Unequal Treatment in the Core

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It is demonstrated that, under regularity assumptions on individuals' preferences, for an open dense set of exchange economies indexed by initial endowments, the core does *not* possess the equal treatment property. The assumptions made on individuals' preferences are subsequently shown to characterize an open dense subset of the space of preferences.

1. INTRODUCTION

THE CORE IS EXTENSIVELY EMPLOYED as a solution concept in game theory and, by extension, in the theory of general economic equilibrium. As a consequence, the question arises whether core allocations possess the equal treatment property: Are identical individuals treated identically? An answer is crucial especially for normative arguments that seek to support core allocations on grounds of fairness. The original attempt to solve the problem was undertaken by Green in [**4**]. He demonstrated that, except for the special case where strong symmetry conditions[1] hold in the distribution of characteristics among the agents in the economy, the core does not possess the equal treatment property for an open dense set of economies.

The object of this note is to give a much simpler proof of Green's results, and to extend them in the following direction: An exchange economy is an assignment of both preferences and endowments and, as such, it must be indexed in terms of both of these indices. Consequently, the space of economies must allow for differences not only in endowments, as in Green's work, but also in preferences. We shall extend the results in [**4**] by showing that the properties of the utility function sufficient for equal treatment to fail are generic.

Our extension of Green's results is similar to the extension by Smale in [**7, 8**] of the results in Debreu [**2**]. Our method of proof is, however, different. In the last section, we shall show how our approach can be used to derive and extend the results of Debreu and Smale.

2. THE MODEL AND RESULTS

In this section we present the model and give precise statements of our results. We begin with some preliminary notation. Let $l+1$ be a finite number of commodities, R^{l+1} the $(l+1)$-dimensional Euclidean space, and $\mathring{R}_+^{l+1}$ the interior of the positive orthant of this space. Let N be the set of positive integers, and for any two vectors of the same dimension let $\gg$, $>$, $\geqslant$ have the obvious meaning. Let the commodities be indexed by a subscript $h (h=0, 1, \ldots, l)$ and let there be T types of consumers with $\bar{s}^t$ individuals of each type, where $\bar{s}^t \in N$, $t=1, \ldots, T$. Thus the total number of consumers is $\Sigma_{t=1}^T \bar{s}^t$, say τ. An individual

[1] Specifically, g.c.d. $\{\bar{s}_t\}_{t=1}^T \neq 1$ where $\bar{s}_t$ is the number of traders of type t; see below.

of type t is identified with the ordered pair (u^t, ω^t), where u^t is his utility function defined on his consumption set X^t, and ω^t his endowment. We shall index an individual of type t by superscripts i, t $(i = 1, \ldots, \bar{s}^t; t = 1, \ldots, T)$. Thus x_h^{it} is the amount of the hth commodity allocated to the ith individual of type t. We shall make the following assumptions:

ASSUMPTION 1: $X^t = \mathring{R}_+^{l+1}$, $t = 1, \ldots, T$.

ASSUMPTION 2: *u^t belong to $C^2(\mathring{R}_+^{l+1}, R)$; it is strictly conçave; for all $x \in \mathring{R}_+^{l+1}$, $Du^t(x) \gg 0$, and the closure of the indifference hypersurface through x is contained in $\mathring{R}_+^{l+1}$, $t = 1, \ldots, T$.*

ASSUMPTION 3: $\omega^t \in \mathring{R}_+^{l+1}$, $t = 1, \ldots, T$.

ASSUMPTION 4: *The greatest common divisor (g.c.d.) of $\{\bar{s}^1, \ldots, \bar{s}^t, \ldots, \bar{s}^T\}$ is 1.*

We let U denote the subset of $C^2(\mathring{R}_+^{l+1}, R)$ satisfying Assumption 2 and endowed with the C^2 norm, on a compact subset K of $\mathring{R}_+^{l+1}$, and U^* the subspace of U of functions of nowhere vanishing Gaussian curvature on K. An *economy* is a point $(\bar{s}, u, \omega) = (\bar{s}^1, \ldots, \bar{s}^T, u^1, \ldots, \bar{u}^T, \omega^1, \ldots, \omega^T)$ in the space $[N \times U \times \mathring{R}_+^{l+1}]^T$. Given an economy $(\bar{s}, u, \omega)$, a *subeconomy* is an economy (s, u, ω) where $s^t \leq \bar{s}^t$ and s^t equals number of elements in $S^t \subseteq \{1, 2, \ldots, \bar{s}^t\}$, $t = 1, \ldots, T$, with strict inequality and strict inclusion for at least one t. A *complementary subseconomy* to the subeconomy (s, u, ω) is an economy (s_c, u, ω) such that s_c^t is the number of elements in $S_c^t = \{\{1, 2, \ldots, \bar{s}^t\}/S\}^t\}$ for all $t = 1, \ldots, T$. An *allocation* for $(\bar{s}, u, \omega)$ is a point $x = (x^{it} | i = 1, \ldots, s^t; t = 1, \ldots, T)$ in $(\mathring{R}_+^{l+1})^\tau$. Given an economy $(\bar{s}, u, \omega)$, we denote the set of *price equilibria* of $(\bar{s}, u, \omega)$ by $E_{\bar{s}}(u, \omega)$ where the latter consists of a price system[2] p in $\mathring{R}_+^{l+1}$, such that:

(i) For all i, t, $u^t(x^{it})$ is maximal over the set $\{x \in \mathring{R}^{l+1} | px \leq p\omega^t\}$.

(ii) $\sum_{t=1}^T \sum_{i=1}^{\bar{s}^t} x^{it} = \sum_{t=1}^T \bar{s}^t \omega^t$.

We shall denote the set of price equilibria of the subeconomy (s, u, ω) by $E_s(u, \omega)$ and the set of price equilibria of the complementary subeconomy by $E_{s_c}(u, \omega)$. Let $C_{\bar{s}}(u, \omega)$ denote the set of *core allocations*[3] of $(\bar{s}, u, \omega)$ and say that the core of $(\bar{s}, u, \omega)$ possesses the *equal treatment property* if and only if for all allocations $x \in C_{\bar{s}}(u, \omega)$, x^{it} is independent of i, for all t.

We shall demonstrate the following:

THEOREM 1: *Given a distribution of types $\bar{s}$ in $[N]^T$ satisfying Assumption 4 and an assignment of utility functions u in $[U^*]^T$, there exists an open dense subset*

[2] We take the price system to be the set

$$\{(p_0, p_1, \ldots, p_h, \ldots, p) | p_0 = 1, p_h > 0, h = 1, \ldots, l\}.$$

[3] See, for example, [4, 5] for a definition.

$\Omega_{(\bar{s},u)}$ of[4] *$[\mathring{R}^{l+1}_+]^T$, such that for every economy $(\bar{s}, u, \omega)$ in $[\bar{s}xux\Omega_{(\bar{s},u)}]$, the core $C_{\bar{s}}(u, \omega)$ does not possess the equal treatment property.*

THEOREM 2: *$[U^*]^T$ is open and dense in $[U]^T$.*[5]

Before proceeding with the proofs, we make a few remarks concerning our assumptions.

REMARK 1: Assumptions 1, 2, and 3 are standard. Furthermore, they guarantee the existence of a competitive equilibrium, and hence a nonempty core. The reader should note, however, that Assumption 1 can be relaxed and the proofs modified in the spirit of Section 1 of Smale [**8**].

REMARK 2: Assumption 4 is natural in that if one enlarged the space of economies from $[U \times \mathring{R}^{l+1}_+]^T$ to $[U \times \mathring{R}^{l+1}_+]^\tau$, then for an open dense set of economies no two individuals would be identical.[6] Furthermore, as demonstrated by Green [**4**], the core does possess the equal treatment property if Assumption 4 does not hold. In the case of a large economy where the core equals the set of Walrasian equilibria, the equal treatment property is, of course, satisfied.

REMARK 3: Note that Theorem 1 is precisely Green's principal result in [**4**]. Whereas we assume that u^t has non-zero Gaussian curvature, he assumes that the demand functions are continuously differentiable in both prices and income. Debreu [**3**] has shown that our assumption implies that of Green.

REMARK 4: The assumption of concavity (as opposed to quasi-concavity) of the utility function, u, can be relaxed with no difficulty. It simplifies, however, the proof of Theorem 2.

3. PROOFS

Pick any subeconomy (s, u, ω) of the economy $(\bar{s}, u, \omega)$ and consider the map

$$\phi: \mathring{R}^{T(l+1)}_+ \times \mathring{R}^{\tau(l+1)}_+ \times \mathring{R}^{\tau}_+ \times \mathring{R}^{l}_+ \to R^{\tau(l+1)} \times R^{\tau} \times R^{2l}$$

[4] $[\mathring{R}^{l+1}_+]^T$ is endowed with the product topology.

[5] $[U]^T$ is endowed with the product C^2 topology on compact sets.

[6] A different justification is offered by Green in Appendix A of [**4**].

given by

$$\phi(\omega, x, \lambda, \hat{p}) =$$

$$a^{it} = Du^{it}(x^{it}) - \lambda^{it} p \qquad (i = 1, \ldots, \bar{s}^t;\ t = 1, \ldots, T),$$

$$b^{it} = p\omega^{it} - px^{it} \qquad (i = 1, \ldots, \bar{s}^t;\ t = 1, \ldots, T),$$

$$c = \sum_{t=1}^{T} \sum_{i \in S^t} x^{it} - \sum_{t=1}^{T} s^t \omega^t,$$

$$d = \sum_{t=1}^{T} \sum_{i \in S_c^t} x^{it} - \sum_{t=1}^{T} (\bar{s}^t - s^t)\omega^t,$$

where $p_0 \equiv 1$ and $p = (1, \hat{p})$, $\hat{p} \in \mathring{R}^l_+$.

