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ON THE STRUCTURE OF THE SET OF FIXED PRICE EQUILIBRIA*

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Working in the framework suggested by Drèze, this paper studies the number of fixed price equilibria and their continuity with respect to the price system. In an exchange economy, the concept of a rationing scheme is introduced, which specifies how shortages are shared among agents. For given utility functions and a given rationing scheme, under standard assumptions, an existence theorem is recalled, and it is shown that the graph of the equilibrium correspondence, when prices and initial endowments vary, is a piecewise continuously differentiable manifold. Moreover, generically, the number of equilibria for an economy, at given prices, is finite and the set of equilibria varies continuously with the price system and the initial endowments.

A number of recent works have addressed the problem of describing the allocation of resources in an economy when relative prices are fixed [Benassy (1975), Drèze (1975), Hahn (1976), Heller–Starr (1976), Malinvaud–Younès (1975), and Younès (1975)]. While the precise relationships between the three basic approaches of Benassy, Drèze, Younès and Malinvaud–Younès, respectively, have not yet been established [see, however, Grandmont (1977) for a comparison of the models of Benassy and Drèze], the basic conceptual difficulties seem to have been solved in a satisfactory manner, and, by and large, agreement has been reached as to the properties that an equilibrium allocation must possess when prices are fixed. These models open up interesting possibilities as to the microeconomic foundations of macroeconomics [Grandmont–Laroque (1976), Malinvaud (1977)] and the dynamics of price adjustments [Veendorp (1975)].

The above studies are essentially concerned with the existence of a fixed-price equilibrium. Our paper, working in the framework suggested by Drèze (1975), studies the number of these equilibria and their continuity with respect to the price system. Our approach is in the tradition of Debreu (1970, Dierker (1974) and Smale (1974a, b). In an exchange economy, we introduce the concept of a rationing scheme which specifies how shortages are shared among the agents. For given utility functions and a given rationing scheme, under standard assum-

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ptions, we recall an existence theorem and we show that the graph of the equilibrium correspondence, when prices and initial endowments vary, is a piecewise continuously differentiable manifold. Moreover, generically, the number of equilibria for an economy, at given prices, is finite and the set of equilibria varies continuously with the price system and the initial endowments.

The description of the model, the statement of the theorems, and examples are presented in the first section. The proofs are given in the second section.

1. Model and results

We consider an exchange economy with $(l+1)$ commodities indexed by an index h $(h=0,1,\dots,l)$, and m agents indexed by an index i $(i=1,\dots,m)$. An agent i is characterized by a consumption set X_i, a convex subset of R^{l+1}, a quasiconcave utility function u_i defined on X_i, and a vector of initial endowment ω_i in X_i. The net trade of agent i is a vector z_i in R^{l+1}, which is equal to the difference between his consumption bundle x_i, an element of X_i, and his initial endowment ω_i. A price system is a vector $p=(p_0,p_1,\dots,p_l)$ in the set $P=\{p\in R^{l+1}, p_0=1, p_h>0 \text{ for } h=1,\dots,l\}$. In addition to the price system, a trader perceives for each commodity h, other than commodity 0, quantitative constraints, $\underline{z}_{ih}\leqq 0$ and $\bar{z}_{ih}\geqq 0$, that set lower and upper bounds on the amount of commodity h he can trade. Given a price system p and constraints $(\underline{z}_i,\bar{z}_i)$ agent i chooses a consumption bundle x_i in X_i that maximizes his utility function subject to the budget constraints $p\cdot x=p\cdot\omega_i$ and the quantitative constraints $\underline{z}_{ih}\leqq x_h-\omega_{ih}\leqq\bar{z}_{ih}$ for $h=1,\dots,l$. For the price system p and the constraints $(\underline{z}_i,\bar{z}_i)$ agent i is said to be *constrained in market* k, if there exists a consumption bundle x in X_i satisfying the budget constraint $p\cdot x=p\cdot\omega_i$ and the quantitative constraints $\underline{z}_{ih}\leqq x_h-\omega_{ih}\leqq\bar{z}_{ih}$ for all h different from 0 and k, which gives a higher utility than $x_i: u_i(x)>u_i(x_i)$.

For a given economy, following Drèze (1975) and Grandmont (1977), we introduce the following definition:

Definition 1.1. An equilibrium with quantity rationing at price p is a set of consumption bundles $x=(x_1,\dots,x_m)$ such that there exist $\underline{z}_{ih}\leqq 0$, $\bar{z}_{ih}\geqq 0$, $i=1,\dots,m$, $h=1,\dots,l$ satisfying the following conditions:

(a) $\sum_{i=1}^{m}(x_i-\omega_i)=0$;

(b) x_i maximizes u_i on $\{x\in X_i,\ p\cdot x=p\cdot\omega_i,\ \underline{z}_{ih}\leqq x_h-\omega_{ih}\leqq\bar{z}_{ih}, \text{ for } h=1,\dots,l\}$;

(c) For $h=1,\dots,l$:

$x_{ih}-\omega_{ih}=\bar{z}_{ih}$ for some i implies $x_{jh}-\omega_{jh}>\underline{z}_{jh}$ for $j=1,\dots,m$,
$x_{ih}-\omega_{ih}=\underline{z}_{ih}$ for some i implies $x_{jh}-\omega_{jh}<\bar{z}_{jh}$ for $j=1,\dots,m$.

We recall the following features of an equilibrium according to our definition. First, the fact that there are no constraints associated with commodity 0 combined with condition (c) rules out, in general, the trivial no-trade allocation.

Second, condition (c), associated with the quasi-concavity of the utility functions, implies that all the agents who are constrained in commodity h belong to the same side of the market, they are either suppliers or demanders; that is, all their net trades in commodity h have the same sign. We shall say that a market $h, h \neq 0$, is *cleared* when no individual is constrained in this particular market.

The foregoing definition of equilibrium does not specify how shortages are distributed among agents. Consequently, we may expect that there will be, in general, a continuum of equilibria. As an example, take the case where $l=1$. Let x_i^* be the most preferred bundle by agent i in the set $\{x_i \in X_i,\ p \cdot x_i = p \cdot \omega_i\}$. Suppose $\sum_{i=1}^m (x_{i1}^* - \omega_{i1}) > 0$. Then the set of equilibria with quantity rationing at price p is the set of consumption bundles (x_i) such that $\sum_{i=1}^m (x_i - \omega_i) = 0$, $p \cdot x_i = 0$ for all $i = 1, \ldots, m$, $x_{i1} = x_{i1}^*$ for all i such that $x_{i1}^* - \omega_{i1} \leqq 0$ and $\omega_{i1} \leqq x_{i1} \leqq x_{i1}^*$ for all i such that $x_{i1}^* - \omega_{i1} > 0$. Thus, except for competitive prices, the economy possesses a continuum of equilibria with quantity rationing at price p. To get a specific theory, we must select one of these equilibria: this is the reason for introducing the concept of a rationing scheme. For each possible set of constrained agents and for each amount of commodity available to them in the process of exchange, the rationing scheme must specify how this amount of commodity is to be shared among the constrained agents, as well as the maximum amount each of the unconstrained agents is entitled to. However, a general formulation of this type would in general display some undesirable features for our purpose: suppose a group of agents C is constrained, sharing quantity z^r where r is a positive integer, and z^r tends to z as r goes to infinity. Suppose furthermore that, when the quantity to share is equal to z, an agent in C, say i, is unconstrained, so that, at point z, $C - \{i\}$ has to share $z - z_i$, the limit of $z^r - z_i^r$. Then it is necessary for our analysis that the distribution of $(z - z_i)$ among $C - \{i\}$ is equal to the limit of the distribution obtained with z^r when i was constrained. This requirement leads to the following formalization: a *rationing scheme* f is a set of $2lm$ functions $\bar{f}_{ih}(\bar{q}_h)$ and $\underline{f}_{ih}(\underline{q}_h)$ for $i = 1, \ldots, m$, and $h = 1, \ldots, l$, mapping R_+ into R_+ and R_- into R_-, respectively, which describes the allocation of resources on each market $h = 1, \ldots, l$, in the case of excess demand and of excess supply, respectively.[2] We describe now briefly well-known examples and show how they are formalized in our framework.

