Identifiability of the von Neumann–Morgenstern Utility Function from Asset Demands

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If the demand for risky assets is determined by the maximization of an analytic von Neumann–Morgenstern utility function, and if these demands are known as a function of the assets' prices, then this utility function can be constructed without ambiguity.

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The theory of revealed preference posed the question of the properties that, if satisfied by a correspondence, allow it to be characterized as the demand correspondence derived from the maximization of a monotone, quasi-concave utility function subject to a budget constraint. A converse to this question can be phrased as follows: Consider a demand correspondence and suppose that it is indeed derived from utility maximization subject to a budget constraint. Does the demand correspondence contain sufficient information to identify the particular utility function (or preferences) in the family of quasi-concave, monotone utility functions (or convex, monotone complete, preference preorders) from which it has been derived? In the case of maximization of an ordinal utility function the answer is clear: If nominal prices and income vary independently, the range of the demand correspondence can be assumed to contain an open subset of the consumption set.

The utility function over this range can be identified up to a monotone transformation, given some very mild regularity conditions on demand behavior [see Mas-Colell (1977)].

The question of identifiability acquires additional complexity in the case of assets demanded by an investor who maximizes a von Neumann–Morgenstern utility function. As the prices of assets vary, and, as is likely to be the case in a framework of incomplete markets, the number of states of nature exceeds the number of assets, only a lower dimensional subspace of the space of state-dependent values of terminal wealth is attainable. No direct observations can be made on preferences for state-contingent wealth patterns outside this subspace. The standard argument for identifiability thus fails. On the other hand, the utility function can be assumed to have the special additively separable form implied by the axioms of expected utility theory. It is the purpose of this paper to demonstrate that knowledge of the asset demands is sufficient to identify the von Neumann–Morgenstern utility function, provided the latter can be assumed to be independent of the state of nature and analytic over the nonnegative real line.

The problem of identifiability of the utility function is of empirical as well as theoretical interest. From a theoretical point of view, identifiability is necessary if the theory of choice under uncertainty is to possess explanatory power. From an empirical viewpoint, changes in the tax structure, introduction of new assets, and the acquisition of previously unavailable information are instances of changes in the economic environment faced by economic agents. Knowledge of the agent's von Neumann–Morgenstern utility function is necessary if, for example, we want to evaluate the impact of these changes on his market behavior or on his welfare. By virtue of the main result of this paper, these effects can be ascertained by observations related to one system of asset returns and need not be reassessed after the environment has been altered.

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We consider an investor who must divide his initial wealth among m assets. The quantity of his asset holdings is denoted $x = (x_1, \ldots, x_m)$. Each asset j has a random gross return $\mathbf{r}_j$. The collection of random variables[1] $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_m)$ describes their joint distribution, and hence the distribution of the value of any portfolio $\mathbf{r} \cdot x$.

We assume that $\mathbf{r}$ satisfies:

r(i) $\mathbf{r}$ is nonnegative with probability 1;

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[1] These are understood to be defined on an underlying measure space, which need not be mentioned explicitly.

r(ii) for each j, $\mathbf{r}_j$ is not zero with probability 1;

r(iii) for each j, $\mathbf{r}_j$ cannot be written as a linear combination of $\{\mathbf{r}_k\}_{k \neq j}$, with probability 1;

r(iv) for each j, and each postive integer l, $E\mathbf{r}_j^l < \infty$.

By virtue of r(i), r(ii), and r(iv) we can choose the units of measurement of each asset so that

$$E\mathbf{r}_j = 1 \qquad \text{for each} \quad j. \tag{1}$$

Condition r(i) is not objectionable because the $\mathbf{r}_j$ represents gross returns.[2] Conditions r(ii) and r(iii) are technical in nature and are designed to rule out redundancies and assets that will never be purchased in positive quantity.[3] Condition r(iv) is of some importance and is surely not innocuous in practice. It is assured in any model with a finite set of states (i.e., where $\mathbf{r}$ can take only finitely many values).

The investor is assumed to be a von Neumann–Morgenstern utility maximizer. He is assumed to know the distribution of $\mathbf{r}$ and to have a von Neumann–Morgenstern utility function u. We assume that u satisfies

u(i) u is defined over the domain of all nonnegative real numbers;
u(ii) u is increasing;
u(iii) u is concave;
u(iv) u is analytic.[4]

For example, the function $u(w) = -e^{-\rho w}$, $\rho > 0$, $w \geqq 0$, satisfies u(i)–u(iv).