We now prove the following lemma:

LEMMA 1: $\phi \pitchfork 0$; *i.e.*, ϕ *is transversal to* 0.

PROOF: We associate with each component of $D\phi$ one variable so as to obtain a square submatrix.

To the $(l+1)$ components a_h^{it}, $h = 0, \ldots, l$, we associate $l+1$ of the variables $(x_h^{it}, \lambda^{it}, h = 0, \ldots, l)$. Observe that by differentiating components of the type $Du^{it}(x^{it}) - \lambda^{it} p$ with respect to (x^{it}, λ^{it}) at a point where $Du^{it}(x^{it}) - \lambda^{it} p = 0$, we obtain, up to the positive constant $\lambda^{it} = Du_0^{it}(x^{it})$, the first $(l+1)$ rows of the determinant

$$\begin{vmatrix} D^2 u^{it} & Du \\ Du & 0 \end{vmatrix}.$$

This determinant is equal, up to a positive multiple λ, to the negative of the Gaussian curvature of the indifference hypersurface through x^{it}. Hence, since $u^t \in U^*$, we can choose a subset of the variables $(x_h^{it}, \lambda^{it}, h = 0, \ldots, l)$ to obtain $(l+1)$ linearly independent columns.

The budget constraint b^{it} is differentiated with respect to ω_0^{it}. Finally, the components c and d are differentiated with respect to $(\omega_h^{it}, h = 1, \ldots, l)$ and $(\omega_h^{i't'}, h = 1, \ldots, l)$, respectively, where (i, t) and (i', t') have been chosen so that $\bar{s}_t / s_t \neq \bar{s}_{t'} / s_{t'}$.

We now show that the matrix associated with the above choice of variables is non-singular. First, we notice that the columns corresponding to the variables ω_0^t, $t = 1, \ldots, T$, are unit vectors and can be used through suitable column operations to put zeroes on all lines corresponding to budget constraints. The columns corresponding to the variables $(\omega_h^{it}, h = 1, \ldots, l)$ and $(\omega_h^{i't'}, h = 1, \ldots, l)$ can now be used to put zeroes in the columns corresponding to x^{it} and $x^{i't'}$ at the level of c and d. Hence the submatrix under consideration has full rank (i.e., $\tau(l+1) + \tau + 2l$) if and only if the $2l$ columns corresponding to $(\omega_h^{it}, h = 1, \ldots, l)$ and $(\omega_h^{i't'}, h = 1, \ldots, l)$ are linearly independent. The problem reduces to the question whether the matrix

$$\begin{vmatrix} s^t I & s^{t'} I \\ (\bar{s}^t - s^t)I & (\bar{s}^{t'} - s^{t'})I \end{vmatrix}$$

is of full rank, which is indeed the case if and only if $\bar{s}^t/s^t \neq \bar{s}^{t'}/s^{t'}$.

To complete the proof of the lemma we must show that it is indeed possible to find t and t' such that $\bar{s}^t/s^t \neq \bar{s}^{t'}/s^{t'}$. Suppose this is not the case, i.e., there exist a proper fraction $\bar{q}/q$ with $\bar{q} > q \geq 1$, such that $\bar{s}^t/s^t = \bar{q}/q$ for all $t = 1, \ldots, T$. But then $\bar{s}^t q = s^t \bar{q}$ for $t = 1, \ldots, T$, and hence a contradiction to g.c.d. $\{s^t\} = 1$ by Lemma 3 in Green [**4**]. *Q.E.D.*

We can now prove the following lemma.

PRINCIPAL LEMMA: *For any subeconomy* (s, u, ω) *of* $(\bar{s}, u, \omega)$, *there exists an open dense subset* Ω *of* $[\mathring{R}_+^{l+1}]^T$ *such that for all* $\omega \in \Omega$,

$$E_s(u, \omega) \cap E_{s_c}(u, \omega) = \emptyset.$$

PROOF: By Assumptions 1, 2, and 3, $E_s(u, \omega) \cap E_{s_c}(u, \omega) = \{p = (1, \hat{p}) | p \in \mathring{R}_+^{l+1}$ and $\hat{p} \in \phi^{-1}(0)\}$. From Lemma 1, $\phi \pitchfork 0$; since