[1] R_+ and R_- stand, respectively, for the positive and negative half line, origin included.

[2] Note that the concept of rationing scheme thus defined depends only on the set of agents and commodities present in the economy. More formally, a general rationing scheme on a given market would be described by a set of m functions $f_i(C, z)$ defined on $\mathcal{M} \times R_+$, where $\mathcal{M}$ is the set of subsets of $M = \{1, \ldots, m\}$, where C is the set of constrained agents and z the quantity they share between themselves. Thus $f_i(C, z)$ is the quantity received by agent i for i in C and the maximum quantity available to him for i not in C. Consequently,

$$\sum_{i \in C} f_i(C, z) = z .$$

If we ask the condition that the distribution of commodity is continuous when the set of constrained agents varies (as described above), we get, considering the situation where all the agents not in C

Examples 1.2. The *uniform* rationing scheme on market h obtains when all agents are equally treated, i.e.,

$$\bar{f}_{ih}(\bar{q}_h)=\bar{q}_h/m, \quad \underline{f}_{ih}(\underline{q}_h)=\underline{q}_h/m, \qquad i=1,\dots,m.$$

Rationing can also follow from some *queueing* process. Let $\succ_h$ be an order relation defined on the set of agents $\{1,\dots,m\}$. This order specifies which agent should be served first, who second, and so on. It is not possible to identify this queueing process with a function of f independent of the economy. However, for all the economies such that $\sum_{i=1}^m \omega_{ih}<c$, where c is a positive constant, the equilibrium allocation will satisfy the queueing process if it is an equilibrium relative to the following function f: let i be the agent whose rank according to the order $\succ_h$ is r; then

$$\bar{f}_{ih}(\bar{q}_h)=0, \qquad \underline{f}_{ij}(\underline{q}_h)=0, \qquad \text{for} \quad |q_h|\leqq rc,$$

$$\bar{f}_{ih}(\bar{q}_h)=\bar{q}_h-rc, \quad \underline{f}_{ih}(\underline{q}_h)=\underline{q}_h+rc, \quad \text{for} \quad rc<|q_h|.$$

However, it may be worth noting at this stage that the results obtained in this paper do not apply to the queueing process, since the functions f are not strictly increasing.

Given an economy, we are now led to the following definition:

Definition 1.3. An equilibrium with quantity rationing at price p relative to the rationing scheme f is a set of consumption bundles $x=(x_1,\dots,x_m)$ such that there exist $\underline{z}_{ih}\leqq 0$, $\bar{z}_{ih}\geqq 0$, $\bar{q}_h\geqq 0$, $\underline{q}_h\leqq 0$ for $i=1,\dots,m$, $h=1,\dots,l$, satisfying the following conditions:

(a) $\sum_{i=1}^m (x_i-\omega_i)=0$;

(b) x_i maximizes u_i in $\{x\in X_i,\ p\cdot x=p\cdot\omega_i,\ \underline{z}_{ih}\leqq x_h=\omega_{ih}\leqq \bar{z}_{ih} \text{ for } h=1,\dots,l\}$;

(c) For $h=1,\dots,l$:
$x_{ih}-\omega_{ih}=\bar{z}_{ih}$ for some i implies $x_{jh}-\omega_{jh}>\underline{z}_{jh}$ for $j=1,\dots,m$,
$x_{ih}-\omega_{ih}=\underline{z}_{ih}$ for some i implies $x_{jh}-\omega_{jh}<\bar{z}_{jh}$ for $j=1,\dots,m$;

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become constrained,

$$f_i(C,z)=f_i\left(M,z+\sum_{j\notin C} f_j(C,z)\right), \qquad i=1,\dots,m.$$

Thus our basic concept reduces to the functions $f_i(M,\cdot)$. Now under the additional assumption that the functions $f_i(M,\cdot)$ are monotone increasing, that $f_i(M,0)=0$, and that $f_i(M,q)$ goes to infinity with q for all i, the data of the functions $f_i(M,\cdot)$ determine unambiguously the underlying general rationing scheme. There is a unique solution $q(C,z)$ to the equation $z=\sum_{i\in C} f_i(M,q)$, and hence $f_i(C,z)=f_i(M,q(C,z))$.

In the paper, we will deal only with the functions $f_i(M,\cdot)$. The preceding interpretation suggests that they satisfy the identity $\sum_{i=1}^m f_i(M,q)=q$. We shall not, however, make this assumption explicit, since we do not use it in our argument.

(d) For $h=1,\ldots,l$:

$x_{ih}-\omega_{ih}=\bar{z}_{ih}$ for some i implies $\bar{z}_{jh}=\bar{f}_{jh}(\bar{q}_h)$ for $j=1,\ldots,m$,

$x_{ih}-\omega_{ih}=\underline{z}_{ih}$ for some i implies $\underline{z}_{jh}=\underline{f}_{jh}(\underline{q}_h)$ for $j=1,\ldots,m$.

For the rest of the paper, we hold fixed the consumption sets and the utility functions of the agents as well as the rationing scheme. Thus an economy is represented by a vector of initial endowments $\omega=(\omega_1,\ldots,\omega_m)$. For short, we shall use the term equilibrium to refer to an equilibrium with quantity rationing relative to the rationing scheme f. For a given economy ω at a price p, the equilibrium correspondence is the set of all equilibria for the economy ω at price p. Our main purpose is to study the graph of the equilibrium correspondence when the endowments and the prices vary. We shall use the following assumptions throughout:

Assumption 1. For all i ($i=1,\ldots,m$), X_i is equal to X, the interior of the positive orthant of R^{l+1}: $X=\{x\in R^{l+1}, x_h>0 \text{ for } h=0,1,\ldots,l\}$. The utility function u_i is a strictly increasing, strictly quasi-concave, twice continuously differentiable function from X into R. For every x_i in X, the indifference hypersurface through x_i, $\{x\in X, u_i(x)=u_i(x_i)\}$, has a non-zero Gaussian curvature at x_i, and its closure relative to R^{l+1} is contained in X.