Among these conditions, only the first and the last require comment.

The fact that u is defined at zero rules out such functions as the logarithm which are unbounded. These can perhaps be ruled out on other grounds.[5] Analyticity is a much stronger condition, which cannot be justified behaviorally, except that an arbitrary function can always be approximated

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[2] In the absence of limited liability corporations, some $\mathbf{r}_j$ might be negative with positive probability. If these could be bounded below; our problem would be essentially unchanged, although some modifications would have to be made in the treatment of the domain of von Neumann–Morgenstern utility.

[3] In a general equilibrium model the price of such an asset is sure to be zero; so if present, it can be deleted *ab initio* without loss of generality.

[4] To define $u^{(l)}(0)$, we extend u to an open set containing the nonnegative real line and use the derivative of this extension at zero.

[5] Menger (1934) was perhaps the first to note that bounded utility is required if expected utility is to order all distributions of returns in a consistent fashion. Arrow (1971), Ryan (1974), Arrow (1974), and Fishburn (1976) have explored the relationship between the class of distributions to be ordered and the restrictions that can be placed on the von Neumann–Morgenstern utility function. The interested reader might consult Aumann (1977) and Shapley (1977a,b) for discussions, in a closely related vein, of the St. Petersburg paradox.

pointwise by an analytic function.[6] It implies, of course, that derivatives of all orders exist. In particular, it implies that $u'(0)$ exists and is finite, a fact that will be used heavily in the proof of our main theorem. [For example $u(w) = (1/\alpha)w^{\alpha}$, $\alpha < 1$, would fail to satisfy this property.]

The investor chooses $x \in X$ facing prices $p = (p_1, \ldots, p_m)$ where $X = \{x | \mathbf{r} \cdot x \geqq 0 \text{ with probability } 1\}$. Without loss of generality we can take his initial level of wealth to be unity. Thus his problem is

$$\max_{x \in X} Eu(\mathbf{r} \cdot x) \qquad \text{subject to} \quad p \cdot x \leqq 1. \tag{2}$$

PROPOSITION 1 The objective function $Eu(\mathbf{r} \cdot x)$ is concave in x and is defined over a domain that includes all nonnegative $x \in \mathbb{R}^m$.

Proof Concavity follows directly. The domain of the expected utility X includes all x for which $\mathbf{r} \cdot x$ is within the domain of definition of u with probability 1. Assumptions r(i) and u(i) guarantee this whenever $x \in \mathbb{R}^m$ is nonnegative. ■

PROPOSITION 2 $dEu(\mathbf{r} \cdot x)/dx_j$ exists and is given by

$$E\mathbf{r}_j u'(\mathbf{r} \cdot x) \qquad \text{for all} \quad x \in X.$$

Proof Let $h^j = (0, \ldots, h, \ldots, 0)$ where the h is in the jth place. By definition,

$$\frac{dEu(\mathbf{r} \cdot x)}{dx_j} = \lim_{h \to \infty} \frac{Eu(\mathbf{r} \cdot x + h^j) - Eu(\mathbf{r} \cdot x)}{h}.$$

We now show that this limit exists for any $x \in X$ and for any sequence $\langle h_k^j \rangle$ with $\lim_k h_k^j = 0$. Consider the sequence of functions

$$f_k(\mathbf{r}) = \frac{u(\mathbf{r} \cdot x + h_k^j) - u(\mathbf{r} \cdot x)}{h_k}.$$

Monotonicity of u and nonnegativity of r imply that $f_k(\mathbf{r}) \geqq 0$. Concavity of u implies that $f_k(\mathbf{r}) \leqq \mathbf{r}_j u'(0)$, which is finite by analyticity of u. $E\mathbf{r}_j u'(0)$ is

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[6] There are several open questions concerning the approximation problem. Let u be a smooth, concave, increasing utility generating demands $\xi(p)$. If u is approximated by a sequence of concave analytic utility functions, does the corresponding sequence of demands converge to $\xi(p)$ in an appropriate sense? Furthermore if the demands are observed with error and are perceived to be $\tilde{\xi}(p)$, can the utility generating $\tilde{\xi}(p)$ be used as an approximation to u? This last question is made more complicated by the observation that even if $\tilde{\xi}(p)$ satisfies revealed preference conditions, it may not be generated by any von Neumann–Morgenstern utility at all. Indeed the precise necessary and sufficient conditions on demands to ensure compatibility with the axioms of expected utility theory are unknown to us at present.