$$\phi; \{\mathring{R}_+^{T(l+1)} \times \mathring{R}_+^{\tau(l+1)} \times \mathring{R}_+^{\tau} \times \mathring{R}_+^{l}\} \to \{R^{\tau(l+1)} \times R^{\tau} \times R^{2l}\}$$

there exists an open dense subset Ω of $[\mathring{R}_+^{l+1}]$, such that for $\omega \in \Omega$, $\phi(x, \lambda, p | \omega) \pitchfork 0$. This follows immediately from Thom's transversal density theorem; see [**1**, p. 48]. Finally,

$$\phi(x, \lambda, p | \omega): \{[\mathring{R}_+^{l+1}]^{\tau} \times [\mathring{R}_+]^{\tau} = \mathring{R}_+^{l}\} \to \{[\mathring{R}_+^{l+1}]^{\tau} \times [\mathring{R}_+]^{\tau} = \mathring{R}_+^{2l}\},$$

$\phi(x, \lambda, p | \omega) \pitchfork 0$ implies $\phi_\omega^{-1}(0) = \emptyset$ by a dimension argument. *Q.E.D.*

We can now complete the proof of Theorem 1.

PROOF OF THEOREM 1: By Lemmata 1 and 2 in [**4**], it suffices to demonstrate that there exists an open dense subset Ω of $[\mathring{R}_+^{l+1}]^T$ such that for all subeconomies (s, u, ω) of $(\bar{s}, u, \omega)$, $E_s(u, \omega) \cap E_{s_c}(u, \omega) = \emptyset$. Since the number of possible subeconomies is finite, the proof is completed by appealing to the Principal Lemma. *Q.E.D.*

PROOF OF THEOREM 2: Openness is clear. Density can be proved as follows. Let $u \in [U/U^*]$. We must find $u^* \in U^*$ such that u^* is arbitrarily close to u in the C^2 norm. Let $\hat{u}$ be an element of U^* and let $u^* = (1 - \varepsilon)u + \varepsilon\hat{u}$, $\varepsilon > 0$. Clearly, as $\varepsilon \to 0$, $u^* \xrightarrow[\|\ \|_2]{} u$. We must show that $u^* \in U^*$ for any $\varepsilon > 0$. The only property that has to be checked is that for any $x \in K$, the Gaussian curvature of the u^*-indifference hypersurface through x does not vanish; i.e., that the matrix

$$\begin{vmatrix} D^2u^* & Du^* \\ Du^* & 0 \end{vmatrix}$$

is strictly negative definite everywhere on R_+^{l+1}. Let Y be an $(l+2)$ vector. Then

$$y'\begin{pmatrix} D^2u^* & Du^* \\ Du^* & 0 \end{pmatrix}y = y'\left[(1-\varepsilon)^{(l+1)}\begin{pmatrix} D^2u & Du \\ Du & 0 \end{pmatrix} + \varepsilon^{(l+1)}\begin{pmatrix} D^2\hat{u} & D\hat{u} \\ D\hat{u} & 0 \end{pmatrix}\right]y$$
$$= \left\{(1-\varepsilon)^{(l+1)}y'\begin{pmatrix} D^2u & Du \\ Du & 0 \end{pmatrix}y\right\} + \left\{\varepsilon^{(l+1)}y'\begin{pmatrix} D^2\hat{u} & D\hat{u} \\ D\hat{u} & 0 \end{pmatrix}y\right\}.$$

Since u is strictly quasi concave, the first term is non-positive. Since $\hat{u}$ is strictly quasi concave and its hypersurfaces have a nowhere vanishing Gaussian curvature, the second term is strictly negative. Hence the sum is strictly negative, which implies $u^* \in U^*$. *Q.E.D.*

4. FURTHER REMARKS

The technique we employed in proving Theorem 1 is similar to the technique in Laroque and Polemarchakis [**6**] and can be adopted in a straightforward way to derive the results in Debreu [**2**]. One simply needs to replace components c and d of the function ϕ by a single component of the form $\Sigma_i\Sigma_t x^{it} - \Sigma_i\Sigma_t\omega^{it}$ and employ an identical argument to prove transversality at the origin. Furthermore, our methods show clearly the connection between the results in Debreu [**2**] with those in Smale [**7, 8**]. Our proof of Theorem 2 yields a result economically more relevant than Smale's Proposition (4) in [**7**]. Finiteness and continuous behavior (with respect to $[u, \omega]$) of the equilibrium set are derived for a set of economies which is open and dense in the space of utilities and endowments which do satisfy assumptions 1, 2, and 3, i.e., the space of economically meaningful environments.

Finally, note that in [**5**], Green has presented a process which converges to a point in the core with probability one in a finite number of iterations provided that *strong super additivity*[7] holds for at least one point in the core. Note that our Principal Lemma generalizes the applicability of Green's process to a larger class of economies.

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[7] See [**4, 5**] for a definition.

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