Assumption 2. For all i and for all h, $i=1,\ldots,m$, $h=1,\ldots,l$, the functions $\bar{f}_{ih}$ and $\underline{f}_{ih}$ map, respectively, R_+ into R_+ and R_- into R_-. They are strictly increasing and continuously differentiable.[3] Moreover, $\bar{f}_{ih}(0)=\underline{f}_{ih}(0)=0$, $\bar{f}_{ih}(\bar{q}_h)$ tends to $+\infty$ with $\bar{q}_h$, $\underline{f}_{ih}(\underline{q}_h)$ tends to $-\infty$ with $\underline{q}_h$, for $i=1,\ldots,m$, $h=1,\ldots,l$.

As already noted, Assumption 2 rules out the queueing process. Removing the condition of strict monotonicity should be the subject of further research.

We first recall an existence theorem which is a well-known result [see Drèze (1975) or Grandmont (1977)]:

Theorem 1.4. Let p be a price system in P and ω_i be a vector of initial endowments in X for all i ($i=1,\ldots,m$). Then there exists an equilibrium at price p for the economy ω. Moreover, the graph of the equilibrium correspondence [i.e., the set of $(p,(\omega_i),(x_i))$ where (x_i) is an equilibrium at price p for the economy ω] is closed in $P\times X^m\times X^m$.

The behavior of consumer i, as defined by (b) in Definitions 1.1 or 1.3, implies under our assumptions that his most preferred bundle (i.e., its demand) is a continuous function of the price system p, the quantity constraints $(\underline{z}_i,\bar{z}_i)$ and the initial endowment ω_i, but is most likely not to be differentiable at points where the

[3]We recall that a function f defined on a closed subset S of R^n is said to be r times continuously differentiable if there is an open neighborhood $\mathcal{O}$ of S and a r times continuously differentiable extension of f on $\mathcal{O}$.

consumer switches from being constrained to being unconstrained on some market. This remark shows that the graph of the equilibrium correspondence has 'kinks', and this will generally happen not only for a value of the price system at which some market is cleared (in particular, at a competitive price system[4]), but also at points where some individual demand changes regime while the aggregate market structure stays unchanged. We are thus led to the following theorem:

Theorem 1.5. The graph of the equilibrium correspondence in $P \times X^m \times X^m$ *is a piecewise continuously differentiable manifold of dimension* $m(l+1)+l$. *Moreover, if the utility functions* u_i, $i=1,\dots,m$, *are* $(l+2)$ *times continuously differentiable and the rationing scheme* $\bar{f}_{ih}$, $\underline{f}_{ih}$, $i=1,\dots,m, h=1,\dots,l$, *is* $(l+1)$ *times continuously differentiable, then, except for a closed set of economies* ω *of Lebesgue measure zero in* X^m, *the graph of the equilibrium correspondence for the economy* ω *in* $P \times X^m$ *is a piecewise continuously differentiable manifold of dimension* l.[5]

Finally, we study how the equilibrium correspondence varies with the price system and the economy:

Theorem 1.6. There exists an open set of prices and economies (p,ω) *of full measure in* $P \times X^m$ *where the following properties hold:*

(a) *For every equilibrium* $x=(x_1,\dots,x_m)$ *of the economy* ω *at price* p, *the vector* (p,ω,x) *lies in a continuously differentiable region of the manifold of equilibria;*
(b) *The number of equilibria of the economy* ω *at price* p *is finite, and there exists a neighborhood of* (p,ω) *in* $P \times X^m$ *where the set of equilibria is represented by a finite number of continuously differentiable functions of* (p,ω).

This last result, unfortunately, does not say anything on the equilibria where a change of regime occurs. These equilibria are, of course, the most interesting ones from an economic point of view and they should be the subject of further research.[6]

To complete this set of results, we give now some properties of some particular

[4]The phenomenon can be visualized clearly in an Edgeworth box ($l=1, m=2$) where the two agents have identical preferences $u(x)=\log x_0+\log x_1$ and initial endowments $\omega_1=(1,0)$, $\omega_2=(0,1)$. The representation of the fixed price equilibria exhibits a right-angle corner at the competitive price system $p=(1,1)$.

[5]A piecewise continuously differentiable manifold is a continuous manifold which is a finite union of continuously differentiable manifolds (some possibly of lower dimensions).

[6]Since there is no differentiability at those points, not very much can be hoped. We can, however, state the following conjecture. We recall that a price system p is competitive for the economy ω if and only if there exists a fixed price equilibrium x at price p for the economy ω with $\bar{z}_{ih}=+\infty$ and $\underline{z}_{ih}=-\infty$ for all i and h. Then we can expect the following to be true.

Conjecture: Except for a closed set of economies and prices (ω,p) of measure zero in the manifold of competitive prices, given a competitive equilibrium (x,p), there exist 2^l subsets of $X^m \times P$ locally homeomorphic at (x,p) to a neighborhood of the origin in $R^{m(l+1)} \times R^l_+$ and 2^l continuously differentiable functions defined on these subsets which describe completely the set of equilibria in a neighborhood of the competitive equilibrium allocation.

economies, and an example of an economy which exhibits a continuum of fixed price equilibria for some price system.

First, it may be worth noting that for $l=1$, all the economies have a unique fixed price equilibrium for all prices. It is obtained by applying the rationing scheme to distribute either excess demand or excess supply.

Second, when the initial endowment is a Pareto-optimal allocation (that is, it is impossible to make someone better off without making someone worse off by redistributing resources), the fixed price equilibrium is independent of the price and equal to the initial endowment. This comes from the fact that, at a fixed price equilibrium, everyone is at least as well off as in the initial situation.

Finally, we describe an economy with three commodities, $l=2$, and two agents, $m=2$, which exhibits a continuum of fixed price equilibria. The utility functions satisfy all our assumptions, except that the closure of the indifference hypersurfaces is not contained in X. This does not alter the validity of the example, since everything will take place in a compact set inside X, and the result still holds if the utility functions are changed outside this compact set. The characteristics of the agents are as follows:

$$u_1(x_1)=4(x_{10})^{1/2}+4(x_{11})^{1/2}+x_{12}, \qquad \omega_1=(4,\omega_{11},\omega_{12}),$$

$$u_2(x_2)=4(x_{20})^{1/2}+x_{21}+4(x_{22})^{1/2}, \qquad \omega_2=(4,\omega_{21},\omega_{22}).$$

Suppose that ω_{11} and ω_{22} are strictly lower than 4, and that $\omega_{11}+\omega_{21}$ and $\omega_{12}+\omega_{22}$ are strictly greater than 4. Then, at price $p=(1,1,1)$, it is easy to check that the set of fixed price equilibria is described by

$$x_1=(4,\omega_{11}+\alpha,\omega_{12}-\alpha),$$

$$x_2=(4,\omega_{21}-\alpha,\omega_{22}+\alpha),$$

where α belongs to the closed interval $[0,4-\mathrm{Max}\,(\omega_{11},\omega_{22})]$. For $\alpha<4-\mathrm{Max}\,(\omega_{11},\omega_{22})$, demands in commodities 1 and 2 are both constrained. $\omega_{11}=\omega_{22}$ and $\alpha=4-\omega_{11}$ corresponds to a competitive equilibrium allocation. The degeneracy disappears when the initial endowment in commodity 0 is slightly changed.