finite by virtue of r(iv). Since u is differentiable for all values of its argument,

$$f_k(\mathbf{r}) \to \mathbf{r}_j u'(\mathbf{r} \cdot x)$$

for all $\mathbf{r} = 0$. Therefore the sequence $f_k(\mathbf{r})$ satisfies the hypotheses of the Lebesque dominated convergence theorem. Hence,

$$\begin{aligned} E\mathbf{r}_j u'(\mathbf{r} \cdot x) &= \lim_k \frac{Eu(\mathbf{r} \cdot x + h_k^j) - Eu(\mathbf{r} \cdot x)}{h_k} \\ &= \frac{dEu(\mathbf{r} \cdot x)}{dx_j}. \quad \blacksquare \end{aligned}$$

Let us write the vector of marginal rates of substitution of assets $j = 1, \ldots, m$ for asset k by

$$s_{jk}(x) = \frac{dEu(\mathbf{r} \cdot x)/dx_j}{dEu(\mathbf{r} \cdot x)/dx_k},$$

which is well defined by Proposition 2.

PROPOSITION 3 For all $x \in \mathbb{R}^m_+$, $s_{jk}(x)$ is a nonzero real number for all pairs j, k.

Proof Differentiability, monotonicity, and concavity of u imply that $u'(\cdot)$ is bounded. Finiteness of mean returns $[r(\text{iv})]$ implies that

$$0 < E\mathbf{r}_j u'(\mathbf{r} \cdot x) < \infty$$

for each j. ■

For each $p \in \mathbb{R}^m_+ \backslash \{0\}$, the investor tries to solve (2). At some prices the maximum may fail to exist.[7]

PROPOSITION 4

(i) For each $x \in \mathbb{R}^m_+ \backslash \{0\}$ there exists $p \in \mathbb{R}^m_+ \backslash \{0\}$ such that x solves (2) at prices p.

(ii) For each $x \in \mathbb{R}^m_+ \backslash \{0\}$ there is a unique $p \in \mathbb{R}^m_+ \backslash \{0\}$ so that x is demanded at p.

Proof

(i) follows by choosing $p \in \mathbb{R}^m$ for each $x \in \mathbb{R}^m$, so that $(p_2, \ldots, p_m)$ is proportional to $(s_{21}, \ldots, s_{m1})$ and so that $p \cdot x = 1$. (Note $x = 0$ is not observed at any finite price system.)

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[7] For example, if for two assets j and k, $(r_j/p_j) > (r_k/p_k)$ with probability 1, then a sure profit can be gained by buying j and selling k short. Thus no maximum can be found unless further restrictions are placed on x, such as a prohibition of short sales.

(ii) follows from the differentiability of u, and the fact that x has at least one positive element. ■

Let the demand correspondence be denoted by $\xi : P \to R^m$ where P is the set of prices for which a solution to (2) exists.

The correspondence ξ is observable. We want to be able to deduce from ξ the individual's von Neumann–Morgenstern utility function, of course up to a positive linear transformation. In addition to ξ we assume a knowledge of the distribution of $\mathbf{r}$. Given $\mathbf{r}$ and ξ, a quantity is called *observable* if it can be deduced without ambiguity.

PROPOSITION 5 The marginal rates of substitution $s_{jk}(x)$ are observable for any pair of assets j, k and any $x \in \mathbb{R}^m_+\{0\}$.

Proof By Proposition 4, there is a unique p such that $x \in \xi(p)$. The s_{jk} can be computed as ratios of the components of this p. ■

PROPOSITION 6 $s_{jk}(0) = 1$ for all j, k, independent of u.

Proof

$$\frac{(d/dx_j)Eu(\mathbf{r} \cdot x)}{(d/dx_k)Eu(\mathbf{r} \cdot x)} = \frac{E\mathbf{r}_j u'(\mathbf{r} \cdot x)}{E\mathbf{r}_k u'(\mathbf{r} \cdot x)}$$

and by Proposition 2, at $x = 0$ we have

$$s_{jk}(0) = E\mathbf{r}_j / E\mathbf{r}_k = 1$$

by virtue of our normalization (1). ■

Thus the functions $s_{jk}(\cdot)$ are observable throughout $\mathbb{R}^m_+$. If their derivatives exist, these quantities are also observable because they are defined by the functions $s_{jk}(\cdot)$ themselves.

PROPOSITION 7 The derivatives of $s_{jk}(\cdot)$ at $x = 0$ of all orders exist and are observable.

