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MORE ON ECONOMIES WITH A FINITE SET OF WALRASIAN EQUILIBRIA

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In the context of a pure exchange economy, I allow explicitly for boundary equilibria, and I demonstrate the following:

Proposition. *The graph of the Walrasian equilibrium correspondence is a piecewise continuously differentiable manifold. Furthermore, there exists an open dense set of economies, Ω, such that for all w in Ω (a) the number of equilibria of the economy w is finite; and (b) there exists a neighborhood V(w) on which the set of equilibria is represented by a finite family of piecewise continuously differentiable functions.*

1. Introduction

Debreu (1970) addressed the problem of local uniqueness of the Walrasian equilibria for a pure exchange economy, while Smale (1974a, b) extended the results to allow for boundary equilibria, for variations in the utility functions of agents as well as in their endowments, and for the introduction of production possibilities. The method employed by Smale is to enlarge the set of equilibria by considering only the first-order necessary conditions for an optimum, and then argue that the local uniqueness and continuous behavior demonstrated for this extended set carry over to the set of genuine equilibria. As far as local uniqueness is concerned the point is clear; continuity and manifold structure, however, require further analysis. Furthermore, Smale is forced to exclude equilibria where a change of regime occurs, i.e., equilibria where the weak inequality first-order conditions for a boundary equilibrium are satisfied with equality.

The purpose of this paper is to present an alternative approach to the problem. Unlike Smale, I shall take explicitly into consideration the necessary and sufficient Kuhn–Tucker conditions for an optimum, while still allowing for boundary equilibria. I shall demonstrate the local uniqueness and continuous behavior of the equilibrium set, and I shall derive its manifold structure.

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2. The model and results

Consider an exchange economy with l commodities indexed by h and m agents indexed by i. An agent is characterized by its consumption set X^i, a subset of R^l, its utility function u^i defined on X^i, and its vector of initial endowments w^i in X^i. A price system is a vector $p = (p_1, ..., p_l)$ in the set $S = \{p \in R^l | \Sigma_{h=1}^l (p_h)^2 = 1, p_h \geqslant 0, h = 1, ..., l\}$. An allocation is a vector $x = (x^1, ..., x^m)$ in $\Pi_{i=1}^m X^i$. The following assumptions will be made:

Assumption 1. For all i, $X^i = X$, the closure of the positive orthant in R^l, and w^i belongs to the interior of X, $\overset{\circ}{X}$.

Assumption 2. For all i, the utility function u^i is a strictly quasi-concave and twice continuously differentiable function from X to R. For every $\alpha \subset \{1, ..., l\}$, and for every $x \in [\alpha]$, the coordinate subspace of X defined by α, $\{x \in X | x_h = 0$ for all h not in $\alpha\}$, the indifference hypersurface through x^i of $u^i|_{[\alpha]}$ has non-zero Gaussian curvature at x^i. Furthermore, $Du^i|_{[\alpha]}(x^i) \neq 0$ and $Du^i_j|_{[\alpha]}(x^i) > 0$ for some $j \in \alpha$. An economy consists of a specification of the endowment vector $w = (w^1, ..., w^m)$; it is a point in $(\overset{\circ}{X})^m$. A Walrasian equilibrium for the economy w is a price vector p in S and an allocation x in $(X)^m$ such that, for each commodity, h, either excess demand, $\Sigma_{i=1}^m (x^i_h - w^i_h)$, equals zero, or excess demand is negative and the price, p_h, is equal to zero. The following can now be demonstrated:

Proposition. *The graph of the Walrasian equilibrium correspondence is a piecewise continuously differentiable manifold of dimension (lm). Furthermore, there exists an open dense set of economies, Ω, in $(\overset{\circ}{X})^m$ such that for all w in Ω (a) the number of equilibria of the economy w is finite; and (b) there exists a neighborhood $V(w)$ on which the set of equilibria is represented by a finite family of piecewise continuously differentiable functions.*

3. Proofs

Let $(H, P) = \{H(i), i = 1, ..., m; P\}$ be a family of sets where $H(i)$ is a subset of the set $N_l = \{1, ..., l\}$ and P is a proper subset of N_l. Let $E(H, P)$ be the set of (p, w, x) in $S \times (\overset{\circ}{X} \times X)^m$ such that (p, x) is an equilibrium for the economy w with $x^i_h = 0$ for all $h \in H(i)$ and $p_h = 0$ for all $h \in P$. Associate with (H, P) the map $F(H, P)$ defined by its components (a, b, c) as follows:

$$\begin{aligned}
&(1) \quad a^i_h = Du^i_h(x^i) - \lambda^i p_h, \qquad h = 1, ..., l, \qquad i = 1, ..., m,\\
&(2) \quad b^i = p^t(x^i - w^i), \qquad i = 1, ..., m,\\
&(3) \quad c_h = \sum_{i=1}^{m} (x^i_h - w^i_h), \; h \in N_l / \{h | h \in N_l \text{ and for } h' \notin P, h' > h\},
\end{aligned} \tag{1}$$

where λ^i, $i = 1, ..., m$, are strictly positive real numbers. Furthermore, let $F(H, P)$ be the piecewise smooth submanifold of $R^{m(l+1)} \times R^{l-1}$ defined by the conditions