As a conclusion, let us indicate possible directions for further research.

First, it must be possible to extend the foregoing results to economies with general consumption sets and production, following the lines initiated by Smale (1974b).

Second, it would be interesting to weaken the assumptions imposed on the rationing scheme (or to find a counterexample) so as to be able to deal with the queueing.

Third, it is a conjecture that the fixed price equilibrium manifold is connected (or, even stronger, homeomorphic to $R^{m(l+1)+l}$). Balasko (1975) has shown a similar property for the competitive equilibrium manifold. However, this fact does not follow from our analysis and this should be the subject of further research.

Finally, the equilibria where a change of regime occurs are of particular importance for economic analysis. Our results in this area are very preliminary and additional analysis is needed in this direction.

2. Proofs

We begin with the proof of the existence theorem and the closed graph property.

Lemma 2.1 (existence and continuity of demand). *Let p be a price system in P, ω_i a vector of initial endowment in X and $(\underline{z}_i, \bar{z}_i)$ a system of constraints in $(R^l_- \times R^l_+)$. Then there exists a unique x_i that maximizes u_i on the set*

$$B_i(p, \omega_i, \underline{z}_i, \bar{z}_i) = \{x \in X, p \cdot x = p \cdot \omega_i, \underline{z}_{ih} \leqq x_h - \omega_{ih} \leqq \bar{z}_{ih}, h = 1, \ldots, l\}.$$

Furthermore, x_i is a continuous function of $(p, \omega_i, \underline{z}_i, \bar{z}_i)$ on $(P \times X \times R^l_- \times R^l_+)$.

Proof of Lemma 2.1. The argument is fairly standard, so we shall only sketch a proof.

We observe that x_i maximizes u_i on the set $B_i(p, \omega_i, \underline{z}_i, \bar{z}_i)$ if and only if it maximizes u_i on the set $\{x \in B_i(p, \omega_i, {}_i, \bar{z}_i), u_i(x) \geqq u_i(\omega_i)\}$. This set is bounded. It is closed, by Assumption 1 on the closure of the indifference hypersurfaces. The existence of a maximizing x_i follows from the compactness of this set and the continuity of the function u_i. Uniqueness of the maximum follows from the convexity of the budget set and the strict quasi-concavity of the function u_i.

Continuity is proved as follows: Let $(p^r, \omega_i^r, \underline{z}_i^r, \bar{z}_i^r)$, $r = 1, \ldots$, be a sequence of prices, endowments and constraints converging to $(p, \omega_i, \underline{z}_i, \bar{z}_i)$ in $(P \times X \times R^l_- \times R^l_+)$. Let x_i^r, $r = 1, \ldots$, be the sequence of corresponding demands, which is bounded, and let x_i be a cluster point of this sequence. For continuity, it suffices to show that $u_i(x_i) \geqq u_i(x)$ for any x in the limit budget set $B_i(p, \omega_i, \underline{z}_i, \bar{z}_i)$. Take x in the limit budget set and observe that $x_0 > 0$. Let x^r be a sequence converging to x and defined by

$$x_h^r = \mathrm{Min}\,(\mathrm{Max}\,(\omega_{ih}^r + \underline{z}_{ih}^r, x_h), \omega_{ih}^r + \bar{z}_{ih}^r) \quad \text{for} \quad h = 1, \ldots, l,$$

$$x_0^r = p^r \cdot \omega_i^r - \sum_{h=1}^{l} p_h^r x_h^r.$$

Since $x_0 > 0$, for r large $x_0^r > 0$ and thus, by definition of x_i^r, $u_i(x_i^r) \geqq u_i(x^r)$. By continuity of u_i, we get that, in the limit, $u_i(x_i) \geqq u_i(x)$. Q.E.D.

Proof of Theorem 1.4. We begin with the existence proof which is a straightforward adaptation of the argument in Drèze (1975).

Let c be a constant strictly greater than $\sum_{i=1}^{m} \omega_{ih}$, for all $h=1,\dots,l$. Since for all $i=1,\dots,m$, and $h=1,\dots,l$, $\bar{f}_{ih}(\bar{q}_h)$ tends to $+\infty$, there exists $\bar{q}>0$ such that for all $i=1,\dots,m$, and $h=1,\dots,l$, $\bar{f}_{ih}(\bar{q})>c$. Similarly, there exists $\underline{q}<0$ such that for all $i=1,\dots,m$, and $h=1,\dots,l$, $\underline{f}_{ih}(\underline{q})<-c$. Let $Q=[\underline{q},\bar{q}]^l$.

We define a mapping $z(q)$ from Q to R^l, the aggregate excess demand function, as follows:

$$\text{For } \underline{q}\leqq q_j\leqq 0, \ \underline{z}_{ih}(q_h)=\underline{f}_{ih}(\underline{q}-q_h),\ \bar{z}_{ih}(q_h)=\bar{f}_{ih}(\bar{q}),$$

$$\text{for } 0\leqq q_h\leqq \bar{q},\ \underline{z}_{ih}(q_h)=\underline{f}_{ih}(\underline{q}),\ \bar{z}_{ih}(q_h)=\bar{f}_{ih}(\bar{q}-q_h).$$

Then $z(q)=\sum_{i=1}^{m}(x_i-\omega_i)$, where x_i maximizes u_i in the set $B_i(p,\omega_i,\underline{z}_i(q),\bar{z}_i(q))$. By Lemma 2.1, z is a continuous function of q.

Let Z be a compact convex subset of R^l containing the image of Q under z. For z in Z, let $g(z)$ be the set $\{q\in Q,\ q\cdot z\geqq q'\cdot z \text{ for all } q' \text{ in } Q\}$. The correspondence $z(q)\times g(z)$ from $Q\times Z$ into itself satisfies the assumptions of the Kakutani fixed point theorem and, hence, has a fixed point (q^*, z^*). We show that $z^*=0$. Suppose $z_h^*>0$, for some h (a similar argument holds in case $z_h^*<0$): then by the construction of g, $q_h^*=\bar{q}$; thus $\bar{z}_{ih}=0$ for all $i=1,\dots,m$, which in turn implies $z_h^*\leqq 0$, a contradiction. Observe that by Walras' law $z_0^*=\sum_{i=1}^{m}(x_{i0}-\omega_{i0})=0$. Finally, note that condition (c) of Definition 1.3 is indeed satisfied: by our choice of the constant c, only one side of the market is constrained.

We turn now to the proof that the graph of the equilibrium correspondence is closed.

