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A THEOREM ON THE IDENTIFIABILITY OF THE VON NEUMANN–MORGENSTERN UTILITY FUNCTION FROM ASSET DEMANDS *

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If the demand for risky assets is determined by the maximization of an analytic von Neumann–Morgenstern utility function, and if these demands are known as a function of the assets' prices, then this utility function can be constructed without ambiguity.

1. Introduction

In the theory of demand, revealed preferences may be used to reconstruct a unique transitive, complete, preference relation under very mild conditions of regularity. Models of asset demands under uncertainty differ from ordinary demand models in two respects. The assets' returns typically do not span the space of returns over states of nature; and the axioms of von Neumann–Morgenstern utility may be used to restrict the nature of preferences over alternative portfolios beyond the continuity and convexity postulates that are common to demand theories for consumption. Therefore, even though preferences may not be observed over the whole space, they may be inferred from the available observations because of these restrictions.

We give conditions under which this program can be carried out. That is, postulating that demand correspondences for assets are generated by expected utility maximization for some u, we give conditions on u and on the assets' random returns,

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such that no von Neumann–Morgenstern utility function v, different from u (and not just a positive linear transformation of u) can generate these demands.

2. Model

The investor's problem is

$$\max_{x \in X} Eu(r \cdot x) \quad \text{subject to} \quad p \cdot x \leqslant 1. \tag{1}$$

Prices, p, are assumed to vary over $\mathbf{R}_+^m \backslash \{0\}$. Demands, x, must lie in the subset of $\mathbf{R}^m$ for which Eu is well defined. Since u: $[0, \infty) \to \mathbf{R}$, this means that $X = \{x \in \mathbf{R}^m \mid \text{prob}(r \cdot x \geqslant 0) = 1\}$. Thus X cannot be defined independent of the distribution of the assets' returns, r, in $\mathbf{R}^m$.

We assume:

r. (i) r is non-negative with probability one.
r. (ii) r_j is not zero with probability one for any $j = 1, ..., m$.
r. (iii) for each j, r_j cannot be written as a linear combination of $\{r_k\}_{k \neq j}$, with probability one.
r. (iv) for each j, and each positive integer l, $Er_j^l < \infty$.

Therefore $\mathbf{R}_+^m \subseteq X$.

With respect to u we assume:

u. (i) u is defined over the domain of all non-negative real numbers.
u. (ii) u is increasing.
u. (iii) u is concave.
u. (iv) u is analytic.

3. Results

These conditions allow us to derive a demand correspondence ξ_u: $P \to \mathbf{R}^m$ where P is the subspace of $\mathbf{R}_+^m \backslash \{0\}$ for which (1) is well-defined.

One can prove that ξ_u satisfies the following properties:

(i) P is non-empty.
(ii) The range of ξ_u includes $\mathbf{R}_+^m \backslash \{0\}$.
(iii) For each $x \in \mathbf{R}_+^m \backslash \{0\}$, there exists a unique $p \in \mathbf{R}_+^m \backslash \{0\}$ such that $x \in \xi_u(p)$.

Let $s_{jk}(x)$ be the marginal rate of substitution between assets j and k, when the choice of a portfolio is $x \in X$,

$$s_{jk}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}r_j} Eu(r \cdot x)\right) \Big/ \left(\frac{\mathrm{d}}{\mathrm{d}x_k} Eu(r \cdot x)\right).$$

We can prove that:

(iv) Knowledge of the demand function is therefore sufficient to derive $s_{jk}(x)$ for all j, k, at every $x \in \mathbf{R}_+^m \backslash \{0\}$.

(v) The value of $s_{jk}(0)$ is independent of u.

(vi) All derivatives of $s_{jk}(\cdot)$ exist at $x = 0$ and can be computed from the derivatives of u and the moments of r.

4. Theorem

Let u and v satisfy the assumptions u.(i)–u.(iv) made on u above and let r satisfy r.(i)–r.(iv). If $\xi_v = \xi_u$, then v is a positive linear transformation of u.

5. Method of proof

If $\xi_v = \xi_u$ then $s_{jk}(x)$ computed from v must be the same as that computed from u. In particular, the derivatives of s_{21} at $x = 0$ with respect to x_1 and x_2 must be equal. By identifying the expressions derived in (vi) we can recover the value of $v^{(l)}(0)$ for all l, provided that either

$$Er_1^l - Er_1^{l-1} r_2 \neq 0 \quad \text{or} \quad Er_2^l - Er_2^{l-1} r_1 \neq 0 \,.$$

The fact that for each l at least one of these relations must hold is proven using r.(i)–r.(iv) and applying Hölder's inequality repeatedly.

6. Implications

This permits the derivation of the von Neumann–Morgenstern utility from data on asset demands. If the distribution of r were to change, due perhaps to the introduction of new assets, changing physical uncertainties, changing tax provisions or changes in the behavior of other agents (e.g. firms' supply of assets), the computed u could be used to generate the new expected utility function. Hence both predictive statements about portfolio behavior and welfare conclusions could be made without the need to re-estimate demand functions in the new regime.

7. Open questions

(1) We do not know if the requirement of analyticity of u can be dropped.

(2) Our method requires observations at all $p \in \mathbf{R}_+^m \backslash \{0\}$ for which (1) is well-defined. If p is allowed to range only over a bounded set, our method could not be

employed directly. Extrapolations would be required to implement it, and their uniqueness has not been proven.

(3) When r can take only finitely many values, the probability of various events, viewed here as objective, may also be derivable from demands. This would allow simultaneous identification of utility and subjective probability.