$$\begin{aligned}
&(1) \quad a_h^i = 0 \quad \text{for} \quad h \in \bar{H}(i), \\
&\qquad\quad \leqslant 0 \quad \text{for} \quad h \in H(i), \qquad i = 1, ..., m, \\
&(2) \quad b^i = 0, \qquad i = 1, ..., m, \\
&(3) \quad c_h = 0 \quad \text{for} \quad h \in \bar{P}/h(P), \\
&\qquad\quad \leqslant 0 \quad \text{for} \quad h \in P,
\end{aligned} \tag{2}$$

where $\bar{H}(i) = N_l/H(i)$, $\bar{P} = N_l/P$, and $h(p) = \{h | h \in N_l$ and for $h' \notin P, h' > h\}$. The proof of the proposition can now be outlined in the following sequence of lemmata:

Lemma 1. *The projection of the space* $[(\text{Proj}_{\bar{P}}\ S) \overset{i=1,...,m}{\times} (\mathring{X} \times \mathring{R}_+ \times \text{Proj}_{\bar{H}(i)}\ X)]$ *on the space* $[(\text{Proj}_{\bar{P}}\ S) \overset{i=1,...,m}{\times} (\mathring{X} \times \text{Proj}_{\bar{H}(i)}\ X)]$ *of prices, endowments and consumption bundles, restricted to* $F^{-1}_{(H,P)}(F_{(H,P)})$ *is a diffeomorphism* $G_{(H,P)}$ *of* $F^{-1}_{(H,P)}(F_{(H,P)})$ *onto* $E_{(H,P)}$.

Lemma 2. *The map* $F_{(H,P)}$ *is transverse to the submanifold* $F_{(H,P)}$. *Furthermore, the map* $F_{(H,P)}$ *is transverse to the substrata of* $F_{(H,P)}$.

Lemma 3. *There exists an open dense set* $\Omega_{(H,P)}$ *in* $(\mathring{X})^m$ *such that for all* $w \in \Omega_{(H,P)}$ *the map* $F_{(H,P)}$ $(w;\ \ )$ *is transverse to the submanifold* $F_{(H,P)}$, *as well as to the substrata of* $F_{(H,P)}$.

Lemma 4. $E_{(H,P)}$ *is a continuously differentiable manifold with corners of dimension m. More precisely, let* $(p, (w^i, x^i), i = 1, ..., m))$ *be a point in* $E_{(H,P)}$ *and let*

$$r = [\#\{h | p_h = 0 \text{ and } \sum_{i=1}^{m} (x_h^i - w_h^i) = 0\} + \sum_{i=1}^{m} \#\{h | x_h^i = 0$$

$$\text{and } Du^i(x^i) - \lambda^i p_h = 0\}] \ .$$

Then a neighborhood of $(p, (w^i, x^i), i = 1, ..., m))$ *in* $E_{(H,P)}$ *is diffeomorphic to a neighborhood of the origin in* $R^{lm-r} \times R^r$.

To complete the proof of the first part of the proposition, we have to glue together the various $E_{(H,P)}$. If at $(p, (w^i, x^i), i, ..., m))$ $r = \bar{r}$, then $(p, (w^i, x^i), i = 1, ..., m))$ belongs to the boundary of $2^{\bar{r}}$ manifolds with corners described in Lemma 4. Hence, there is the right number of orthants to yield $R^{\bar{r}}$. To construct the desired homeomorphism between a neighborhood of $(p, (w^i, x^i), i = 1, ..., m))$ in $E = \bigcup_{(H,P)} E_{(H,P)}$ and a neighborhood of the origin in R^{lm}, we only have to modify the $2^{\bar{r}}$ homeomorphisms so that they agree on the boundaries of the manifolds. This is done by applying the following lemma:

Lemma 5. *Let A be a subset of a finite dimensional vector space, and suppose there exists a homeomorphism ϕ between a neighborhood of a point a in A and a neighborhood of the origin in $R^s \times R^t_-$. Let ∂A be the inverse image under ϕ of $R^s \times \partial R^t_-$, and let $\partial\psi$ be a given homeomorphism between a neighborhood of a in ∂A and a neighborhood of the origin in $R^s \times \partial R^t_-$. Then $\partial\psi$ can be extended to a homeomorphism ψ between a neighborhood of a in A and $R^s \times R^t_-$.*

To complete the proof of the second part of the proposition we observe that by Lemma 3, for each (H, P), there exists $\Omega_{(H,P)}$ such that for all $w \in \Omega_{(H,P)}$ the map $F_{(H,P)}$ is transverse to $F_{(H,P)}$. Since transversal maps preserve codimension, for all $w \in \Omega_{(H,P)}$, $E_{(H,P)}(w)$ is finite and is described locally by a finite family of continuously differentiable functions. Taking the finite union $\bigcup_{(H,P)} \Omega_{(H,P)}$ gives the desired result.

4. Extensions

It is straightforward to demonstrate that the assumptions made on consumer preferences are characteristic of a dense open subset. Furthermore, the results demonstrated in the previous section hold for economies with production, and the method of proof as well as the set of assumptions remain essentially the same.

References

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