Let $(p^r, (\omega_i^r), (x_i^r))$, $r=1,\dots$, be a sequence of points belonging to this graph and suppose that they converge to $(p,(\omega_i),(x_i))$. We want to show that (x_i) is an equilibrium at price p for the economy $\omega=(\omega_i)$. We thus have to find $(\underline{z}_{ih}, \bar{z}_{ih}, \bar{q}_h, \underline{q}_h)$ such that conditions (a), (b), (c), (d) of Definition 1.3 are satisfied. Observe that condition (a) is always satisfied.

Let c be a constant strictly greater than $\sum_{i=1}^{m}\omega_{ih}$, for all $h=1,\dots,l$. Let $\bar{q}>0$ and $\underline{q}<0$ be two constants such that $\bar{f}_{ih}(\bar{q})>c+1$ and $\underline{f}_{ih}(\underline{q})<-(c+1)$ for all $i=1,\dots,m$, and $h=1,\dots,l$. Clearly, for r large enough, we can associate to $(p^r, (\omega_i^r), (x_i^r))$ quantities $(\underline{z}_{ih}^r, \bar{z}_{ih}^r, \underline{q}_h^r, \bar{q}_h^r)$ such that (a), (b), (c), (d) of Definition 1.3 are satisfied, and, moreover:

$$\underline{z}_{ih}^r\geqq-(c+1),\quad \bar{z}_{ih}^r\leqq c+1 \quad \text{for} \quad i=1,\dots,m,\quad h=1,\dots,l;$$

$$\underline{q}_h^r\geqq\underline{q},\ \bar{q}_h^r\leqq\bar{q} \quad \text{for} \quad h=1,\dots,l;$$

$$x_{ih}^r-\omega_{ih}^r=\bar{z}_{ih}^r \quad \text{for some } i \text{ implies}$$

$$x_{jh}^r-\omega_{jh}^r>\underline{z}_{jh}^r+1 \quad \text{for all} \quad j=1,\dots,m;$$

$$x_{ih}^r - \omega_{ih}^r = \underline{z}_{ih}^r \quad \text{for some } i \text{ implies}$$
$$x_{jh}^r - \omega_{jh}^r < \bar{z}_{jh}^r - 1 \quad \text{for all} \quad j = 1, \ldots, m;$$

$$x_{ih}^r - \omega_{ih}^r < \bar{z}_{ih}^r \quad \text{for all} \quad i = 1, \ldots, m \quad \text{implies}$$
$$x_{ih}^r - \omega_{ih}^r < \bar{z}_{ih}^r - 1 \quad \text{for all} \quad i = 1, \ldots, m;$$

$$x_{ih}^r - \omega_{ih}^r > \underline{z}_{ih}^r \quad \text{for all} \quad i = 1, \ldots, m \quad \text{implies}$$
$$x_{ih}^r - \omega_{ih}^r > \underline{z}_{ih}^r + 1 \quad \text{for all} \quad i = 1, \ldots, m.$$

The sequence $(\underline{z}_{ih}^r, \bar{z}_{ih}^r, \underline{q}_h^r, \bar{q}_h^r)$ belongs to a compact set: Let $(\underline{z}_{ih}, \bar{z}_{ih}, \underline{q}_h, \bar{q}_h)$ be one of its accumulation points. We claim that $(p, (\omega_i), (x_i))$ associated with $(\underline{z}_{ih}, \bar{z}_{ih}, \underline{q}_h, \bar{q}_h)$ satisfies conditions (b), (c) and (d) of Definition 1.3. (b) follows from Lemma 2.1. By our choice of $(\underline{z}_i^r, \bar{z}_i^r)$, either $x_{ih} - \omega_{ih} \leqq \bar{z}_{ih} - 1$ for all $i = 1, \ldots, m$, or $x_{ih} - \omega_{ih} \geqq \underline{z}_{ih} + 1$ for all $i = 1, \ldots, m$; thus (c) is satisfied. Finally, if $x_{ih} - \omega_{ih} = \bar{z}_{ih}$ for some i, for r large enough, $x_{jh}^r - \omega_{jh}^r = \bar{z}_{jh}^r$ for some j along the converging subsequence, and (d) follows from the continuity of $\bar{f}$. Q.E.D.

We proceed now to the study of our main object, that is, the structure of the fixed price equilibrium correspondence. Unlike the competitive equilibria, the fixed price equilibria cannot be characterized as the zeroes of a function in a natural manner. We are thus led to the following construction:

We consider 2^l different cases. Each case is defined by a partition of $\{1, \ldots, l\}$ into two subsets $\bar{H}$ and $\underline{H}$ such that for h in $\bar{H}$, supply is *not* constrained on market h, while for h in $\underline{H}$, demand is *not* constrained on market h. These cases are overlapping: An equilibrium where r markets are cleared will be obtained in 2^r different cases. Let λ_i be a real number for $i = 1, \ldots, m$. We associate with the case $H = (\bar{H}, \underline{H})$ the function $F_H(p, q, (\omega_i, \lambda_i, x_i)_{i=1,\ldots,m})$ defined on $P \times R^l \times (X \times R \times X)^m$, taking its values in $R^{m(l+2)} \times R^{ml} \times R^l$ and described by its components (a, b, c) as follows:

$$\left.\begin{aligned} a_{ih} &= Du_{ih}(x_i) - \lambda_i p_h, && h = 0, 1, \ldots, l, \\ a_{il+1} &= p \cdot (x_i - \omega_i), && \\ b_{ih} &= \bar{g}_{ih}(q_h) - x_{ih} + \omega_{ih}, && h \text{ in } \bar{H}, \\ b_{ih} &= \underline{g}_{ih}(q_h) - x_{ih} + \omega_{ih}, && h \text{ in } \underline{H}, \end{aligned}\right\} \text{ for } i = 1, \ldots, m,$$
$$c_h = \sum_{i=1}^{m} (x_{ih} - \omega_{ih}), \qquad h = 1, \ldots, l,$$

where $\bar{g}_{ih}$ and $\underline{g}_{ih}$ are continuously differentiable monotone strictly increasing extensions of $\bar{f}_{ih}$ and $\underline{f}_{ih}$, respectively, on the real line.

Let $\mathscr{E}_H$ be the set of (p, ω, x) in $P \times (X \times X)^m$ such that x is a fixed-price equilibrium for the economy ω at price p, where for h in $\bar{H}$ (resp. h in $\underline{H}$) supply (resp. demand) is not constrained on market h.

Finally, let $\mathscr{F}_H$ be the set in $R^{m(l+2)} \times R^{ml} \times R^l$ defined by the following conditions:

(1) for $i = 1, \ldots, m$,

$$a_{i0} = 0,$$

$$a_{ih} \geqq 0, \quad b_{ih} \geqq 0, \quad a_{ih} b_{ih} = 0 \quad \text{for } h \text{ in } \bar{H},$$

$$a_{ih} \leqq 0, \quad b_{ih} \leqq 0, \quad a_{ih} b_{ih} = 0 \quad \text{for } h \text{ in } \underline{H},$$

$$a_{il+1} = 0;$$

(2) if $a_{ih} = 0$ for all $i = 1, \ldots, m$, then

$$b_{jh} = 0 \quad \text{for some } j, \quad h = 1, \ldots, l;$$

(3) $c_h = 0$ for $h = 1, \ldots, l$.