Proof Consider the first expression

$$\frac{E\mathbf{r}_k u'(\mathbf{r} \cdot x) E\mathbf{r}_j \mathbf{r}_i u''(\mathbf{r} \cdot x) - E\mathbf{r}_j u'(\mathbf{r} \cdot x) \cdot E\mathbf{r}_k \mathbf{r}_i u''(\mathbf{r} \cdot x)}{(E\mathbf{r}_k u'(\mathbf{r} \cdot x))^2}, \tag{3}$$

which, if well defined, would be the value of $ds_{jk}(x)/dx_i$. At $x = 0$ we have

$$\begin{aligned} \frac{ds_{jk}(0)}{dx_i} &= \frac{u'(0)u''(0)(E\mathbf{r}_k E\mathbf{r}_j \mathbf{r}_i - E\mathbf{r}_j E\mathbf{r}_k \mathbf{r}_i)}{(u'(0))^2 (E\mathbf{r}_k)^2} \\ &= \frac{u''(0)}{u'(0)} (E\mathbf{r}_j \mathbf{r}_i - E\mathbf{r}_k \mathbf{r}_i). \end{aligned}$$

In order for this to be well defined, it is required that the two expectations in the last expression exist. Note that, by Hölder's inequality[8]

$$(E\mathbf{r}_j\mathbf{r}_i)^2 \leqq (E\mathbf{r}_j^2)(E\mathbf{r}_i^2),$$

which is finite by virtue of r(iv).

Higher order derivatives follow by successive differentiation of (3) and repeated application of Hölder's inequality. ■

We now come to our main result: the uniqueness of the von Neumann–Morgenstern utility explaining the demand ξ, within the class of all analytic utility functions.

THEOREM Let v be analytic and generate the demand correspondence ξ. Then $v \equiv u$, up to a positive linear transformation.

Proof Without loss of generality we can suppose $u(0) = v(0) = 0$ and $u'(0) = v'(0) = 1$. We will show how to construct the higher derivatives of v at zero recursively, thus defining v uniquely within the class of analytic functions. That $v \equiv u$ follows directly.

If v generates ξ, the marginal rates of substitution implied by r must be the same as those that are observable through knowing ξ, by Proposition 5. Moreover, the derivatives of the marginal rates of substitution under v must be the same as those observable at zero through knowing ξ, by Propositions 6 and 7.

Proposition 7 holds for any j, k, and any higher derivative of $s_{jk}(x)$ at $x = 0$. We need only two assets and a particular sequence of derivatives. Let us consider assets 1 and 2 and the derivatives

$$\frac{d}{dx_1} s_{21}(0), \frac{d^2}{dx_1^2} s_{21}(0), \ldots, \frac{d^l}{dx_1^l} s_{21}(0), \ldots$$

and

$$\frac{d}{dx_2} s_{12}(0), \frac{d^2}{dx_2} s_{12}(0), \ldots, \frac{d^l}{dx_2^l} s_{12}(0), \ldots$$

all of which are observable by virtue of Proposition 7.

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[8] Hölder's inequality states that if z_1 and z_2 are nonnegative random variables, and if $(1/p) + (1/q) = 1$, then

$$Ez_1z_2 \leqq (Ez_1^p)^{1/p}(Ez_2^q)^{1/q}$$

with equality holding if and only if $z_1 = cz_2$ for some c, with probability 1. [See, e.g., Royden (1963), pp. 95 and 202).]

If v explains the same demands as u, then

$$\left(\frac{d}{dx_1} Ev(\mathbf{r} \cdot x)\right) \cdot s_{21}(x) = \left(\frac{d}{dx_2} Ev(\mathbf{r} \cdot x)\right) \tag{4}$$

holds as an identity in x. Let us identify successive derivatives of each side in x_1, at $x = 0$. Differentiating (4) we have

$$\begin{aligned} s_{21}(x)E\mathbf{r}_1^2 v''(\mathbf{r} \cdot x) &+ E\mathbf{r}_2\mathbf{r}_1 v''(\mathbf{r} \cdot x) \\ &= -\left(\frac{d}{dx_1} s_{21}(x)\right)(E\mathbf{r}_1 v'(\mathbf{r} \cdot x)). \end{aligned}$$

Substituting $x = 0$ we have

$$v''(0)(E\mathbf{r}_1^2 - E\mathbf{r}_2\mathbf{r}_1) = -\frac{d}{dx_1} s_{21}(0).$$

Since the right-hand side is observable, a unique value of $v''(0)$ compatible with the demand function ξ exists provided

$$E\mathbf{r}_1^2 - E\mathbf{r}_2\mathbf{r}_1 \neq 0. \tag{5}$$

Similarly, by reversing the roles of commodities 1 and 2 we can conclude that

$$E\mathbf{r}_2^2 - E\mathbf{r}_2\mathbf{r}_1 \neq 0 \tag{6}$$

would be sufficient to recover $v''(0)$ uniquely.

