at the origin. Then, if u and v generate the same demand correspondence for assets satisfying Assumptions* $R(1)$–$R(3)$, *they can differ only by a positive linear transformation.*

Both the counterexample to recoverability as well as the conjecture stated above indicate that the behaviour of the marginal utility at the origin is central to the problem. We shall conclude with an argument demonstrating that, under Assumptions R and U the boundedness, or unboundedness, of the marginal utility at the origin is observable, provided some risky asset, l, has discrete support.

Restrict attention to portfolio of the form $x_0 r_0 + x_l r_l$, where l is the asset with discrete support. Let $\bar{r}$ be the maximum value attained with non-zero probability by the discrete random variable r_l, and define $r_{l^*} = r_0 - (r_0/r_l) r_l$. Since the random variable r_{l^*} is spanned by r_0 and r_l and since the demand correspondence for assets 0 and l is known, so is the demand correspondence over assets 0 and l^*. Consequently, the marginal rate of substitution

$$s_{0l^*}(x_0, x_{l^*}) = \frac{E r_0 u'(x_0 r_0 + x_{l^*} r_{l^*})}{E r_{l^*} u'(x_0 r_0 + x_{l^*} r_{l^*})} \tag{21}$$

is also known. For fixed $x_{l^*} > 0$ consider the limit of $s_{0l^*}(x_0, x_{l^*})$ as $x_0 \downarrow 0$. The claim is that $\lim_{x_0 \downarrow 0} s_{0l^*}(x_0, x_{l^*}) < \infty$ if and only if $\lim u'(\alpha) < \infty$. Necessity is clear since, if $\lim_{\alpha \downarrow 0} u'(\alpha) < \infty$, s_{0l^*} is continuous and the limit is the same as the value at $x_0 = 0$, which is finite. If $\lim_{\alpha \downarrow 0} u'(\alpha) = \infty$, s_{0l^*} is unbounded, since r_l attains the value r^* with positive probability, while the denominator maintains a finite value.

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