Lemma 2.2. The projection of $P \times R^l \times (X \times R \times X)^m$ on the space $P \times (X \times X)^m$ of prices, endowments and consumption bundles, restricted to $F_H^{-1}(\mathscr{F}_H)$, is a homeomorphism G_H of $F_H^{-1}(\mathscr{F}_H)$ onto $\mathscr{E}_H$. Moreover, G_H^{-1} is differentiable at every point (p, ω, x) of $\mathscr{E}_H$ whose image under $F_H \circ G_H^{-1}$ in $\mathscr{F}_H$ is such that for all $h = 1, \ldots, l$, there exists some i $(i = 1, \ldots, m)$ with $a_{ih} \neq 0$. Furthermore, market h is cleared at (p, ω, x) in $\mathscr{E}_H$ if and only if $\mathscr{F}_H(G_H^{-1}(p, \omega, x))$ has all its components a_{ih} equal to zero, for $i = 1, \ldots, m$.

Proof. First, we show that the image of $F_H^{-1}(\mathscr{F}_H)$ under G_H is contained in $\mathscr{E}_H$.

Let $(p, q, (\omega_i, \lambda_i, x_i))$ be a point in $F_H^{-1}(\mathscr{F}_H)$. We observe that for h in $\bar{H}$ (resp. h in $\underline{H}$), q_h is non-negative (resp. non-positive): Otherwise the conditions $b_{ih} \geqq 0$ (resp. $b_{ih} \leqq 0$) would imply $x_{ih} - \omega_{ih} < 0$ (resp. $x_{ih} - \omega_{ih} > 0$) for all $i = 1, \ldots, m$, which is incompatible with $c_h = 0$. This enables us to define $\bar{z}_{ih} = \bar{f}_{ih}(q_h)$ for h in $\bar{H}$ (resp. $\underline{z}_{ih} = \underline{f}_{ih}(q_h)$ for h in $\underline{H}$) for all $i = 1, \ldots, m$, and we choose $\underline{z}_{ih}$ negative (resp. $\bar{z}_{ih}$ positive) large in absolute value. Thus conditions (c) and (d) of Definition 1.3 are satisfied. (1) of the definition of $\mathscr{F}_H$ is interpreted as the Kuhn–Tucker necessary and sufficient conditions for the consumer's maximization problem.[7] Thus (b) is satisfied. Finally, (3), with the conditions $a_{il+1} = p \cdot (x_i - \omega_i) = 0$ for $i = 1, \ldots, m$, leads to (a) by Walras' law.

[7]Sufficiency follows by Theorem 10.1.1 in Mangasarian (1969). In fact, u_i is pseudoconcave since it is strictly quasi-concave and, by the assumption of non-zero Gaussian curvature, $Du_i \neq 0$.

Conversely, we now show that G_H^{-1} exists and is a continuous function which maps $\mathscr{E}_H$ in $F_H^{-1}(\mathscr{F}_H)$. We define $\lambda_i = Du_{i0}(x_i)$, and q_h as the unique solution of the equation $\mathrm{Min}_{i=1,\ldots,m}(\bar{f}_{ih}(q_h) - x_{ih} + \omega_{ih}) = 0$ for h in $\bar{H}$ (resp. $\mathrm{Max}_{i=1,\ldots,m}(\underline{f}_{ih}(q_h) - x_{ih} + \omega_{ih}) = 0$ for h in $\underline{H}$). By the conditions $a_{i0} = 0$ for $i = 1, \ldots, l$, and (2) of the definitions of $\mathscr{F}_H$, this definition gives clearly the only possible inverse, which is continuous. Moreover, $F_H(G_H^{-1}(p, \omega, x))$ belongs to $\mathscr{F}_H$: (a) gives (3), (b) and the Kuhn–Tucker conditions lead to (1), and (2) is warranted by our choice of q.

Furthermore, if for some i, $a_{ih} \neq 0$, then the equation $b_{ih} = 0$ gives q_h, which is thus a differentiable function in the neighborhood. Finally if, for some h, $a_{ih} = 0$ for all $i = 1, \ldots, m$, by Kuhn–Tucker no agent is constrained on market h, and thus market h is cleared. Q.E.D.

To prove Theorem 1.5, we are now led to a two-step procedure. First we will show that each $F_H^{-1}(\mathscr{F}_H)$ is a piecewise continuously differentiable manifold with corners (i.e., is locally diffeomorphic to $R^{m(l+1)} \times R_+^{\bar{H}} \times R_-^{\underline{H}}$, except along a finite number of submanifolds of lower dimensions). Through the preceding lemma, this will also apply to the $\mathscr{E}_H$. Then we will show that the various $\mathscr{E}_H$ patch together along the sets of equilibria where some market is cleared.

To derive the structure of $F_H^{-1}(\mathscr{F}_H)$, we will work locally around a point $(p, q, (\omega_i, \lambda_i, x_i))$ and build a chart for a neighborhood of that point in $F_H^{-1}(\mathscr{F}_H)$. We first remark that if $(a, b, c) = F_H(p, q, (\omega_i, \lambda_i, x_i))$, the shape of the neighborhood of $(p, q, (\omega_i, \lambda_i, x_i))$ is entirely determined by the coordinates which are zero in (a, b, c), since by continuity the others are not binding. More precisely, let $r(h)$ be the number of agents i such that $a_{ih} = b_{ih} = 0$, and $r = \sum_{h=1}^{l} r(h)$. Let proj_{abc} be the projection of $R^{m(l+2)+ml+l}$ on the subspace $R^{m(l+2)+l} \times R^r$ generated by the zero coordinates of (a, b, c). Then $F_H^{-1}(\mathscr{F}_H)$ is locally equal to $\mathrm{proj}_{abc} F_H^{-1}(\mathrm{proj}_{abc} \mathscr{F}_H)$.

This reduction of dimension is essential for our analysis: In fact the statement that $\mathrm{proj}_{abc} F_H$ is transverse to the origin is exactly the statement that F_H is transverse to $\mathscr{F}_H$.

Lemma 2.3. $\mathrm{Proj}_{abc} F_H$ *is transverse to the origin (that is, at any* $(p, q, (\omega_i, \lambda_i, x_i))$ *such that* $(a, b, c) = F_H(p, q, (\omega_i, \lambda_i, x_i))$ *and* $\mathrm{proj}_{abc} F_H = 0$, *the jacobian of* $\mathrm{proj}_{abc} F_H$ *is of full rank).*

Proof. We associate to each of the components of $\mathrm{proj}_{abc} F_H$ one variable, obtaining thus a square matrix.