Assume that (5) and (6) were both false. Multiplying equals by equals we would have

$$(E\mathbf{r}_2\mathbf{r}_1)^2 = (E\mathbf{r}_2^2)(E\mathbf{r}_1^2). \tag{7}$$

From Hölder's inequality we know that (7) can occur only if $r_2 = cr_1$ with probability 1 for some constant c. This would clearly violate r(iii). Therefore either (5) or (6) must hold, and $v''(0)$ is uniquely recoverable.

Differentiating (4) again in x_1 we find

$$\begin{aligned} s_{21}(x)E\mathbf{r}_1^3 v'''(\mathbf{r} \cdot x) &+ E\mathbf{r}_2\mathbf{r}_1^2 v'''(\mathbf{r} \cdot x) \\ &= \left(\frac{-d}{dx_1} s_{21}(x)\right)(2E\mathbf{r}_1^2 v''(\mathbf{r} \cdot x)) - \left(\frac{d^2}{dx_1^2} s_{21}(x)\right) E\mathbf{r}_1 v'(\mathbf{r} \cdot x). \end{aligned}$$

Evaluating at $x = 0$, we have

$$v'''(0)\{E\mathbf{r}_1^3 - E\mathbf{r}_2\mathbf{r}_1^2\} = \frac{-d}{dx_1} s_{21}(0)(E\mathbf{r}_1^2)(\mathbf{r}''(0)) - \frac{d^2}{dx_1^2} s_{21}(0).$$

Again the right-hand side is known, either by direct observation or by deduction at the previous stage. Therefore $v'''(0)$ will be uniquely recoverable whenever

$$E\mathbf{r}_1^3 - E\mathbf{r}_2\mathbf{r}_1^2 \neq 0.$$

Similarly, to recover the lth derivative $v^l(0)$ for any $l \geqq 3$ by this method, we must have either

$$E\mathbf{r}_1^l - E\mathbf{r}_2\mathbf{r}_1^{l-1} \neq 0 \tag{8}$$

or

$$E\mathbf{r}_2^l - E\mathbf{r}_1\mathbf{r}_2^{l-1} \neq 0. \tag{9}$$

Suppose both of these fail; then,

$$(E\mathbf{r}_1\mathbf{r}_2^{l-1})(E\mathbf{r}_2\mathbf{r}_1^{l-1}) = (E\mathbf{r}_2^l)\,(E\mathbf{r}_1^l). \tag{10}$$

By Hölder's inequality

$$E\mathbf{r}_1\mathbf{r}_2^{l-1} \leqq (E\mathbf{r}_1^l)^{1/l}(E\mathbf{r}_2^l)^{l-1/l} \tag{11}$$

and

$$E\mathbf{r}_2\mathbf{r}_1^{l-1} \leqq (E\mathbf{r}_1^l)^{l-1/l}(E\mathbf{r}_2^l)^{1/l} \tag{12}$$

with equality in both relations if and only if $\mathbf{r}_1 = c\mathbf{r}_2^{l-1}$ and $\mathbf{r}_2 = d\mathbf{r}_1^{l-1}$, with probability 1, for two constants c and d. Therefore, since all the terms in (10)–(12) are nonnegative,

$$(E\mathbf{r}_1\mathbf{r}_2^{l-1})(E\mathbf{r}_2\mathbf{r}_1^{l-1}) < (E\mathbf{r}_2^l)(E\mathbf{r}_1^l) \tag{13}$$

unless $\mathbf{r}_1 = c\mathbf{r}_2^{l-1}$ and $\mathbf{r}_2 = d\mathbf{r}_1^{l-1}$ with probability 1. From the latter we would have $r_1 = d^{-1/l-1}r_2^{1/l-1}$ with probability 1. Combining this with the former,

$$1 = cd^{1/l-1}\mathbf{r}_2^{(l-1)} - (1/l - 1)$$

with probability 1. Since $l \geqq 3$, this implies that $\mathbf{r}_2$ is a constant, almost surely. Reversing the roles of assets 1 and 2 in this argument, we would have that r_1 is a constant almost surely, and hence that there are two safe assets—which would violate r(iii). Therefore (13) holds and at least one of (8) and (9) is true, so that $v^l(0)$ is uniquely determined given $v^{(l-l')}(0)$ for $l' = 1, \ldots, l$. Since $v \equiv u$ clearly suffices, this uniqueness argument completes the proof. ■