We observe that, for a given i, the components of the type $(Du_{ih} - \lambda_i p_h)$, when differentiated with respect to (x_i, λ_i) at a point where $Du_{ih} - \lambda_i p_h = 0$, give, up to the positive constant $\lambda_i = Du_{i0}(x_i)$, a subset of the first $(l+1)$ rows of the determinant:

$$\begin{vmatrix} D^2 u & Du \\ Du & 0 \end{vmatrix}.$$

This determinant is equal to the opposite of the Gaussian curvature of the indifference hypersurface through x_i [see Debreu (1972, p. 612)]. Thus, using Assumption 1, we can choose a subset of the variables (x_i, λ_i) (the number of these variables being equal to the number of components a_{ih}, $h=0,1,\ldots,l$, equal to zero) such that the corresponding columns of the jacobian of $\mathrm{proj}_{abc}F_H$ are independent.

To the components b_{ih}, $h=1,\ldots,l$, which are equal to zero, we assign the variable ω_{ih}. The budget constraint a_{il+1} is differentiated with respect to ω_{i0}. Finally, with the l components c_h we associate the corresponding variables q_h.

We claim that the matrix associated with the above choice of variables is non-singular. The following argument shows that by suitable manipulation of its columns one obtains a matrix whose columns are clearly linearly independent.

First we notice that the columns corresponding to the variables ω_{i0}, $i=1,\ldots,m$, are unit vectors with the entry (-1) on the principal diagonal of the matrix. Hence, by a suitable column operation, we can put zeroes on all the lines corresponding to the budget constraints, except for their intersection with the principal diagonal. Second we consider the columns associated with the last l components c_h, $h=1,\ldots,l$. Their only non-zero entries are equal to $f'_{ih}(q_h)$ and appear at the rows b_{ih} [such a row always exists by (1) and (2) of the definition of $\mathscr{F}_H$]. Furthermore, the columns associated with the variable ω_{ih} have an entry $(+1)$ on row b_{ih} and (-1) on row c_h. Multiplying each of these columns by $-f'_{ih}(q_h)$ and adding the result to the column in question, we obtain a column vector whose only non-zero entry is on the principal diagonal and equals $\sum_{(i \text{ such that } b_{ih}=0)} f'_{ih}(q_h)$. By a suitable combination, we can now put zeroes on all the lines corresponding to the components c, except for their intersection with the principal diagonal. The resulting matrix is block diagonal and hence clearly non-singular. Q.E.D.

Corollary 2.4. (i) *Suppose that the utility functions u_i, $i=1,\ldots,m$, are $(l+2)$ times continuously differentiable and the rationing scheme $\bar{f}_{ih}$, $\underline{f}_{ih}$, $i=1,\ldots,m$, $h=1,\ldots,l$, is $(l+1)$ times continuously differentiable. Then, except for a closed set of economics ω of Lebesgue measure zero in X^m, $\mathrm{proj}_{abc}F_H(\omega;\cdot)$ is transverse to the origin.*

(ii) *Except for a closed set of economies and prices (ω, p) of Lebesgue measure zero in $X^m \times P$, $\mathrm{proj}_{abc}F_H(\omega, p;\cdot)$ is transverse to the origin.*

Proof. It follows from a simple generalization of a theorem of Dierker (1974, p. 91) which is an adaptation of the transversal density theorem [Abraham and Robbin (1967, p. 48)] in finite dimensional spaces.

If we consider $\mathrm{proj}_{abc}F_H$ as the evaluation map of $\mathrm{proj}_{abc}F_H(\omega;\cdot)$ which maps a set of dimension $m(l+2)+2l$ into a set of dimension $m(l+2)+l+r$ [and thus where the codimension of the origin is equal to $m(l+2)+l+r$], we obtain (i).

If we consider $\mathrm{proj}_{abc}F_H$ as the evaluation map of $\mathrm{proj}_{abc}F_H(\omega, p;\cdot)$ which maps a set of dimension $m(l+2)+l$ into a set of dimension $m(l+2)+l+r$, we obtain (ii). Q.E.D.

Given $(p, q, (\omega_i, \lambda_i, x_i))$ and $(a, b, c) = F_H(p, q, (\omega_i, \lambda_i, x_i))$, let us denote $\bar{K}$ the set of h in $\bar{H}$ such that $\sum_{i=1}^m a_{ih} = 0$, and $\underline{K}$ the set of h in $\underline{H}$ such that $\sum_{i=1}^m a_{ih} = 0$.

Lemma 2.5. A neighborhood of the origin in $\text{proj}_{abc}\mathscr{F}_H$ *is locally homeomorphic to a neighborhood of the origin in*

$$\prod_{h \text{ in } \bar{K}} (R_+ \times R^{r(h)-1}) \prod_{h \text{ in } \underline{K}} (R_- \times R^{r(h)-1}) \prod_{h \text{ not in } \bar{K} \cup \underline{K}} (R^{r(h)}).$$

[*it reduces to an isolated point when* $r(h) = 0$ *for all* $h = 1, \ldots, l$].

Proof. For every h such that $r(h) \neq 0$, we consider the following change of coordinates, which applies to agents i such that $a_{ih} = b_{ih} = 0$:

$$s_{ih} = a_{ih} - b_{ih}$$

and

$$t_{ih} = \text{Min}\,(a_{ih}, b_{ih}) \qquad \text{if } h \text{ is in } \bar{H},$$

or

$$t_{ih} = \text{Max}\,(a_{ih}, b_{ih}) \qquad \text{if } h \text{ is in } \underline{H}.$$

These functions define a homeomorphism of $R^{2r(h)}$ onto itself. Under this homeomorphism, $\text{proj}_{abc}\mathscr{F}_H$ is locally defined by[8]

– for h not in $\bar{K} \cup \underline{K}$: $t_{ih} = 0$ for all i;

– for h in $\bar{K}$: $t_{ih} = 0$ for all i,

and (s_{ih}, i such that $a_{ih} = b_{ih} = 0$) belongs to $R^{r(h)} - \text{int}\, R_-^{r(h)}$;

– for h in $\underline{K}$: $t_{ih} = 0$ for all i,

and (s_{ih}, i such that $a_{ih} = b_{ih} = 0$) belongs to $R^{r(h)} - \text{int}\, R_+^{r(h)}$.

Since $R^{r(h)} - \text{int}\, R_-^{r(h)}$ is homeomorphic to $R_+ \times R^{r(h)-1}$, we obtain the desired result. Q.E.D.

We are now able to complete the first step of the proof of Theorem 1.5:

Lemma 2.6. (i) $\mathscr{E}_H$ *is a piecewise continuously differentiable manifold with corners of dimension* $m(l+1)+l$. (ii) *Except for a closed set of economies* ω *of measure zero in* X^m, *the set* $\mathscr{E}_H(\omega) = \{(p, x)$ *in* $P \times X^m$, (ω, p, x) *belongs to* $\mathscr{E}_H\}$ *is a*

[8]Given a set S, $\text{int}\,S$ stands for the interior of S and ∂S for its boundary.

piecewise continuously differentiable manifold with corners of dimension l.[9]

More precisely, let (ω, p, x) be a point in $\mathscr{E}_H$ [resp. (p, x) a point in $\mathscr{E}_H(\omega)$] where markets h in $\bar{K}$ and h in $\underline{K}$, respectively subsets of $\bar{H}$ and $\underline{H}$, are cleared. Then there exists a homeomorphism between a neighborhood of (ω, p, x) in $\mathscr{E}_H$ [resp. (p, x) in $\mathscr{E}_H(\omega)$] and a neighborhood of the origin in $R^{m(l+1)} \times R^{l-\bar{K}-\underline{K}} \times R^{\bar{K}}_{+} \times R^{\underline{K}}_{-}$ (resp. $R^{l-\bar{K}-\underline{K}} \times R^{\underline{K}}_{-}$). Moreover, except for points (ω, p, x) on a finite number of submanifolds of lower dimensions, this homeomorphism can be chosen to be a diffeomorphism.