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We have shown in our main theorem that it is possible to identify uniquely the von Neumann–Morgenstern utility function from an asset demand correspondence defined for all $p \in \mathbb{R}_+^m \backslash \{0\}$ provided that the joint distribution of gross returns is known and satisfies certain conditions and provided that the utility function is analytic and concave on the nonnegative real line. From the structure of our proof, it is clear that if the utility function is state dependent, then there is in general no possibility of unique identification of each of the state-dependent utility functions.

There are a number of open questions concerning the possible relaxation of the assumptions of the main theorem that remain to be investigated.

First, it is of interest to examine the case in which the asset demand correspondence is defined only on a bounded subset of $\mathbb{R}_+^m + \{0\}$. The implication of this restriction on the domain of the asset demand correspondence is that the range of the asset demand correspondence can no longer be assumed to include all of $\mathbb{R}_+^m + \{0\}$. In particular, the origin may be disjoint from the range of the asset demand correspondence. Our proof, which is based on an analytic expansion of the utility function at the origin, will no longer apply.

Second, it is of interest to examine the case in which the joint distribution of the returns is unknown. The question is whether one can determine the joint distribution of returns and the von Neumann–Morgenstern utility function uniquely from an asset demand correspondence. The finiteness of the states of nature may make a difference in this regard.

Third, it is of interest to examine the case in which the utility function is not analytic on the nonnegative real line. There are at least two possible directions of relaxation. First, one may assume that the utility function is analytic only over the positive real line, thus admitting the case in which marginal utility is unbounded at the origin. Second, one may investigate the possibility of approximation of an arbitrary concave function defined on the nonnegative real line by a concave and analytic function. The sense in which this approximation is taken will be related to the sense in which the demand correspondence can be approximated by one derived from an analytic utility function.[9]

Resolution of each of the three open questions will enhance the usefulness of our main result derived here.

REFERENCES

Arrow, K. J. (1971). "Essays in the Theory of Risk Bearing." North-Holland Publishing, Amsterdam.

Arrow, K. J. (1974). The use of unbounded utility functions in expected utility maximization: Response, *Q. J. Econ.* **88**, 136–138.

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[9] There exists an increasing concave function that is analytic everywhere except at zero; for example,

$$\alpha e^{-1/w} - e^{-\rho w}, \qquad w \geqq 0, \quad \alpha, \rho > 0, \quad \text{and} \quad \alpha \quad \text{sufficiently small.}$$

For this utility function the method of recovering if used in our proof above would not work—it would recover only the $e^{-\rho w}$ part. But since these preferences do not coincide, the identity of the marginal rates of substitution does not hold globally. Whether analyticity can be dropped if concavity is maintained remains therefore an open question.

Aumann, R. J. (1977). The St. Petersburg paradox: A discussion of some recent comments, *J. Econ. Theory* **14**, 443–445.

Fishburn, P. (1976). Unbounded utility functions in expected utility theory, *Q. J. Econ.* **90**, 163–168.

Mas-Colell, A. (1977). The recoverability of consumers' preferences from market demand behavior, *Econometrica* **45**, 1409–1430.

Menger, K. (1934). The role of uncertainty in economics, *in* "Essays in Mathematical Economics in Honor of Oskar Morgenstern," (Martin Shubik, ed.), pp. 211–231. Princeton University Press, Princeton, N.J., translated by Wolfgang Schoellkopf from the *Zeitschrift fur National Okonomie*, *5*, pp. 349–485.

Royden, H. L. (1963). "Real Analysis." Macmillan, New York.

Ryan, T. M. (1974). The use of unbounded utility functions in expected utility Maximization: Comment, *Q. J. Econ.* **88**, 133–135.

Shapley, L. S. (1977a). The St. Petersburg paradox—Acon game? *J. Econ. Theory* **14**, 439–442.

Shapley, L. S. (1977b). Lotteries and menus: A comment on unbounded utilities, *J. Econ. Theory* **14**, 446–453.

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