Proof. We begin with (i).

Let $(p, q, (\omega_i, \lambda_i, x_i))$ be a point in $F_H^{-1}(\mathscr{F}_H)$ and $(a, b, c) = F_H(p, q, (\omega_i, \lambda_i, x_i))$. By Lemma 2.3, $\mathrm{proj}_{abc} F_H$ is a submersion from $P \times R^l \times (X \times R \times X)^m$ in $R^{m(l+2)} \times R^l \times R^r$ at $(p, q, (\omega_i, \lambda_i, x_i))$. We can complete it by $(ml + l + m - r)$ real linear independent functions so as to obtain a local diffeomorphism from $P \times R^l \times (X \times R \times X)^m$ into $R^{2l} \times R^{m(2l+3)}$ at the point $(p, q, (\omega_i, \lambda_i, x_i))$. From our previous remark, $F_H^{-1}(\mathscr{F}_H)$ is locally equal to the inverse image by F_H of $\mathrm{proj}_{abc}\mathscr{F}_H \times R^{ml+l+m-r}$. The composition of this diffeomorphism with the homeomorphism of Lemma 2.5 on one side, and with the projection of Lemma 2.2 on the other side gives the desired result. Moreover, when $a_{ih} + b_{ih} \neq 0$ for all i and h, $r = 0$ and we stay with the initial diffeomorphism composed with the projection. By Lemma 2.5, we lose this diffeomorphism at the points where simultaneously $a_{ih} = b_{ih} = 0$ for some i. These points lie on submanifolds by Lemma 2.3, since they are obtained as the inverse image of the origin by a transversal mapping.

(ii) is proved by the same argument, using Corollary 2.4 (ii), since a finite union of closed sets of measure zero is a closed set of measure zero. Q.E.D.

We proceed now to the second part of the proof of Theorem 1.5. We will need the following technical lemma:

Lemma 2.7. Let A be a subset of a finite dimensional vector space and suppose that there exists a homeomorphism ϕ between a neighborhood of a point a in A and a neighborhood of the origin in $R^s \times R^t_{+}$. Let ∂A be the inverse image under ϕ of $R^s \times \partial R^t_{+}$, and let $\partial\psi$ be a given homeomorphism between a neighborhood of a in ∂A and a neighborhood of the origin in $R^s \times \partial R^t_{+}$. Then $\partial\psi$ can be extended in a local homeomorphism ψ between A and $R^s \times R^t_{+}$.

Proof. Without loss of generality, we can work with the whole $R^s \times R^t_{+}$. Let $\partial\alpha = \partial\psi \circ \partial\phi^{-1}$ be the homeomorphism of change of chart on $R^s \times \partial R^t_{+}$. The problem amounts to extending this homeomorphism to a homeomorphism of $R^s \times R^t_{+}$ into itself.

[9]A (continuously differentiable) manifold with corners of dimension n is a manifold homeomorphic (diffeomorphic) at every point to an open subset of R^n_{+}. A piecewise continuously differentiable manifold with corners is a continuous manifold with corners, which is a finite union of continuously differentiable manifolds with corners.

Any point b in $R^s \times R^t_+$ can be written in a unique way as

$$b = \partial b + \lambda v,$$

where ∂b belongs to $R^s \times \partial R^t_+$, v is a vector of components 0 in R^s and $+1$ in R^t and λ is a positive scalar. Define

$$\alpha(b) = \partial\alpha(\partial b) + \lambda v;$$

α is clearly a homeomorphism of $R^s \times R^t_+$ into itself. $\psi = \alpha \circ \phi$ is the extension we are looking for. Q.E.D.

Proof of Theorem 1.5. We have to patch together the various $\mathscr{E}_H$ along their boundaries.

By Lemma 2.6, if markets $\bar{K}$ and $\underline{K}$ are cleared at (p, ω, x), (p, ω, x) belongs to the boundary of $2^{|\bar{K}|+|\underline{K}|}$ manifolds with corners of the type described in Lemma 2.6: This gives us the right number of orthants to obtain $R^{\bar{K}} \times R^{\underline{K}}$. As previously remarked, these manifolds are disjoint, except for their boundaries along which they intersect. Thus to build up the desired homeomorphism between a neighborhood of (p, ω, x) and the neighborhood of the origin in $R^{m(l+1)} \times R^l$, we have simply to modify the $2^{|\bar{K}|+|\underline{K}|}$ homeomorphisms, so that they agree on the boundaries of the manifolds. This is done by induction, using Lemma 2.7: First take the points where all markets $\bar{K}$ and $\underline{K}$ are cleared. This is a submanifold of dimension $m(l+1)+l-|\bar{K}|-|\underline{K}|$ (by Lemma 2.6). Select one chart for this submanifold. Now take all the possible cases where all markets are cleared except one, which is either in excess supply or in excess demand. We obtain subsets of $\mathscr{E}_H$ which are homeomorphic to $R^{m(l+1)} \times R^{l-\bar{K}-\underline{K}} \times R_+$ or $R^{m(l+1)} \times R^{l-\bar{K}-\underline{K}} \times R_-$. Pick homeomorphisms for these subsets. Apply Lemma 2.7 to modify them so that they agree on the points where all markets are cleared. We can thus increase dimension one by one, applying Lemma 2.7 to get the desired result.

The second part of the theorem is proved along the same lines, using (ii) of Lemma 2.6 and the fact that a finite union of closed sets of measure zero is of measure zero. Q.E.D.

Proof of Theorem 1.6. By Corollary 2.4(ii), taking the union of the closed sets (ω, p) of measure zero for all possible proj_{abc} and all possible H, we know that $\text{proj}_{abc} F_H$ is transverse to the origin at any equilibrium $(q, x_i, \lambda_i))$. $\text{proj}_{abc} F_H$ is a mapping from a set of dimension $m(l+2)+l$ into a set of dimension $m(l+2)+l+r$ [where we recall that r is the number of (ih) such that $a_{ih}=b_{ih}=0$]. Transversality then implies that for the set of endowments and prices that we are studying, all the equilibria are such that for all i and h, $a_{ih}+b_{ih} \neq 0$. Thus, by Lemma 2.6, these equilibria lie in a differentiable region of the manifold.

Moreover, when $r=0$, the set of $(q, (x_i, \lambda_i))$ is a discrete set. Since it is bounded, it is a finite set. The implicit function theorem applied at each of its points gives (b) of Theorem 1.6. Q.E.D.

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